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Tensor products of almost r-summing maps

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Abstract

If T_1 and T_2 are continuous linear maps on locally convex spaces, we prove that $T_1 \otimes T_2$ is almost r-summing if and only T_1 and T_2 so are. We also obtain a sufficient condition under which the unique extension of $T_1 \otimes T_2$ to the complete ϵ-tensor product is almost r-summing

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0. Introduction and notation

The purpose of this note is to study tensor products of almost r-summing maps. In 1970, Holub [3] considered tensor product mappings on Banach spaces, obtaining the following result: "Let E_1, E_2, F_1, and F_2 be Banach spaces. If $T_i : E_i \to F_i$ $(i = 1, 2)$ are continuous linear maps, then $T_1 \otimes T_2 : E_1 \tilde{\otimes}_\epsilon E_2 \to F_1 \tilde{\otimes}_\epsilon F_2$ is r-summing if and only if T_1 and T_2 so are". We prove an analogous statement for almost r summing maps on locally convex spaces, and we obtain a sufficient condition under which the unique extension of $T_1 \otimes T_2$ to the complete ϵ tensor product is almost r-summing.

We use in this paper the notation from [2]. Let E be a Hausdorff locally convex space, following K. Floret and J. Wloka [2]. A sequence (x_n) from E is called weakly r-summable if$(\langle x_n, x'\rangle) \in \ell^r$ whenever $x' \in E'$ $(1 \leq r < +\infty)$. We put

$$\epsilon_{r,U}(x_n) = \sup\left\{ \left(\sum |\langle x_n, x'\rangle|^r\right)^{1/r} : x' \in U^\circ \right\}. \qquad (1)$$

U running through the family $\mathcal{U}_E(0)$ of absolutelyconvex and closed neighbourhoods of E. The space of all weakly r-summable sequences from E is denoted by $\ell^r_w(E)$.

The finite section $\hat{x}(P)$ of $\hat{x} = (x_n)$ is the sequence defined by

$$x_n(P) = \begin{cases} x_n, & \text{if } n \in P; \\ 0, & \text{if } n \notin P, \end{cases}$$

where $P \subset \mathbb{N}$ is finite. A sequence $\hat{x} = (x_n)$ is called r-summable if

$$\hat{x} = \epsilon_r - \lim_P \hat{x}(P) \qquad \text{in } \ell^r_w(E).$$

The subspace of $\ell^r_w(E)$ formed by all r-summable sequences from E is denoted by $\ell^r_s(E)$.

A sequence (x_n) from E is called absolutely r-summable if $(p_U(x_n)) \in \ell^r$ whenever $U \in \mathcal{U}_E(0)$. The class of all absolutely r summable sequences from E is denoted by $\ell^r_a(E)$. If we put

$$\pi_{r,U}(x_n) = \left[\sum p_U(x_n)^r\right]^{1/r} \qquad \text{for all } U \subset \mathcal{U}_E(0) \tag{2}$$

the system of all seminorms (2) defines a natural topology π_r on $\ell^r_a(E)$.

In [2], a continuous linear map $T : E \to F$ is called almost r-summing ($1 \leq r < +\infty$) if it takes each r-summable sequence (x_n) from E into an absolutely r-summable sequence (Tx_n) from F. If this is the case, it is known that a linear map can be defined by

$$\hat{T} : (x_n) \in \ell^r_w(E) \longrightarrow (Tx_n) \in \ell^r_a(F)$$

mapping bounded subsets of $\ell^r_w(E)$ into bounded subsets of $\ell^r_a(F)$, but it is not necessarily continuous [5, p. 36]. In [2] T is called r-summing when $\hat{T}$ is continuous. If E is a metric or nuclear locally convex space, then almost r-summing maps defined on E are r-summing.

1. Tensor products of almost r-summing maps

Let E_1, E_2, F_1, and F_2 be locally convex spaces. If $T_i : E_i \to F_i$ ($i = 1, 2$) are continuous linear maps such that $T_1 \otimes T_2$ from $E_1 \otimes_\epsilon E_2$ into $F_1 \otimes_\epsilon F_2$ is almost r-summing and $T_i \neq 0$ ($i = 1, 2$), then simple modifications of the proof of [3, Proposition 3.1] show that T_1 and T_2 are almost r summing, because the class of all almost r-summing maps on locally convex spaces is an operator ideal [2]. For the converse

Theorem 1

If $T_1 : E_1 \to F_1$ and $T_2 : E_2 \to F_2$ are two almost r-summing maps, then $T_1 \otimes T_2 : E_1 \otimes_\epsilon E_2 \to F_1 \otimes_\epsilon F_2$ is almost r-summing.

Proof. If (z_n) belongs to $\ell_s^r(E_1 \otimes_\epsilon E_2)$ we must prove that

$$\sum_{n=1}^{+\infty} \left(p_{V_1} \otimes_\epsilon p_{V_2}[(T_1 \otimes T_2)z_n]\right)^r < +\infty. \tag{3}$$

where V_1, resp. V_2, run through 0-neighbourhoods of F_1, resp. F_2. If $z_n = \sum_i x_{in} \otimes y_{in}$ for each $n \in \mathbb{N}$, the inequality (3) is equivalent to prove that there exists a constant $M > 0$ such that

$$\sum_{n=1}^{+\infty} \left| \sum_i \langle T_1 x_{in}, x_n' \rangle \langle T_2 y_{in}, y_n' \rangle \right|^r \leq M \qquad \text{for } x_n' \in V_1^\circ \; y_n' \in V_2^\circ. \tag{4}$$

Since

$$\sum_i \langle T_1 x_{in}, x_n' \rangle \langle T_2 y_{in}, y_n' \rangle = \left\langle \sum_i \langle y_{in}, {}^tT_2 y_n' \rangle x_{in}, {}^tT_1 x_n' \right\rangle$$
$$= \left\langle T_1 \left(\sum_i \langle y_{in}, {}^tT_2 y_n' \rangle x_{in} \right), x_n' \right\rangle.$$

(4) is equivalent to the following

$$\sum_{n=1}^{+\infty} \left(p_{V_1}[(T_1 \circ A({}^tT_2 y_n'))z_n]\right)^r \leq M \qquad \text{for } y_n' \in V_2^\circ. \tag{5}$$

where, for each $u \in E_2'$, $A(u)$ is the continuous linear map defined by

$$\sum_{i=1}^m x_i \otimes y_i \in E_1 \otimes_\epsilon E_2 \longmapsto \sum_{i=1}^m \langle y_i, u \rangle x_i \in E_1.$$

Now we shall prove that the set

is bounded in $\ell_s^r(E_1)$. Indeed, if r^* is the conjugate exponent of r, for each $x' \in E_1'$, $P \in \mathcal{F}(\mathbb{N})$ and (α_n) in the unit ball of $\ell^{r^*}(\mathbb{N})$, we have

$$\left|\left\langle \sum_{n\in P} \alpha_n A({}^tT_2 y_n') z_n, x' \right\rangle\right| = \left|\sum_{n\in P} \alpha_n \left\langle T_2 \left(\sum_i \langle x_{in}, x'\rangle y_{in}\right), y_n' \right\rangle\right|. \tag{6}$$

So if $B(x')$ denotes the linear map defined by

$$\sum_i x_i \otimes y_i \in E_1 \otimes_\epsilon E_2 \longmapsto \sum \langle x_i, x'\rangle y_i \in E_2,$$

from (6) we obtain the following estimate:

$$\begin{aligned}\left|\left\langle \sum_{n\in P} \alpha_n A({}^tT_2 y_n') z_n, x' \right\rangle\right| &\leq \left(\sum_{n\in P} |\alpha_n|^{r^*}\right)^{1/r^*} \left(\sum_{n\in P} |\langle (T_2 \circ B(x')) z_n, y_n'\rangle|^r\right)^{1/r} \\ &\leq \left(\sum_{n=1}^{+\infty} (p_{V_2}((T_2 \circ B(x')) z_n))^r\right)^{1/r} \\ &< +\infty,\end{aligned}$$

because $T_2 : E_2 \longrightarrow F_2$ is almost r-summing and $(B(x')z_n)$ belongs to $\ell_s^r(E_2)$. This proves that H is bounded in $\ell_s^r(E_1)$. Hence, as the map T_1 is almost r-summing, there exists $M > 0$ so that (5) is valid. □

2. Almost r-summing maps on dense subspaces

Let E, F and G be locally convex spaces such that E is a dense subspace of F and G is complete. If $T : F \longrightarrow G$ is a continuous linear map so that its restriction to E is almost r-summing, it seems to be unknown if T is always almost r summing. We have obtained the following results.

DEFINITION 2. A subspace F of a space E is said to be large if every bounded set in E is contained in the closure in E of a bounded set in F [1].

Proposition 3

Let E, F and G be locally convex spaces such that $\ell_s^r(E)$ is a large subspace of $\ell_s^r(F)$ and G is complete. If $T : F \longrightarrow G$ is a continuous linear map, then its

Proof. If T is almost r-summing, it is clear that T_E is almost r-summing. Assume then that T_E is almost r-summing. If $\hat{x} = (x_n)$ belongs to $\ell_s^r(F)$, the set $A = \{\hat{x}(n) : n \in \mathbb{N}\}$ is bounded in $\ell_s^r(F)$ (here $\hat{x}(n)$ denotes the finite section $(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$). By assumption there exists a bounded subset B of $\ell_s^r(E)$ such that A is contained in the closure in $\ell_s^r(F)$ of B. Since $T_E : E \to G$ is almost r-summing, for each continuous seminorm $q(x)$ on G, there exists a constant $M > 0$ so that

$$\sum_{n=1}^{+\infty} q(z_n)^r \leq M^r \qquad \text{for } (z_n) \in B.$$

On the other hand, there is a continuous seminorm $p(x)$ on F such that $q(Tx) \leq p(x)$ for all $x \in F$. Now we shall see that

$$\sum_{n=1}^{m} q(Tx_n)^r \leq (1+M)^r \qquad \text{for } m \in \mathbb{N}.$$

Indeed, given $m \in \mathbb{N}$, we can choose $\hat{z} = (z_n) \in B$ so that

$$\epsilon_{r,V_p}(\hat{x}(m) - \hat{z}) < m^{-1/r}.$$

Hence we have $p(x_n - z_n) < m^{-1/r}$ for all $n \leq m$. Thus we can obtain

$$\begin{aligned}
\left(\sum_{n=1}^{m} q(Tx_n)^r\right)^{1/r} &\leq \left(\sum_{n=1}^{m} (q(Tx_n - Tz_n) + q(Tz_n))^r\right)^{1/r} \\
&\leq \left(\sum_{n=1}^{m} q(Tx_n - Tz_n)^r\right)^{1/r} + \left(\sum_{n=1}^{m} q(Tz_n)^r\right)^{1/r} \\
&\leq \left(\sum_{n=1}^{m} p(x_n - z_n)^r\right)^{1/r} + \left(\sum_{n=1}^{\infty} q(Tz_n)^r\right)^{1/r} \\
&\leq 1 + M
\end{aligned}$$

for all $m \in \mathbb{N}$. This proves that $\sum_{n=1}^{\infty} q(Tx_n)^r < +\infty$ and the proof is complete because $q(x)$ is an arbitrary continuous seminorm on G. □

Obviously, if $\ell_s^r(E)$ is a large subspace of $\ell_s^r(F)$, then E is a large subspace of

DEFINITION 4. Let E be a subspace of F. We say that E has the property (P) if there exists an equicontinuous net $(T_\alpha)_{\alpha\in\Lambda}$ from $\mathcal{L}(F,E)$ so that

$$\lim_\alpha T_\alpha(x) = x \qquad \text{for } x \in F. \tag{7}$$

Proposition 5

If E is a subspace of F which has the property (P), then $\ell_s^r(E)$ is a large subspace of $\ell_s^r(F)$.

Proof. Let A be a bounded subset of $\ell_s^r(F)$. If we put

$$A_0 = \{\hat{x}(n) : n \in \mathbb{N},\ \hat{x} \in A\},$$

then A_0 is bounded in $\ell_s^r(F)$. By the property (P), there is an equicontinuous net $(T_\alpha)_{\alpha\in\Lambda} \subset \mathcal{L}(F,E)$ such that (7) is valid.

Hence the set

$$B = \bigcup_\alpha \hat{T}_\alpha(A_0)$$

is bounded in $\ell_s^r(E)$. We shall see that A is contained in the closure in $\ell_s^r(F)$ of B. In fact, if $\hat{x} = (x_n) \in A$ and $p(x)$ is a continuous seminorm on F, given $\epsilon > 0$ we can choose $n_0 \in \mathbb{N}$ such that

$$\epsilon_{r,V_p}(\hat{x} - \hat{x}(n)) < \epsilon\, 2^{-1/r} \qquad \text{for } n \geq n_0.$$

By (7), there exists $\alpha \in \Lambda$ so that

$$p(T_\alpha(x_n) - x_n) \leq \epsilon(2n_0)^{-1/r} \qquad \text{for } n \leq n_0.$$

Then, if $x' \in V_p^\circ$, we have

$$\sum_{n=1}^{n_0} |\langle x_n - T_\alpha(x_n), x'\rangle|^r + \sum_{n>n_0} |\langle x_n, x'\rangle|^r < \frac{\epsilon^r}{2} + \frac{\epsilon^r}{2} = \epsilon^r.$$

Hence $\epsilon_{r,V_p}(\hat{x} - T_\alpha(\hat{x}(n_0))) \leq \epsilon$. This proves that $\hat{x}$ belongs to $\bar{B}$. □

Remark 6. a) If E is a dense subspace of F which has the bounded approximation property [4], then E has property (P). Indeed, if $(T_\alpha)_\alpha \in \Lambda$ is an equicontinuous net from $\mathcal{F}(E,E)$ which is pointwise convergent to the identity mapping, then for each $\alpha \in \Lambda$ the continuous linear map $x \in E \to T_\alpha(x) \in T_\alpha(E)$ has a unique extension to F which is denoted also by T_α. Easy arguments prove that $\{T_\alpha : \alpha \in \Lambda\} \subset \mathcal{L}(F,E)$ is equicontinuous and $\lim_\alpha T_\alpha(x) = x$ for all $x \in F$.

b) It is well known that the identity mapping from ℓ^1 into ℓ^2 is 1-summing. We can generalize this result to spaces of sequences whose terms are elements of a locally convex space: "The identity mapping from $\ell_s^1(E)$ into $\ell_s^2(E)$ is almost 1-summing if and only if the identity mapping on E so is". (Note that $\ell^1 \otimes_\epsilon E$ is a dense subspace

Now we turn our attention to the complete ϵ-tensor product $E\tilde{\otimes}_\epsilon F$. Simple modifications of the proof of [1, Proposition 4] prove the following

Proposition 7

Let E and F be locally convex spaces such that E has the bounded a. p. and F is complete. Then $E\otimes_\epsilon F$ is a dense subspace of $E\tilde{\otimes}_\epsilon F$ which has property (P).

Finally, by combining the above results we obtain

Theorem 8

Let E_1, E_2, F_1 and F_2 be locally convex spaces such that E_1 has the bounded a. p. and E_2 is complete. If $T_i : E_i \to F_i$ $(i = 1, 2)$ are almost r summing, then $T_1\tilde{\otimes}T_2 : E_1\tilde{\otimes}_\epsilon E_2 \to F_1\tilde{\otimes}_\epsilon F_2$ so is.

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