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On the relation between heterogeneous uniform 2-algebraic closure operators and heterogeneous algebras

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Abstract

In this paper it is shown that the heterogeneous translation of a theorem due to Birkhoff and Frink, proved not to be true by Mathiessen, is valid for a restricted class of heterogeneous algebraic closure operators.

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The theorem of Herbrand-Schmidt-Wang [5, 8, 10] states that every heterogeneous theory can be mechanically replaced by an equally powerful homogeneous theory. As a consequence, the heterogeneous theories are considered as "inessential variations" of the corresponding homogeneous theories [9], moreover, replaceability has lead to believe that by using the heterogeneous theories nothing relevant can be accomplished. However, as has been pointed out by Hook [6], for consistency proofs the heterogeneous theories are fundamentally indipensable; besides that Goguen and Meseguer [4] have shown, among other interesting things, that the deduction rules of the ordinary (or homogeneous) equational logic, when translated naively, are not sound for the heterogeneous deduction, i.e., heterogeneous equational logic is not an inessential variation of homogeneous equational logic.

On the other hand, it happens that some theorems of ordinary (or homogeneous) universal algebra can not be automatically generalized to heterogeneous universal algebra, e.g., Mathiessen [7] shows that there exist heterogeneous algebraic closure systems that can not be concretely represented as the set of the subalgebras of a heterogeneous algebra. This is not the case for the homogeneous algebraic closure systems, according to a theorem of Birkhoff and Frink [1].

The aim of this paper is to specify a class of heterogeneous algebraic closure operators, the so-called uniform 2-algebraic closure operators, that allows us to obtain a concrete representation for them. In order to obtain such a thing, we have studied, firstly, the behaviour of the heterogeneous algebraic closure operator $\mathrm{Sg}^{(\underline{A})}$ (that gives, when applied to an $X \subseteq_S A$, the subalgebra of the heterogeneous algebra $\underline{A}$ generated by X) with respect to the supports of the subsets of A; secondly, we have taken into account the relation between the homogeneous topological closure systems and the operator domains, as can be found, e.g., in Cohn [3].

Terminology. Every set we consider will be member of a Grothendieck universe U, fixed once and for all.

Let S be a set (of sorts or types), then an S-sorted set A is a family of component sets A_s for each index s in S, i.e., a mapping $A = (A_s : s \in S)$ from S to U, and the support of A, denoted by $\mathrm{supp}(A)$, is the set $\{s \in S : A_s \neq \emptyset\}$; $P_S(A)$ is the set whose members are those S-sorted sets X such that $X \subseteq_S A$, i.e., for every s in S, $X_s \subseteq A_s$; $\underline{P}_S(A)$ is the Boolean set algebra of subsets of A; $K(\underline{P}_S(A))$ is the set of all compacts in $\underline{P}_S(A)$, i.e., the set of all X in $P_S(A)$ such that card $(\sqcup X) < \aleph_0$, being $\sqcup X$ the coproduct of $X = (X_s : s \in S)$; and $2 - K(\underline{P}_S(A))$ is the set of all 2-compacts in $\underline{P}_S(A)$, i.e., the set of all X in $P_S(A)$ such thast card $(\sqcup X) < 2$.

For s in S, we denote by $\ulcorner s \urcorner$ the mapping from 1 to S whose range is $\{s\}$, i.e., the image of s under the canonical mapping η_S from S to the underlying set of the free monoid, $\underline{M}(S)$, on S.

Finally, given a set I and an I-indexed family $(A_i : i \in I)$ of S-sorted sets, we denote by $\bigcup_S (A_i : i \in I)$ the S-sorted set such that, for every s in S,

$$\bigcup_S (A_i : i \in I)(s) = \bigcup (A_i(s) : i \in I).$$

DEFINITION. Given a set S, an S-sorted signature is an ordered triple $\underline{\Sigma} = (\Sigma, d, cd)$ consisting of a set Σ, a mapping d from Σ into the underlying set of the free monoid on S, and a mapping cd from Σ to S.

Remark. To give an S-sorted signature $\underline{\Sigma} = (\Sigma, d, cd)$, amounts to the same as to give a pairwise disjointed $M(S) \coprod S$-indexed family $(\Sigma_{w,s} : (w, s) \in M(S) \coprod S)$.

DEFINITION. A heterogeneous $\underline{\Sigma}$-algebra structure on an S-sorted set A is a member of $\prod(\underline{\mathrm{Set}}(A_{d(\sigma)}, A_{cd(\sigma)}) : \sigma \in \Sigma)$, where for a σ in Σ such that

$$d(\sigma) = (s_i : i \in n),$$

$$A_{d(\sigma)} = \prod(A_{s_i} : i \in n),$$

and $\underline{\mathrm{Set}}(A_{d(\sigma)}, A_{cd(\sigma)})$. is the set of all mappings from $A_{d(\sigma)}$ to $A_{cd(\sigma)}$.

Then a heterogeneous Σ-algebra is an ordered pair $\underline{A} = (A, F)$ consisting of an S-sorted set A and a heterogeneous $\underline{\Sigma}$-algebra structure F on A.

DEFINITION. Given a heterogeneous $\underline{\Sigma}$-algebra $\underline{A} = (A, F)$, let $E^{(\underline{A})}$ be the operator on $P_S(A)$ defined by

$$E^{(\underline{A})}(X) = X \cup_S \left(\bigcup_{\sigma \in \Sigma_{\cdot,s}} F_\sigma(X_{d(\sigma)}) : s \in S \right), \qquad X \in P_S(A),$$

where, for a given s in S, $\Sigma_{\cdot,s} = cd^{-1}(\{s\})$.

Then, given an X in $P_S(A)$ we obtain the $\mathbb{N}$ indexed family $\left(E_n^{(\underline{A},X)} : n \in \mathbb{N}\right)$ defined by recursion as follows:

$$E_0^{(\underline{A},X)} = X,$$

$$E_{n+1}^{(\underline{A},X)} = E^{(\underline{A})}\left(E_n^{(\underline{A},X)}\right).$$

Proposition 3

Let A be a heterogeneous $\underline{\Sigma}$-algebra and $X, Y \in P_S(A)$. If $\mathrm{supp}(X) = \mathrm{supp}(Y)$, *then for every n in $\mathbb{N}$.*

$$\mathrm{supp}\left(E_n^{(\underline{A},X)}\right) = \mathrm{supp}\left(E_n^{(\underline{A},Y)}\right).$$

Proof. The base of the induction being obvious, let us suppose that for $k \geq 0$,

$$\operatorname{supp}\left(E_k^{(\underline{A},X)}\right) = \operatorname{supp}\left(E_k^{(\underline{A},Y)}\right).$$

We want to show that

$$\operatorname{supp}\left(E_{k+1}^{(\underline{A},X)}\right) = \operatorname{supp}\left(E_{k+1}^{(\underline{A},Y)}\right),$$

and for this it suffices to prove, by symmetry, that

$$\operatorname{supp}\left(E_{k+1}^{(\underline{A},X)}\right) \subseteq \operatorname{supp}\left(E_{k+1}^{(\underline{A},Y)}\right).$$

Let $t \in \operatorname{supp}(E_{k+1}^{(\underline{A},X)})$, then, since

$$E_{k+1}^{(\underline{A},X)} = E^{(\underline{A})}\left(E_k^{(\underline{A},X)}\right),$$

we have that ${E_k^{(\underline{A},X)}}_t$ is nonempty or that

$$\bigcup_{\sigma \in \Sigma_{\cdot,t}} F_\sigma\left({E_k^{(\underline{A},X)}}_{d(\sigma)}\right)$$

is nonempty.

If ${E_k^{(\underline{A},X)}}_t \neq \emptyset$, then $t \in \operatorname{supp}(E_k^{(\underline{A},X)})$, hence $t \in \operatorname{supp}\left(E_k^{(\underline{A},Y)}\right)$, therefore $t \in \operatorname{supp}(E_{k+1}^{(\underline{A},Y)})$. If

$$\bigcup_{\sigma \in \Sigma_{\cdot,t}} F_\sigma\left({E_k^{(\underline{A},X)}}_{d(\sigma)}\right) \neq \emptyset,$$

then, for some $(s_j : j \in n) \in M(S)$ and $\sigma \in \Sigma_{(s_j:j\in n),t}$, we have that

$$F_\sigma\left({E_k^{(\underline{A},X)}}_{(s_j:j\in n)}\right) \neq \emptyset.$$

If $(s_j : j \in n)$ is ϵ, the empty word on S, then, obviously, $F_\sigma({E_k^{(\underline{A},Y)}}_\epsilon) \neq \emptyset$, hence $t \in \operatorname{supp}(E_{k+1}^{(\underline{A},Y)})$; if $(s_j : j \in n) \neq \epsilon$, then for every $j \in n$, ${E_k^{(\underline{A},X)}}_{s_j} \neq \emptyset$, i.e., for every $j \in n$, $s_j \in \operatorname{supp}(E_k^{(\underline{A},X)})$, hence for every $j \in n$, $s_j \in \operatorname{supp}(E_k^{(\underline{A},Y)})$, therefore ${E_k^{(\underline{A},Y)}}_{(s_j:j\in n)} \neq \emptyset$, consequently ${E_{k+1}^{(\underline{A},Y)}}_t \neq \emptyset$, i.e., $t \in \operatorname{supp}(E_{k+1}^{(\underline{A},Y)})$. In this way we have proved that

$$\operatorname{supp}\left(E_{k+1}^{(\underline{A},X)}\right) \subseteq \operatorname{supp}\left(E_{k+1}^{(\underline{A},Y)}\right).$$

Now we can affirm that for every n in $\mathbb{N}$,

$$\operatorname{supp}\left(E_n^{(\underline{A},X)}\right) = \operatorname{supp}\left(E_n^{(\underline{A},Y)}\right). \square$$

DEFINITION. Given a heterogeneous $\underline{\Sigma}$-algebra $\underline{A}$, let $\mathrm{Sg}^{(\underline{A})}$ be the operator on $P_S(A)$ defined by:

$$\mathrm{Sg}^{(\underline{A})}(X) = \bigcup_S \left(E_n^{(\underline{A},X)} : n \in \mathbb{N} \right), \qquad X \in P_S(A).$$

Remark. If $\underline{A}$ is a heterogeneous $\underline{\Sigma}$-algebra and $X \in P_S(A)$, $\mathrm{Sg}^{(\underline{A})}(X)$ is the heterogeneous $\underline{\Sigma}$-subalgebra of $\underline{A}$ generated by X.

Proposition 2

Let $\underline{A}$ be a heterogeneous $\underline{\Sigma}$-algebra, then $\mathrm{Sg}^{(\underline{A})}$ is a heterogeneous algebraic closure operator in $\underline{P}_S(A)$, i.e., $\mathrm{Sg}^{(\underline{A})}$ is inflationary, isotone, idempotent and such that for every X in $P_S(A)$,

$$\mathrm{Sg}^{(\underline{A})}(X) = \bigcup_S \left(\mathrm{Sg}^{(\underline{A})}(F) : F \in K(\underline{P}_S(X)) \right).$$

Moreover,

$$\underline{\mathrm{Im}}\left(\mathrm{Sg}^{(\underline{A})}\right) = \left(\mathrm{Im}\left(\mathrm{Sg}^{(\underline{A})}\right), \subseteq_S\right)$$

is an algebraic lattice, and the compacts in $\underline{\mathrm{Im}}(\mathrm{Sg}^{(\underline{A})})$ are the values under $\mathrm{Sg}^{(\underline{A})}$ of the compacts in $\underline{P}_S(A)$.

Proof. Identical to the similar and well known proof for the homogeneous universal algebra. □

Proposition 3

Let $\underline{A}$ be a heterogeneous $\underline{\Sigma}$-algebra and $X \in P_S(A)$. Then

$$\mathrm{supp}\left(\mathrm{Sg}^{(\underline{A})}(X)\right) = \bigcup \left(\mathrm{supp}\left(E_n^{(\underline{A},X)}\right) : n \in \mathbb{N} \right).$$

Proof. If t belongs to $\mathrm{supp}(\mathrm{Sg}^{(\underline{A})}(X))$, then, since

$$\mathrm{Sg}^{(\underline{A})}(X) = \bigcup_S \left(E_n^{(\underline{A},X)} : n \in \mathbb{N} \right),$$

it follows that

$$\bigcup \left({E_n^{(\underline{A},X)}}_t : n \in \mathbb{N} \right) \neq \emptyset,$$

and therefore, for some $k \in \mathbb{N}$, ${E_k^{(\underline{A},X)}}_t \neq \emptyset$, hence $t \in \bigcup(\mathrm{supp}(E_n^{(\underline{A},X)}) : n \in \mathbb{N})$. Conversely, if t belongs to $\bigcup(\mathrm{supp}(E_n^{(\underline{A},X)}) : n \in \mathbb{N})$, then, for some $k \in \mathbb{N}$, $t \in \mathrm{supp}(E_k^{(\underline{A},X)})$, hence ${E_k^{(\underline{A},X)}}_t \neq \emptyset$, but

$${E_k^{(\underline{A},X)}}_t \subseteq \mathrm{Sg}^{(\underline{A})}(X)_t,$$

therefore $\mathrm{Sg}^{(\underline{A})}(X)_t \neq \emptyset$, i.e., $t \in \mathrm{supp}(\mathrm{Sg}^{(\underline{A})}(X))$. □

Corollary 4

Let $\underline{A}$ be a heterogeneous Σ-algebra, and $X, Y \in P_S(A)$. If $\mathrm{supp}(X) = \mathrm{supp}(Y)$, then

$$\mathrm{supp}\left(\mathrm{Sg}^{(\underline{A})}(X)\right) = \mathrm{supp}\left(\mathrm{Sg}^{(\underline{A})}(Y)\right).$$

The property of the heterogeneous algebraic closure operator $\mathrm{Sg}^{(\underline{A})}$, pointed out in the Corollary 4 (a type of uniformity, without interest in the homogeneous case, but not so in the heterogeneous case, as will be seen later in the proof of the Proposition 6) together with the well known concept of homogeneous topological closure operator, has lead us to propose the notion, defined below, of heterogeneous uniform 2-algebraic closure operator in $\underline{P}_S(A)$, that will allow us to obtain the desired concrete representation, mentioned in the introduction of this paper.

DEFINITION. Given an S-sorted set A, a heterogeneous uniform 2-algebraic closure operator in $\underline{P}_S(A)$ is an operator J on $P_S(A)$ subject to the following three axioms:

(i) J is a heterogeneous closure operator.

(ii) J is 2-algebraic, i.e., for every X in $P_S(A)$,

$$J(X) = \bigcup_S (J(L) : L \in 2 - K\,(\underline{P}_S(X))).$$

(iii) J is uniform, i.e., for every X, Y in $P_S(A)$, if $\mathrm{supp}(X) = \mathrm{supp}(Y)$, then $\mathrm{supp}J(X) = \mathrm{supp}J(Y)$.

Remark. If $\underline{A}$ is a heterogeneous Σ-algebra, then $\mathrm{Sg}^{(\underline{A})}$ is a heterogeneous uniform algebraic closure operator in $\underline{P}_S(A)$, by Proposition 2 and Corollary 4.

Proposition 5

Let $\underline{A}$ an S-sorted set and J a heterogeneous uniform closure operator in $\underline{P}_S(A)$. Then there exists a heterogeneous uniform 2-algebraic closure operator J^* in $\underline{P}_S(A)$ such that $J^* \leq J$, i.e., for every X in $P_S(A)$, $J^*(X) \subseteq_S J(X)$, and for every other heterogeneous uniform 2-algebraic closure operator J' in $\underline{P}_S(A)$, if $J' \leq J$, then $J' \leq J^*$.

Proof. It is enough to take J^ω as the mapping of $P_S(A)$ into itself such that

$$J^\omega(X) = \bigcup_S (J_2^n(X) : n \in \mathbb{N}),$$

for every X in $P_S(A)$, where $(J_2^n : n \in \mathbb{N})$ is the unique mapping from $\mathbb{N}$ into $\underline{\mathrm{Set}}(P_S(A), P_S(A))$ such that $J_2^0 = \mathrm{id}_{P_S(A)}$, and for every n in $\mathbb{N}$, $J_2^{n+1} = J_2 \circ J_2^n$, where J_2 is the operator on $P_S(A)$ defined by:

$$J_2(X) = X \cup_S \bigcup_S (J(L) : L \subset 2 - K(\underline{P}_S(X))), \qquad X \in P_S(A). \square$$

Proposition 6

Let A be an S-sorted set and J a heterogeneous uniform 2-algebraic closure operator in $\underline{P}_S(A)$. Then there exists an S-sorted signature $\underline{\Sigma}$ such that for every σ in Σ the length of the string $d(\sigma)$ is at most one, and a heterogeneous $\underline{\Sigma}$-algebra structure F on A such that $J = \mathrm{Sg}^{(\underline{A})}$, where $\underline{A} = (A, F)$.

Proof. In order to prove that, we consider three families of operations on the S sorted set A.

Firstly, for the identity ϵ of the free monoid on S, let $(\Sigma_{\epsilon,s} : s \in S)$ be the S-indexed family (of operation symbols) such that for every s in S, $\Sigma_{\epsilon,s}$ is the set whose members are the ordered cuadruples $\sigma = (\epsilon, (\emptyset : s \in S), a, s)$, where a belongs to the sth component of the S sorted set $J(\emptyset : s \in S)$ and $(\emptyset : s \in S)$ is the S-sorted set whose range is $\{\emptyset\}$.

Given a $\sigma = (\epsilon, (\emptyset : s \in S), a, s)$ in $\Sigma^1 = \bigcup(\Sigma_{\epsilon,s} : s \in S)$, let F_σ be the mapping from A_ϵ, a terminal set, to A_s, whose range is $\{a\}$.

Secondly, let $(\Sigma_{\ulcorner s \urcorner, s} : s \in S)$ be the S-indexed family (of operation symbols) such that for every s in S, $\Sigma_{\ulcorner s \urcorner, s}$ is the set whose members are the ordered cuadruples $\sigma = (\ulcorner s \urcorner, X, a, s)$, where X is a member of $2 - K(\underline{P}_S(A))$ such that $\mathrm{supp}(X) = \{s\}$ and a belongs to the sth component of the S-sorted set $J(X)$.

Given a $\sigma = (\ulcorner s \urcorner, X, a, s)$ in $\Sigma^2 = \bigcup(\Sigma_{\ulcorner s \urcorner, s} : s \in S)$, let F_σ be the mapping from $A_{\ulcorner s \urcorner}$, identified to A_s, into A_s, defined by:

$$F_\sigma : \begin{array}{ccc} A_s & \longrightarrow & A_s \\ x & \longmapsto & F_\sigma(x) = \begin{cases} a, & \text{if } x \in X_s \\ x, & \text{otherwise.} \end{cases} \end{array}$$

Finally, let $(\Sigma_{\ulcorner t \urcorner, s} : (t, s) \in S^2 - \Delta_S)$ be the $(S^2 - \Delta_S)$-indexed family (of operation symbols), where Δ_S is the diagonal of S, such that for every (t, s) in

$S^2-\Delta_S$, $\Sigma_{\lceil t\rceil,s}$ is the set whose members are the ordered cuadruples $\sigma = (\lceil t\rceil, X, a, s)$, where X is a member of $2\text{-}K(\underline{P}_S(A))$ such that $\mathrm{supp}(X) = \{t\}$, and a belongs to the sth component of the S-sorted set $J(X)$.

For $\sigma = (\lceil t\rceil, X, a, s)$ in $\Sigma^3 = \bigcup(\Sigma_{\lceil t\rceil,s} : (t,s) \in S^2 - \Delta_S)$, we have that for every x in A_t there exists an S-sorted set $B_x \in 2-K(\underline{P}_S(A))$ such that $\mathrm{supp}(B_x) = \mathrm{supp}(X)$ and $J(B_x)_s \neq \phi$. Indeed, given a member x of A_t, if x belongs to X_t, let $B_x = X$, and if x does not belong to X_t, let $B_x = \delta_{t,x}$, where $\delta_{t,x}$ is the S-sorted set defined as follows:

$$\delta_{t,x}(s) = \begin{cases} \phi, & \text{if } s \neq t, \\ \{x\}, & \text{if } s = t. \end{cases}$$

It is clear that $\delta_{t,x} \in 2-K(\underline{P}_S(A))$ and that $\mathrm{supp}(\delta_{t,x}) = \mathrm{supp}(X)$; then, because J is uniform, we can affirm that $\mathrm{supp}(J(\delta_{t,x})) = \mathrm{supp}(J(X))$, therefore $J(\delta_{t,x})_s \neq \emptyset$. Then we take a member $(b^\sigma_x : x \in A_t - X_t)$ of the set $\prod J(\delta_{t,x})_s : x \in A_t - X_t)$, and define the mapping F_σ from $A_{\lceil t\rceil}$, identified to A_t, into A_s, as follows:

$$\begin{array}{rccl} F_\sigma : & A_t & \longrightarrow & A_s \\ & x & \longmapsto & F_\sigma(x) = \begin{cases} a, & \text{if } x \in X_t \\ b^\sigma_x, & \text{otherwise.} \end{cases} \end{array}$$

Now, we take Σ as the set $\Sigma^1 \cup \Sigma^2 \cup \Sigma^3$, and define for that set d and cd in the obvious way, and take as heterogeneous $\underline{\Sigma}$-algebra structure F on A the Σ-indexed family $(F_\sigma : \sigma \in \Sigma)$ defined before, it is not difficult to verify that for the heterogeneous $\underline{\Sigma}$-algebra $\underline{A} = (A, F)$ we have $J = \mathrm{Sg}^{(\underline{A})}$.

Proposition 7

Let $\underline{\Sigma}$ be an S-sorted signature, such that for every σ in Σ the length of the string $d(\sigma)$ be at most one, and $\underline{A}$ a heterogeneous $\underline{\Sigma}$-algebra, then $\mathrm{Sg}^{(\underline{A})}$ is a heterogeneous 2-algebraic closure operator in $\underline{P}_S(A)$.

Proof. Without any additional hypothesis on $\underline{\Sigma}$, we know that $\mathrm{Sg}^{(A)}$ is a heterogeneous uniform algebraic closure operator. Now let us suppose that $\underline{\Sigma}$ is such that for every σ in Σ, the length of $d(\sigma)$ is at most one. We want to show that $\mathrm{Sg}^{(\underline{A})}$ is 2-algebraic, i.e., that for every $X \subseteq_S A$,

$$\mathrm{Sg}^{(\underline{A})}(X) = \bigcup_S \left(\mathrm{Sg}^{(\underline{A})}(L) : L \in 2 - K(\underline{P}_S(X))\right).$$

It is obvious that

$$\bigcup_S \left(\mathrm{Sg}^{(\underline{A})}(L) : L \in 2 - K(\underline{P}_S(X))\right) \subseteq_S \mathrm{Sg}^{(\underline{A})}(X).$$

For the dual inclusion, it suffices to take into account that for every $X \subseteq_S A$,

$$\mathrm{Sg}^{(\underline{A})}(X) = \bigcup_S \left(E_n^{(\underline{A},X)} : n \in \mathbb{N} \right). \square$$

Corollary 8

Let A be an S-sorted set and J a heterogeneous uniform algebraic closure operator in $\underline{P}_S(A)$: then the following two conditions are equivalent:

(i) J is 2-algebraic.

(ii) There exists an S-sorted signature $\underline{\Sigma}$ such that for every σ in Σ the length of $d(\sigma)$ is at most one, and a heterogeneous $\underline{\Sigma}$-algebra structure F on A such that $J = \mathrm{Sg}^{(\underline{A})}$, where $\underline{A} = (A, F)$.

Remark. For another time, we undertake the proof of the non validity of Proposition 5 and Proposition 6, when the term "2-algebraic" is replaced by "k-algebraic" for $3 \leq k \leq \aleph_0$, the case $k = \aleph_0$ being solved in the negative way in [7].

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References

1. G. Birkhoff and O. Frink, Representation of lattices by sets, *Trans. Amer. Math. Soc.* 64 (1948), 299–316.
2. G. Birkhoff and J. Lipson, Heterogeneous algebras, *J. of Comb. Theory* 8 (1970), 115–133.
3. P. M. Cohn, *Universal Algebra*, Harper and Row, New York, 1965.
4. J. A. Goguen and J. Meseguer, Completeness of many-sorted equational logic, *Houston J. of Math.* 11 (1985), 307–334.
5. J. Herbrand, Recherches sur la théorie de la démonstration, in *Écrits Logiques*, pp. 35–153, P. U. F., Paris, 1968.
6. J. Hook, A note on interpretations of many-sorted theories, *J. of Symb. Logic* 50 (1985), 372–374.
7. G. Matthiessen, *Theorie der Heterogenen Algebren*, Mathematik-Arbeits-Papiere Nr. 3, Universität Bremen, Bremen, 1976.
8. A. Schmidt, Ueber deduktive Theorien mit mehreren Sorten von Grunddingen, *Math. Ann.* 115 (1938), 485–506.
9. D. Monk, *Mathematical Logic*, Springer, Heidelberg-Berlin-New York, 1976.
10. H. Wang, Logic of many-sorted theories, *J. of Symb. Logic* 17 (1952), 105–116.