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On rosettes and antipodal rosettes

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Abstract

In this paper we introduce the notion of an antipodal set of a rosette, which is a natural generalization of an antipodal pair of an oval. We give a counterpart of the Blaschke-Süss theorem. Moreover, we consider antipodal rosettes.

1. Introduction

In this paper we will consider the family of positively oriented rosettes, i.e., C^2 plane closed curves with positive curvature [3, 2]. A rosette C can be written in the following form

$$z(t) = \int_0^t r(u)e^{iu}\,du \qquad \text{for } 0 \leq t \leq 2\pi j, \tag{1}$$

where $1/r$ is the curvature and j is the index [4, 5].

The Fourier coefficients of r

$$\begin{cases} a_n = \dfrac{1}{j}\displaystyle\int_0^{2\pi j} r(t)\cos\frac{n}{j}t\,dt, \\ b_n = \dfrac{1}{j}\displaystyle\int_0^{2\pi j} r(t)\sin\frac{n}{j}t\,dt \end{cases} \tag{2}$$

for $n = 1, 2, \ldots$, will be called the Fourier coefficients of C. We note that the conditions

$$a_j = b_j = 0. \tag{4}$$

are equivalent (for $j = 1$ see [1]).

1. Antipodal sets

Let us fix a rosette C and a positive integer n.

DEFINITION 1. A set of points

$$\left\{z(t), z\left(t+\frac{j\pi}{n}\right), \ldots, z\left(t+\frac{(2n-1)j\pi}{n}\right)\right\} \tag{5}$$

of C is said to be an n-antipodal set of C if and only if

$$\sum_{l=0}^{n-1}\left[r\left(t+\frac{2lj\pi}{n}\right)-r\left(t+\frac{(2l+1)j\pi}{n}\right)\right]=0. \tag{6}$$

Remark. We note that a 1-antipodal set is an antipodal pair [4, 2, 1].

Theorem 1

If the Fourier coefficient b_n of C is equal to 0, then there exists at least three n-antipodal sets of C.

Proof. Let us consider the function

$$g(t)=\sum_{l=0}^{n-1}\left[r\left(t+\frac{(2l+1)j\pi}{n}\right)-r\left(t+\frac{2lj\pi}{n}\right)\right] \tag{7}$$

for $0 \le t \le 2\pi j$.

We have

$$g\left(t+\frac{j\pi}{n}\right) = -g(t) \qquad \text{for } 0 \le t \le 2\pi j. \tag{8}$$

We may assume that $t_0 = 0$. Then we have

$$\begin{aligned}
\int_0^{j\pi/n} g(\theta)\,\sin\frac{n}{j}\,\theta\,d\theta &= \sum_{l=0}^{n-1}\int_0^{j\pi/n} r\left(\theta+\frac{(2l+1)j\pi}{n}\right)\sin\frac{n}{j}\,\theta\,d\theta \\
&\quad -\sum_{l=0}^{n-1}\int_0^{j\pi/n} r\left(\theta+\frac{2lj\pi}{n}\right)\sin\frac{n}{j}\,\theta\,d\theta \\
&= \sum_{l=0}^{n-1}\int_{(2l+1)j\pi/n}^{(2l+2)j\pi/n} r(t)\,\sin\left(\frac{n}{j}\,t-(2l+1)\pi\right)dt \\
&\quad -\sum_{l=0}^{n-1}\int_{2lj\pi/n}^{(2l+1)j\pi/n} r(t)\sin\left(\frac{n}{j}\,t-2l\pi\right)dt \\
&= -\sum_{l=0}^{n-1}\int_{2lj\pi/n}^{(2l+1)j\pi/n} r(t)\,\sin\frac{n}{j}\,t\,dt \\
&= -\int_0^{2\pi j} r(t)\,\sin\frac{n}{j}\,t\,dt \\
&= -jb_n = 0.
\end{aligned}$$

The same considerations as in the proof of Blaschke-Süss Theorem [4] guarantee the existence of two further zeros of g. It completes our proof. □

From Theorem 1 we immediately obtain

Theorem 2

Each rosette of the index j has at least three j-antipodal sets.

Remark. If $j = 1$, the Theorem 2 reduces to Blaschke-Süss Theorem [4, 2].

2. A geometric meaning of antipodal sets

The length of the arc contained between two points $z(a)$, $z(b)$ for $a < b$ of a rosette C is given by the formula [3, 1]

(2) $\int^{b}$

The function g is a derivative of the function f defined by the formula

$$f(t) = \sum_{l=0}^{n-1} \int_{t+2lj\pi/n}^{t+(2l+1)j\pi/n} r(u)\,du. \tag{10}$$

$f(t)$ is a sum of lengths of disjoint arcs determined by any set of points

$$\left\{z(t), z\left(t+\frac{j\pi}{n}\right), \ldots, z\left(t+\frac{(2n-1)j\pi}{n}\right)\right\}$$

of C. The extremum value of f is attained at a point t_0 such that

$$\left\{z(t_0), z\left(t_0+\frac{j\pi}{n}\right), \ldots, z\left(t_0+\frac{(2n-1)j\pi}{n}\right)\right\}$$

is an n-antipodal set.

3. Antipodal rosettes

Let us fix a rosette C and a positive integer n.

DEFINITION 2. A rosette C is said to be an n-antipodal rosette if and only if for each t the set

$$\left\{z(t), z\left(t+\frac{j\pi}{n}\right), \ldots, z\left(t+\frac{(2n-1)j\pi}{n}\right)\right\}$$

is n-antipodal.

We note that C is an n-antipodal rosette if and only if

$$g \equiv 0. \tag{11}$$

Making use of the Fourier series expansion of r we get

$$g(t) = \sum_{l=0}^{n-1} \sum_{\nu=1}^{\infty} \left[a_\nu \cos\frac{\nu}{j}\left(t+\frac{(2l+1)j\pi}{n}\right) + b_\nu \sin\frac{\nu}{j}\left(t+\frac{(2l+1)j\pi}{n}\right) \right.$$
$$\left. - a_\nu \cos\frac{\nu}{j}\left(t+\frac{2lj\pi}{n}\right) - b_\nu \sin\frac{\nu}{j}\left(t+\frac{2lj\pi}{n}\right)\right]$$
$$= \sum_{l=0}^{n-1} \sum_{\nu=1}^{\infty} \left(a_\nu \left[\cos\frac{\nu}{j}\left(t+\frac{(2l+1)j\pi}{n}\right) - \cos\frac{\nu}{j}\left(\frac{2lj\pi}{n}+t\right)\right]\right.$$

$$+ b_\nu \left[\sin\frac{\nu}{j}\left(t+\frac{(2l+1)j\pi}{n}\right) - \sin\frac{\nu}{j}\left(t+\frac{2lj\pi}{n}\right)\right]\Bigg)$$

$$= 2\sum_{l=0}^{n-1}\sum_{\nu=1}^{\infty}\left[-a_\nu \sin\left(\frac{\nu}{j}t+\frac{(4l+1)\nu\pi}{2n}\right)\right.$$

$$\left. + b_\nu \cos\left(\frac{\nu}{j}t+\frac{(4l+1)\nu\pi}{2n}\right)\right]\sin\frac{\nu\pi}{2n}$$

$$= 2\sum_{\nu=1}^{\infty}\sin\frac{\nu\pi}{2n}\left[-a_\nu\sum_{l=0}^{n-1}\sin\left(\frac{\nu}{j}t+\frac{(4l+1)\nu\pi}{2n}\right)\right.$$

$$\left. + b_\nu\sum_{l=0}^{n-1}\cos\left(\frac{\nu}{j}t+\frac{(4l+1)\nu\pi}{2n}\right)\right].$$

If $2n|\nu$, then $\sin(\nu\pi/2n) = 0$. Moreover if $n \nmid \nu$, then

$$\sum_{l=0}^{n-1}\exp\left[i\left(\frac{\nu}{j}t+\frac{(4l+1)\nu\pi}{2n}\right)\right] = \exp\left[i\left(\frac{\nu}{j}t+\frac{\nu\pi}{2n}\right)\right]\sum_{l=0}^{n-1}\exp\left[\frac{i2l\nu\pi}{n}\right] = 0$$

and if $n|\nu$, then

$$\sum_{l=0}^{n-1}\exp\left[\frac{i2l\nu\pi}{n}\right] = n.$$

Thus we have

$$g(t) = 2\sum_{\substack{\nu=1\\ n|\nu\\ 2n\nmid\nu}}^{\infty}\sin\frac{\nu\pi}{2n}\cdot n\left[-a_\nu\sin\left(\frac{\nu}{j}t+\frac{\nu\pi}{2n}\right)+b_\nu\cos\left(\frac{\nu}{j}t+\frac{\nu\pi}{2n}\right)\right].$$

If $n|v$ and $2n \nmid \nu$, then $\nu = n(2m+1)$, $m = 0, 1, 2, \ldots$, and we have

$$g(t) = 2n\sum_{m=0}^{\infty}(-1)^{2m+1}$$

(12) $$\times\left[a_{n(2m+1)}\cos n(2m+1)\frac{t}{j} + b_{n(2m+1)}\sin n(2m+1)\frac{t}{j}\right].$$

The formula (12) implies that the conditions (11) and

(13) $$a_{n(2m+1)} = b_{n(2m+1)} = 0 \qquad \text{for } m = 0, 1, 2, \ldots$$

Theorem 3

A rosette C is n-antipodal if and only if the Fourier coefficients of C, $a_{n(2m+1)}$, $b_{n(2m+1)}$, for $m = 0, 1, 2, \ldots$, vanish.

Corollary

Let $n_1, n_2, \ldots$ be an arbitrary increasing sequence of positive integers. There exists an n_r-antipodal rosette for each $r = 1, 2, \ldots$.

A rosette C with the index 1 is an oval. Let $n \geq 3$ be a fixed integer. By n-polygon we mean a polygon with n sides.

Theorem 4

If all the n-polygons described on an oval C have the same perimeter, then C is an n-antipodal oval.

Proof. All the n-polygons described on an oval C have the same perimeter if and only if $a_k = b_k = 0$ for $n|k$ [1]. In particular we have $a_{n(2m+1)} = b_{n(2m+1)} = 0$ for $m = 1, 2, \ldots$. Thus C is an n-antipodal oval. □

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