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Weakly c'-compact subsets of non-archimedean Banach spaces over a spherically complete field

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Abstract

Each weakly c'-compact subset of a locally convex space over a spherically complete non-archimedean field with dense valuation is a pure compactoid. This is an answer to an open problem posed by W. H. Schikhof [4].

0. Introduction and preliminaries

0.1. Unless stated otherwise, $\mathbb{K}$ will be a non-archimedean (n.a.), spherically complete valued field with non-trivial valuation $|\cdot|$. We set

$$B(0,1) = \{\lambda \in \mathbb{K} : |\lambda| \leq 1\}$$

and

$$B(0,1^-) = \{\lambda \in \mathbb{K} : |\lambda| < 1\}$$

and denote the residue class field of $\mathbb{K}$ by k and its value group by $|\mathbb{K}^*|$. If the valuation of $\mathbb{K}$ is discrete, there exists $\rho \in B(0,1^-)$ such that

$$|\mathbb{K}^*| = \{|\rho|^n : n \in \mathbb{Z}\}$$

and $B(0,1^-) = B(0,|\rho|)$.

Unless stated otherwise, E will be a Banach space (B.S.) with norm $\|\cdot\|$. If $\mathbb{K}$ is *discretely* valued, we choose $\|\cdot\|$ such that

$$\|E\| = \{\|x\| : x \in E\} \subset |\mathbb{K}|.$$

We denote by E' the topological dual space of E and we assume that $E' \neq \{0\}$. For $S \subset E$, we denote by $\operatorname{co} S$ the absolutely convex (a.c.) hull of S, by $\overline{\operatorname{co}}\, S$ the closure of $\operatorname{co} S$ and by $[S]$ the linear hull of S.

A subset B of E is called absorbing if for every $x \in E$ there exists $\lambda \in \mathbb{K}$ such that $x \in \lambda B$. An a.c. subset B of E is called finite dimensional if it is contained in a finite-dimensional linear subspace of E. Otherwise, it is said to be infinite-dimensional.

0.2. Introduction

In section 1, we recall some general properties of Banach spaces and a few definitions which we need in the sequel.

Section 2 is dedicated to Banach spaces over a trivially valued field.

In section 3, some properties of seminorms and their relation to weakly c'-compact sets in locally convex spaces are given.

Sections 4 and 5, the main parts of our paper, deal with Krein-Milman like theorems in E.

Important results are:

a) If the valuation on $\mathbb{K}$ is discrete, each a.c., closed, weakly c'-compact subset of E is an orthogonal sum of one-dimensional a.c. subsets of E.

b) If the valuation on $\mathbb{K}$ is dense, each a.c., closed, weakly c'-compact subset of E is pure compactoid.

1. Two general lemmas about Banach spaces and orthogonality in Banach spaces

1.1. Remark. The trivial valuation is a case which is not excluded throughout section 1.

1.2. DEFINITION. For $x \in E$ and a subset B of E, we denote

$$\operatorname{dist}(x, B) = \inf_{y \in B} \|x - y\|.$$

1.3. Lemma

Let $D \subset E$, $D \neq E$, be a closed, linear subspace of E. For every $t \in (0,1)$, there exists $x_t \in E \setminus D$ such tat $\|y - x_t\| > t\,\|x_t\|$ for any $y \in D$.

Proof. Choose $x \in E \setminus D$. As D is closed, $\text{dist}(x, D) = r > 0$. So, for $t \in (0, 1)$, there exists $d \in D$ such that $\|x - d\| < r/t$. Put $x_t = x - d$. Then

$$t\|x_t\| = t\|x - d\| < \text{dist}(x, D) = \text{dist}(x_t, D).$$

Hence, for any $y \in D$, $\|y - x_t\| > t\|x_t\|$. $\square$

1.4. Lemma

Suppose there exists $t \in (0, 1)$ such that $\|E\| \subset \{t^n : n \in \mathbb{Z}\} \cup \{0\}$. Let D be a closed linear subspace of E. Then there exists $z \in E \setminus D$, such that $\text{dist}(z, D) = \|z\|$.

Proof. Use lemma 1.3 and choose $z = x_t$. $\square$

1.5. *Remark.* For 1.3 and 1.4, the completeness of E is not required.

1.6. Definition.

1) A subset B of $E \setminus \{0\}$ is called orthogonal if for any $n \in \mathbb{N}_0$, $b_1, \ldots, b_n \in B$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$:

$$\|\lambda_1 b_1 + \cdots + \lambda_n b_n\| = \max\{\|\lambda_1 b_1\|, \ldots, \|\lambda_n b_n\|\}.$$

2) Choose $t \in (0, 1)$. A subset B of $E \setminus \{0\}$ is called t-orthogonal if for any $n \in \mathbb{N}_0$, $b_1, \ldots, b_n \in B$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$:

$$\|\lambda_1 b_1 + \cdots + \lambda_n b_n\| \geq t \max\{\|\lambda_1 b_1\|, \ldots, \|\lambda_n b_n\|\}.$$

1.7. Proposition

Let $(e_i)_{i \in I}$ be a $(t\text{-})$orthogonal subset of E. Then

$$\overline{[\{e_i : i \in I\}]} = \left\{ \sum_{i \in I} \lambda_i e_i : \forall i \in I,\ \lambda_i \in \mathbb{K} \text{ and } \|\lambda_i e_i\| \longrightarrow 0 \right\}.$$

Proof. [5, lemma 6.b]. $\square$

2. Some properties of Banach spaces over a trivially valued field

2.1. *Remark.* Throughout this section, $\mathbb{K}$ will be a field with trivial valuation.

2.2. Lemma

For every $t \in (0,1)$, there exists a norm p on E such that:

$$t\,\|x\| < p(x) \le \|x\|, \qquad \text{for } x \in E \setminus \{0\},$$

and

$$p(E) = \{p(x) : x \in E\} \subset \{t^n : n \in \mathbb{Z}\} \cup \{0\}.$$

Proof. First we establish the definition of p. Choose $t \in (0,1)$.

1) If $x = 0$, we set $p(x) = 0$.

2) If $x \neq 0$, then there exists $n \in \mathbb{Z}$, such that $t^{n+1} \le \|x\| < t^n$. We define $p(x) = t^{n+1}$. The inequalities $t\,\|x\| < p(x) \le \|x\|$, for $x \in E \setminus \{0\}$, follow easily from the definition of p. It is easy to see that $p(x) = 0$ if and only if $x = 0$, and that

$$p(x+y) \le \max\{p(x), p(y)\}, \qquad \text{for } x, y \in E.$$

For $\lambda \in \mathbb{K}^*$ and $x \in E \setminus \{0\}$, we have that $\|\lambda x\| = |\lambda|\,\|x\| = \|x\|$, because the valuation on $\mathbb{K}$ is trivial, and thus: $p(\lambda x) = p(x) = |\lambda|\,p(x)$. □

2.3. Theorem

Suppose that $\|E\| \subset \{t^n : n \in \mathbb{Z}\} \cup \{0\}$, for some $t \in (0,1)$. Then $(E, \|\cdot\|)$ has an orthogonal base.

Proof. Let $\mathcal{P}$ be the set of all the orthogonal subsets W of E such that $0 \notin W$. By a standard application of Zorn's lemma, $\mathcal{P}$ has some maximal element $S = (s_i)_{i\in I}$. Put

$$D := \left\{ \sum_{i\in I} \lambda_i s_i : \lambda_i \in \mathbb{K}\ \forall i \in I;\ \lambda_i s_i \longrightarrow 0 \right\}.$$

D is a complete linear subspace of E, hence D is closed. Now $D = E$.

Indeed, suppose $D \neq E$. According to lemma 1.4, we can find $z \in E \setminus D$ such that $\operatorname{dist}(z, D) = \|z\|$. Hence $S \cup \{z\}$ is an orthogonal subset of E, which contradicts the maximality of S. (The uniqueness of the expansion of an element x of E in terms of $(s_i)_{i\in I}$, follows from the orthogonality of S). □

2.4. *Remark.* In [1], a theorem analogous (but stronger) to 2.3 is formulated. For details we refer to [1].

2.5. Corollary

For every $t \in (0,1)$, $(E, \|\cdot\|)$ has a t-orthogonal base.

Proof. Let $t \in (0,1)$. Choose a norm p as in lemma 2.2. According to theorem 2.3, (E, p) has an orthogonal base $(s_i)_{i\in I}$. Then, for any $n \in \mathbb{N}_0$, $i_1, \ldots, i_n \in I$, $\lambda_{i_1}, \ldots, \lambda_{i_n} \in \mathbb{K}$:

$$\left\|\sum_{k=1}^{n} \lambda_{i_k} s_{i_k}\right\| \geq p\left(\sum_{k=1}^{n} \lambda_{i_k} s_{i_k}\right)$$
$$= \max_{k=1,\ldots,n} p(\lambda_{i_k} s_{i_k})$$
$$\geq t \max_{k=1,\ldots,n} \|\lambda_{i_k} s_{i_k}\|,$$

hence $(E, \|\cdot\|)$ has a t-orthogonal base $(s_i)_{i\in I}$. □

2.6. Proposition

Every a.c. subset of E is a linear subspace of E and conversely.

3. On seminorms and weakly c'-compact subsets in a locally convex space

3.1. *Remark.* Throughout section 3, E is a Hausdorff locally convex space.

3.2. Definition [2]. If A is an a.c. closed subset of E, we put

$$A^i = \bigcup_{\lambda \in B(0,1^-)} \lambda A$$

and $\partial A = A \setminus \overline{A^i}$ where $\overline{A^i}$ is the closure of A^i in E. (Note that ∂A may be empty).

3.3. *Construction.* With the notations of the preceding definition, we denote by

$$\begin{array}{rccc} \pi: & A & \longrightarrow & A/\overline{A^i} \\ & x & \longmapsto & \bar{x} \end{array}$$

and by

$$\begin{array}{rccc} \pi_1: & B(0,1) & \longrightarrow & k \\ & \lambda & \longmapsto & \lambda \end{array}$$

the canonical surjections. Then $A/\overline{A^i}$ is a k-vector space.

3.4. Lemma

Let $A \subset E$ be a.c. and closed and let p be a continuous seminorm on E. (If $\mathbb{K}$ is discretely valued, we assume that $p(E) \subset |\mathbb{K}|$). Then the following are equivalent:

i) $p \leq 1$ *on* A;

ii) $p < 1$ *on* $\overline{A^i}$;

iii) $p < 1$ *on* A^i.

Proof. i) $\Longrightarrow$ ii) For every $y \in A^i$ there is some $z \in A$ and some $\lambda \in B(0, 1^-)$ such that $y = \lambda z$ and thus $p(y) < 1$. For $y \in \overline{A^i}$ with $p(y) \neq 0$, we can find a net $(z_\nu)_{\nu \in \mathbb{N}}$ in A^i such that $z_\nu \longrightarrow y$. There is some $\nu_0 \in \mathbb{N}$ such that

$$p(y) = p(z_\nu) < 1, \qquad \nu \geq \nu_0.$$

ii) $\Longrightarrow$ iii) Obvious.

iii) $\Longrightarrow$ i) Choose $y \in A$. Then $\lambda y \in A^i$ for all $\lambda \in B(0, 1^-)$.

a) If the valuation of $\mathbb{K}$ is discrete, $A^i = \rho A$ and $p(\lambda y) < 1$ for all $\lambda \in B(0, 1^-)$, so $p(\rho y) < 1$ and thus $p(y) < 1/|\rho|$ which means that $p(y) \leq 1$.

b) If the valuation of $\mathbb{K}$ is dense, we have $p(\lambda y) < 1$ for all $\lambda \in B(0, 1^-)$, so

$$p(y) \leq \inf_{\lambda \in B(0,1^-)} \frac{1}{|\lambda|} = 1. \square$$

3.5. Proposition (W. H. Schikof)

Let $A \subset E$ be a.c. and closed, let $x \in E \setminus A$. Then there exists a continuous seminorm p with $p(a) < 1$ for $a \in A$ and $p(x) = 1$. If the valuation of $\mathbb{K}$ is discrete, p can be chosen such that $p(z) \in |\mathbb{K}|$ for any $z \in E$. As $\mathbb{K}$ is spherically complete, we can choose $p = |f|$ with $f \in E'$.

Proof. [3, Proposition 4.2]. $\square$

3.6. Proposition (W. H. Schikof)

Let $A \subset E$ be a.c. For $x \in A$, the following are equivalent:

i) there exists a continuous seminorm p with $p \neq 0$ on A and

$$p(x) = \max_{y \in A} p(y);$$

ii) $x \in \partial A$.

Proof. i) $\Longrightarrow$ ii) Suppose $x \in \overline{A^i}$ and set

$$U = \{z \in E : p(z) < p(x)\}.$$

$x + U$ meets A^i, so $x = u + v$ where $u \in U$ and $v \in A^i$. As $v \in A^i$ and $p \leq p(x)$ on A, we have that $p(v) < p(x)$. So

$$p(x) \leq \max\{p(u), p(v)\} < p(x),$$

which is a contradiction.

ii) $\Longrightarrow$ i) According to proposition 3.5, there is a continuous seminorm p with $p < 1$ on $\overline{A^i}$ and $p(x) = 1$. From lemma 3.4 we deduce that $p \leq 1$ on A so

$$p(x) = 1 = \max_{y \in A} p(y). \ \square$$

3.7. Definition. For $A \subset E$, absorbing, we define

$$p_A(x) = \inf\{|\lambda| : \lambda \in \mathbb{K},\ x \in \lambda A\}.$$

Note that p_A is a seminorm on E. (p_A is the so called Minkowski functional).

3.8. Definition. $A \subset E$ a.c. is called weakly c'-compact, if for each $f \in E'$ there is some $x \in A$ such that

$$|f(x)| = \max_{y \in A} |f(y)|.$$

3.9. Corollary

Let $A \subset E$ be a.c., closed and weakly c'-compact. Then $\partial A \neq \emptyset$ and as a consequence $A/\overline{A^i}$ is not trivial.

3.10. Proposition

Let the valuation on $\mathbb{K}$ be discrete and let $A \subset E$ be a.c. Then the following are equivalent:

i) A is weakly c'-compact;

ii) A is bounded.

Proof. [4, proposition 4.2]. $\square$

3.11. Corollary

If the valuation on $\mathbb{K}$ is dense and if $A \subset E$ is a.c. and weakly c'-compact, then A is bounded.

Proof. [4, proposition 4.2]. $\square$

4. Krein-Milman like theorems in Banach spaces

4.1. Remark. Throughout sections 4 and 5, $A \neq \{0\}$ will be an a.c., *closed* and *weakly c'-compact* subset of E.

4.2. Construction. With the notations of 3.2 and 3.3, we put $V = A/\overline{A^i}$. V is a k-vector space and the formula

$$\|\pi(x)\| = \inf_{t \in \overline{A^i}} \|x - t\| \ (= \operatorname{dist}(x, \overline{A^i}))$$

for $x \in A$, defines a norm on V. This norm induces a topology on V which we will use in the sequel. On A we establish the topology induced by the norm on E. As a consequence, π is continuous and $\|\pi(x)\| \leq \|x\|$ for all $x \in A$.

4.3. Proposition

$(V, \|\cdot\|)$ is complete.

4.4. Proposition

$(V, \|\cdot\|)$ has a t-orthogonal base for any $t \in (0,1)$.

Proof. Corollary 2.5. □

4.5. Proposition

If $A = \overline{\text{co}}\, X$, then $\overline{[\pi(X)]} = V$.

Proof. As π is continuous

$$V = \pi(\overline{\text{co}}\, X) \subset \overline{\pi(\text{co}X)} = \overline{[\pi(X)]} \subset V. \ \square$$

4.6. DEFINITION. Let $B \subset E$ be a.c. and closed. X is called a generating subset of B if $\overline{\text{co}}\, X = B$. It is called a minimal generating subset of B if it is a generating subset of B and if for every $Y \subset X$ with $\overline{\text{co}}\, Y = B$, $Y = X$.

4.7. Corollary

$Y \subset V$ is a generating subset of V if and only if $\overline{[Y]} = V$.

4.8. Proposition

If $Y \subset V$ is a generating subset of V and if T is a subset of A such that $\pi(T) = Y$, then $A = \overline{\text{co}}\, T$.

Proof. Suppose $A \neq \overline{\mathrm{co}}\, T$. Choose $x \in A \setminus \overline{\mathrm{co}}\, T$. There exists a continuous seminorm p (we may even choose $p = |f|$, for some $f \in E'$, because $\mathbb{K}$ is spherically complete) such that $p(x) = 1$ and $p(\overline{\mathrm{co}}\, T) < 1$. As A is weakly c'-compact, there exists $\alpha \geq 1$ and $z \in A$ such that

$$p(z) = \alpha = \max_{y \in A} p(y).$$

Hence, $p\left(\overline{A^i}\right) < \alpha$ and $p(\overline{\mathrm{co}}\, T) < \alpha$. But as $\overline{[\pi(T)]} = V$, we have that

$$A = \overline{\overline{A^i} + \overline{\mathrm{co}}\, T}.$$

Indeed, for $y \in A$ and $\epsilon > 0$, there is some $t \in T$ such that

$$\|\pi(y) - \pi(t)\| = \|\pi(y - t)\| < \epsilon,$$

hence there is some $a \in \overline{A^i}$ such that

$$\|y - (t + a)\| < \epsilon.$$

Hence

$$\max_{z \in A} p(z) < \alpha,$$

which is a contradiction. □

4.9. Corollary

Let $X \subset A$, such that $\overline{\mathrm{co}}\, X = A$ and such that $\pi_{|X}$ is injective. Then X is a minimal generating subset of A if and only if $\pi(X)$ is a minimal generating subset of V.

Proof. "only if": Suppose that there eixsts a proper subset Y of $\pi(X)$ such that $\overline{[Y]} = V$. Then there is a proper subset T of X such that $\pi(T) = Y$. According to 4.8, $\overline{\mathrm{co}}\, T = A$, which is a contradiction with the minimality of X.

"if": Suppose that there exists $Y \subset X$, $Y \neq X$, such that $\overline{\mathrm{co}}\, Y = A$. Then $\overline{[\pi(Y)]} = V$. But obviously, $\pi(Y) \subset \pi(X)$ and $\pi(Y) \neq \pi(X)$, and this contradicts the fact that $\pi(X)$ is a minimal generating subset of V. □

4.10. Corollary

For $t \in (0,1)$, let $(s_i)_{i \in I}$ be a t-orthogonal base of $(V, \|\cdot\|)$. For each $i \in I$, choose $e_i \in A$ such that $\pi(e_i) = s_i$. Then $A = \overline{\mathrm{co}}\,\{e_i : i \in I\}$ and $\{e_i : i \in I\}$ is a minimal generating subset of A. Note that $\{e_i : i \in I\} \subset \partial A$.

4.11. Corollary

If the topology on V is discrete and if S is a minimal generating subset of A, then $\pi(S)$ is an algebraic base of V.

Proof. As $\overline{\mathrm{co}}\, S = A$ and as the topology on V is discrete, it follows that

$$[\pi(S)] = \overline{[\pi(S)]} = V,$$

which means that $\pi(S)$ contains an algebraic base of V. Hence, as $\pi(S)$ is a minimal generating subset of V, $\pi(S)$ is an algebraic base of V. □

4.12. Remark. The topology on V can be discrete. Indeed, consider the following example: Let the valuation of $\mathbb{K}$ be discrete, and set $A = \{x \in E : \|x\| \leq 1\}$. Note that A is closed and weakly c'-compact (corollary 3.11). Then

$$A^i = \{x \in E : \|x\| \leq |\rho|\}.$$

It follows that $\|\pi(x)\| = 1$ for $x \in \partial A$, and $\|\pi(x)\| = 0$ for $x \in A^i$, so the topology on V induced by $\|\cdot\|$ is discrete. □

4.13. Remark. Here we give an example of a situation where the topology on V induced by $\|\cdot\|$ is not discrete. Let the valuation of $\mathbb{K}$ be discrete. Put

$$c_0 = \left\{\alpha = (\alpha_n)_{n\in\mathbb{N}_0} : \alpha_n \in \mathbb{K}\ \forall n \in \mathbb{N}_0,\ \lim_{n\to\infty} \alpha_n = 0\right\}.$$

For $\alpha \in c_0$, we put

$$\|\alpha\| = \max_{n\in\mathbb{N}_0} |\alpha_n|.$$

Let $(a_n)_{n\in\mathbb{N}_0}$ be the canonical base of c_0 and set

$$A = \overline{\mathrm{co}}\left\{\rho^{n-1} a_n : n \in \mathbb{N}_0\right\}.$$

Note that A is weakly c'-compact (corollary 3.11). Then

$$A^i = \rho A = \overline{\mathrm{co}}\left\{\rho b_n : n \in \mathbb{N}_0\right\}$$

where $b_n = \rho^{n-1} a_n$.

For $k \in \mathbb{N}_0$ and $t \in \rho A$ we have

$$\|b_k - t\| = \max_{n\in\mathbb{N}_0}\left\{|t_n|\ (n \neq k),\ |\rho^{k-1} - t_k|\right\}.$$

Since we have $|t_n| \leq |\rho|^k$ for $n \geq k$, it follows that $|\rho^{k-1} - t_k| = |\rho|^{k-1}$ and thus:

$$\|\pi(b_k) - t\| = \max_{n<k}\left\{|t_n|, |\rho|^{k-1}\right\}.$$

So, for each $t \in A^i$, we have that $\|b_k - t\| \geq |\rho|^{k-1}$ and thus $\pi(b_k)\| = |\rho|^{k-1}$. It follows that

$$\|V\| = \{\|\pi(x)\| : x \in A\} = \{|\rho|^{k-1} : k \in \mathbb{N}_0\} \cup \{0\},$$

and hence the topology on V is not discrete.

4.14. Remark. Later on (5.1.6 and 5.2.11), we will see that for infinite dimensional A, the topology on V induced by $\|\cdot\|$ can be discrete only if the valuation of $\mathbb{K}$ is discrete.

4.15. Remark. Looking at 4.9, it would be nice to know wether for some minimal generating subset X of A, $\pi(X)$ is a base of V. However, this is not true in general, although V itself has a base.

EXAMPLE. Let the valuation on $\mathbb{K}$ be discrete and let $E = c_0$ with the max norm. Put $(a_n)_{n\in\mathbb{N}_0}$ the canonical base of E and for $n \in \mathbb{N}_0$, put $x_n = a_1 + \rho^n a_{n+1}$.

Set

$$A = \overline{\text{co}}\left\{\rho^{n-1}a_n : n \in \mathbb{N}_0\right\}.$$

Then

$$A = \overline{\text{co}}\left\{x_n : n \in \mathbb{N}_0\right\}.$$

($\supset$ is obvious and for $\subset$, observe that $a_1 = \lim_{n\to\infty} x_n$).

After some calculation, we find that, for any $n \in \mathbb{N}_0$:

$$\text{dist}\left(x_n, \overline{\text{co}}\left\{x_m : m \neq n\right\}\right) = |\rho|^n > 0,$$

so $\{x_n : n \in \mathbb{N}_0\}$ is a minimal generating subset of A, hence $\{\pi(x_n) : n \in \mathbb{N}_0\}$ is a linearly independent subset of V.

$$\overline{A^i} = \rho A = \overline{\text{co}}\left\{\rho^n a_n : n \in \mathbb{N}_0\right\}.$$

For $t \in \rho A$, we have that

$$t = \sum_{n=1}^{\infty} \lambda_n^t \rho^n a_n$$

and, for any $n \in \mathbb{N}_0$, that $|\lambda_n^t| \le 1$.

So, for any $n \in \mathbb{N}_0$:

$$\|x_n - t\| = \max\left\{|1 - \lambda_1^t\rho|, \max_{m\in\mathbb{N}_0\setminus\{n+1\}} |\lambda_m^t\rho^m|, |\rho|^n |1 - \lambda_{n+1}^t\rho|\right\} \ge 1 = \|x_n\|,$$

hence $\|\pi(x_n\| = 1$.

But $\{\pi(x_n) : n \in \mathbb{N}_0\}$ is not a base of V. Indeed, suppose that $\{\pi(x_n) : n \in \mathbb{N}_0\}$ is a base of V. Then, for each $v \in V$ there exists $\bar{\lambda}_n \in k$ such that

$$v = \sum_{n=1}^{\infty} \bar{\lambda}_n \pi(x_n)$$

and $\lambda_n \pi(x_n) \longrightarrow 0$.

But as $\|\pi(x_n)\| = 1$ for all $n \in \mathbb{N}_0$, there is some $n_0 \in \mathbb{N}_0$ such that $\dot{\lambda}_n = 0$ for $n \geq n_0$, hence $\{\pi(x_n) : n \in \mathbb{N}_0\}$ is an algebraic base of V.

But then there exist $N \in \mathbb{N}_0$, $i_1, \ldots, i_N \in \mathbb{N}_0$ such that

$$\pi(a_1) = \sum_{n=1}^{N} \lambda_{i_n} \pi(x_{i_n}),$$

with $\lambda_{i_n} \neq 0$ for $n \in \{1, \ldots, N\}$.

Hence,

$$\pi(a_1) = \left(\sum_{n=1}^{N} \lambda_{i_n} \right) \pi(a_1) + \sum_{n=1}^{N} \lambda_{i_n} \pi\left(\rho^{i_n} a_{i_n+1}\right).$$

But $\{\pi(\rho^{n-1} a_n) : n \in \mathbb{N}_0\}$ is a linearly independent subset of V (because $(\rho^{n-1} a_n)_{n \in \mathbb{N}_0}$ is a minimal generating subset of A), hence $\lambda_{i_n} = \bar{0}$ for $n \in \{1, \ldots, N\}$ and

$$\sum_{n=1}^{N} \dot{\lambda}_{i_n} = 1,$$

which is a contradiction. In the same way one can prove that a_1 does not have a unique expansion in terms of the $(x_n)_{n \in \mathbb{N}_0}$.

5. A connection between A and V

5.1. The valuation on $\mathbb{K}$ *is discrete*

5.1.1. Remark. Throughout 5.1, the valuation on $\mathbb{K}$ is *discrete*.

5.1.2. Construction. In 5.1.2, we will determine some notations and definitions which are valid throughout 5.1.

1) As $\|E\| \subset |\mathbb{K}|$ and as the valuation on $\mathbb{K}$ is discrete, it follows that $\|V\| \subset |\mathbb{K}|$. Hence, $(V, \|\cdot\|)$ has an orthogonal base $(s_i)_{i \in I}$.

2) Thoughout 5.1, the choice of $(s_i)_{i \in I}$ will *not* be altered.

5.1.3. Lemma

With the notations of 5.1.2, we have that for every $i \in I$ *there exists* $e_i \in \partial A$ *such that* $\|e_i\| = \|s_i\|$ *and* $\pi(e_i) = s_i$.

Proof. For every $i \in I$, there exists $u_i \in \partial A$ such that $\pi(u_i) = s_i$. Then $\mathrm{dist}(u_i, \rho A) = \|s_i\|$. So, there exists $v_i \in \rho A$, such that

$$\|u_i - v_i\| < \frac{\|s_i\|}{|\rho|} .$$

For each $i \in I$, put $e_i = u_i - v_i$. It is easy to see that, for every $i \in I$, e_i has the required properties. □

5.1.4. Proposition

Let $(e_i)_{i\in I}$ be a family in A with the properties mentioned in lemma 5.1.3. Then $(e_i)_{i\in I}$ is an orthogonal subset of E.

Proof. For $J \subset I$, finite, put

$$x = \sum_{j\in J} \lambda_j e_j$$

with $\lambda_j \in \mathbb{K}$ for $j \in J$. We will assume that $x \neq 0$. Put

$$L = \left\{ j \in J : \|\lambda_j e_j\| = \max_{i\in J} \|\lambda_i e_i\| = \beta \right\}$$

and choose $j_0 \in L$ such that

$$|\lambda_{j_0}| = \max_{i\in L} |\lambda_i|$$

($\lambda_{j_0} \neq 0$ as $x \neq 0$).

Then

$$\begin{aligned}
\beta &\geq \left\| \sum_{i\in L} \lambda_i e_i \right\| \\
&= |\lambda_{j_0}| \left\| \sum_{i\in L} \frac{\lambda_i}{\lambda_{j_0}} e_i \right\| \\
&\geq |\lambda_{j_0}| \left\| \sum_{i\in L} \overline{\left(\frac{\lambda_i}{\lambda_{j_0}}\right)} s_i \right\| \\
&= |\lambda_{j_0}| \max_{\{i\in L : |\lambda_i| = |\lambda_{j_0}|\}} \|s_i\| \\
&= \max_{\{i\in L : |\lambda_i| = |\lambda_{j_0}|\}} |\lambda_{j_0}| \, \|e_i\| \\
&= \max_{\{i\in L : |\lambda_i| = |\lambda_{j_0}|\}} \|\lambda_i e_i\| \\
&= \beta .
\end{aligned}$$

and therefore

$$\|x\| = \left\|\sum_{i\subset L} \lambda_i e_i + \sum_{i\notin L} \lambda_i e_i\right\| = \left\|\sum_{i\in L} \lambda_i e_i\right\| = \beta = \max_{i\subset J} \|\lambda_i e_i\|.$$

Hence, $(e_i)_{i\in I}$ is an orthogonal family of E. □

5.1.5. Theorem

There exists an orthogonal family $(e_i)_{i\in I}$ in E such that $A = \overline{\text{co}}\{e_i : i \in I\}$. Hence, there exists a family $(T_i)_{i\in I}$ of one-dimensional a.c. subsets of E such that

$$A = \bigoplus_{i\in I}^{\perp} T_i.$$

5.1.6. Corollary

The following are equivalent:
i) A is open in $\overline{[A]}$;
ii) The topology induced by $\|\cdot\|$ on V is discrete.

Proof. i) $\Longrightarrow$ ii) We have $p_A(x) = 1$ for $x \in \partial A$ and $p_A(x) < 1$ for $x \in \overline{A^i}$. A is open in $\overline{[A]}$, so there is some $\epsilon > 0$ such that

$$B = \left\{x \in \overline{[A]} : \|x\| \leq \epsilon\right\} \subset A.$$

As a consequence $p_A \leq p_B$. Hence, there exists $c > 0$ such that $p_A(x) \leq c\,\|x\|$ for $x \in \overline{[A]}$.

As p_A is continuous on $\overline{[A]}$, it follows that

$$\inf_{t\in \overline{A^i}} p_A(x - t) = 1, \qquad x \in \partial A,$$

hence

$$c\,\text{dist}\left(x, \overline{A^i}\right) \geq 1, \qquad x \in \partial A,$$

so the topology induced by $\|\cdot\|$ on V is discrete.

ii) $\Longrightarrow$ i) Let $(e_i)_{i\in I}$ be an orthogonal family in $\overline{[A]}$ with the properties mentioned in lemma 5.1.3. As $A = \overline{\text{co}}\{e_i : i \in I\}$ it follows that

$$\overline{[A]} = \overline{\left[\{e_i : i \in I\}\right]}.$$

Now, the topology on V is discrete, hence there exists $\alpha > 0$ such that

$$\|e_i\| = \|s_i\| > \alpha, \qquad i \in I.$$

So

$$B := \{x \in \overline{[A]} : \|x\| < \alpha\} \subset A.$$

Indeed, for $y \in B$, put

$$y = \sum_{i \in I} \lambda_i e_i$$

with $|\lambda_i| \to 0$. Then

$$\|y\| = \max_{i \in I} \|\lambda_i e_i\| > \alpha \max_{i \in I} |\lambda_i|.$$

Hence,

$$\max_{i \in I} |\lambda_i| < 1 \implies y \in A. \ \square$$

5.2. The valuation on $\mathbb{K}$ is dense

5.2.1. Remark. Throughout 5.2, the valuation on $\mathbb{K}$ is *dense*. We will also assume that A is *infinite-dimensional*.

5.2.2. Construction. In 5.2.2 we will determine some notations and definitions which will be valid throughout 5.2.

1) Choose $t \in (0, 1)$. In the sequel, t will remain unchanged.

2) On $(V, \|\cdot\|)$ we establish a norm p which has the properties mentioned in lemma 2.2, ie,

$$t\|v\| < p(v) \leq \|v\|, \qquad v \in V \setminus \{0\}$$

and

$$p(V) \subset \{t^n : n \in \mathbb{Z}\} \cup \{0\}.$$

In the sequel, the choice of p will *not* be altered.

3) Let $(s_i)_{i \in I}$ be an orthogonal base of (V, p). In the sequel, the choice of $(s_i)_{i \in I}$ will not be altered.

Of course, there exists for every $i \in I$ an $n_i \in \mathbb{Z}$ such that:

$$t^{n_i+1} \leq p(s_i) \leq \|s_i\| < t^{n_i}.$$

5.2.3. Lemma

With the notations of 5.2.2, we have that, for every $i \in I$, there exists $e_i \in \partial A$ such that

$$\|s_i\| \leq \|e_i\| < t^{n_i}, \qquad \pi(e_i) = s_i.$$

Proof. For every $i \in I$, there exists $u_i \in \partial A$ such that $\pi(u_i) = s_i$. Then

$$\operatorname{dist}\left(u_i, \overline{A^i}\right) = \|s_i\|,$$

so there exists $v_i \in \overline{A^i}$, such that

$$\|u_i - v_i\| < t^{n_i}.$$

For $i \in I$, put $e_i = u_i - v_i$. It is easy to see that e_i has the properties required. □

5.2.4. *Remark.* One could ask the question of wether a family $(e_i)_{i\in I}$ which has the properties mentioned in 5.2.3 (and for which we know that it is a minimal generating subset of A), is a t'-orthogonal subset of $(E, \|\cdot\|)$ for some $t' \in (0,1)$.

I haven't been able (yet) to prove that the family $(e_i)_{i\in I}$ is a t'-orthogonal subset of E, nor have I been able (yet) to find a counterexample for the fact that it is not.

Still, we have the following:

5.2.5. Proposition

Let $(e_i)_{i\in I}$ be a family in A such that, for all $i \in I$, $\|s_i\| \leq \|e_i\| < t^{n_i}$ and $\pi(e_i) = s_i$. The family $(e_i)_{i\in I}$ has the following properties:

i) $(e_i)_{i\in I}$ is a linearly independent subset of E.

ii) For all $i \in I$: $\operatorname{dist}\big(e_i, \overline{\mathrm{co}}\{e_j : j \neq i\}\big) \geq t\,\|e_i\|$.

iii) For all $i, j \in I$ with $i \neq j$, e_i and e_j are t-orthogonal.

Proof. i) First note that for all $i \in I$, $\|e_i\| \geq \|s_i\| > 0$. ($(s_i)_{i\in I}$ is a base of V!). Now, for $J \subset I$, finite, consider $\sum_{i\in J} \lambda_i e_i$ with $\lambda_i \neq 0$ for all $i \in J$. Put

$$|\lambda| = \max_{i\in J} |\lambda_i| \neq 0, \qquad J_1 = \{i \in J : |\lambda_i| = |\lambda|\}.$$

Then

$$\begin{aligned}
\left\|\sum_{i\in J} \lambda_i e_i\right\| &= |\lambda| \left\|\sum_{i\in J} \frac{\lambda_i}{\lambda}\, e_i\right\| \\
&\geq |\lambda| \left\|\sum_{i\in J} \overline{\left(\frac{\lambda_i}{\lambda}\right)} s_i\right\| \\
&\geq |\lambda|\, p\left(\sum_{i\in J} \overline{\left(\frac{\lambda_i}{\lambda}\right)} s_i\right) \\
&\geq |\lambda| \max_{i\in J_1} p(s_i) \\
&= t \max_{i\in J_1} |\lambda_i| \frac{p(s_i)}{t} \\
&\geq t \max_{i\in J_1} |\lambda_i|\, \|e_i\|,
\end{aligned}$$

hence

$$\|\sum_{i\in J}\lambda_i e_i\| > 0.$$

This implies that $(e_i)_{i\in I}$ is a linearly independent family of E.

ii) Choose $i \in I$. For $J \subset I \setminus \{i\}$, finite, and for $(\lambda_j)_{j\in J} \in B(0,1)^J$, we have that

$$\begin{aligned}\left\|e_i - \sum_{j\in J}\lambda_j e_j\right\| &\geq \left\|s_i - \sum_{j\in J}\lambda_j s_j\right\| \\ &\geq p\left(s_i - \sum_{j\in J}\overline{\lambda}_j s_j\right) \\ &= t\max_{i\in J}\left\{\frac{p(s_i)}{t}, \frac{p(\bar{\lambda}_j s_j)}{t}\right\} \\ &\geq t\,\frac{p(s_i)}{t} \\ &\geq t\,\|e_i\|.\end{aligned}$$

iii) Let $i, j \in I$ and $i \neq j$. Choose $\lambda, \mu \in K$. We only have to consider the case that $|\lambda|\,\|e_i\| = |\mu|\,\|e_j\|$. We may assume that $|\lambda| \geq |\mu|$. Then

$$\begin{aligned}\|\lambda e_i + \mu e_j\| &= |\lambda|\left\|e_i + \frac{\mu}{\lambda}e_j\right\| \\ &\geq |\lambda|\left\|s_i + \overline{\left(\frac{\mu}{\lambda}\right)}s_j\right\| \\ &\geq t\,|\lambda|\,\frac{p(s_i)}{t} \\ &\geq t\max\{\|\mu e_j\|, \|\lambda e_j\|\}. \ \square\end{aligned}$$

5.2.6. *Remark.* All the preceding results are also valid if the valuation on $\mathbb{K}$ is discrete, but as we have seen, even stronger results hold for a discretely valued field, so we decided to mention them for a densely valued field.

5.2.7. *Construction.* 1) To lighten the proof of proposition 5.2.8., and as A is bounded (corollary 3.11), we may assume that $\sup_{x\in A}\|x\| < 1$. As a consequence $\sup_{v\in V}\|v\| < 1$.

2) For $n \in \mathbb{N}_0$, put

$$I_n = \{i \in I : p(s_i) = t^n\},$$

and

$$B_n = \{s_i : i \in I_n\}.$$

Then $\bigcup_{n\in\mathbb{N}_0} B_n$ is an orthogonal base of (V,p).

3) Let $(e_i)_{i\in I}$ be a family in A with the properties mentioned in lemma 5.2.3. For $n \in \mathbb{N}_0$, put

$$E_n = \{e_i | i \in I_n\}.$$

Then

$$A = \overline{\text{co}}\left(\bigcup_{n\in\mathbb{N}_0} E_n\right).$$

4) Throughout the rest of 5.2, we only use the family $(e_i)_{i\in I}$ constructed above. □

5.2.8. Proposition

With the notations and assumptions of 5.2.7, we have $\#I_n < \infty$ for all $n \in \mathbb{N}_0$.

Proof. Suppose that there exists $n_0 \in \mathbb{N}_0$ such that $\#I_{n_0}$ is not finite. We are going to split up our proof in seven parts.

1) Put

$$J := \bigcup_{k=1}^{n_0} I_k.$$

Then $(e_i)_{i\in J}$ is a t^{n_0}-orthogonal subset of $(E, \|\cdot\|)$.

First, note that $p(s_j) \geq t^{n_0}$ for all $j \in J$. Let $J" \subset J$ be finite and consider $(\lambda_i)_{i\in J"} \in K^{J"}$. Put

$$J_1 = \left\{i \in J" : |\lambda_i| = \max_{j\in J"} |\lambda_j|\right\}$$

and choose $\lambda \in K$ such that

$$|\lambda| = \max_{j\in J"} |\lambda_j|.$$

We may assume that $\lambda \neq 0$.

Then

$$\begin{aligned}\|\sum_{i\in J"} \lambda_i e_i\| &= |\lambda| \left\|\sum_{i\in J"} \frac{\lambda_i}{\lambda} e_i\right\| \\ &\geq |\lambda| \left\|\sum_{i\in J"} \overline{\left(\frac{\lambda_i}{\lambda}\right)} s_i\right\|\end{aligned}$$

$$
\begin{aligned}
&\geq |\lambda| p\left(\sum \overline{\left(\frac{\lambda_i}{\lambda}\right)} s_i\right) \\
&= |\lambda| \max_{i \in J_1} p(s_i) \\
&\geq t^{n_0} |\lambda| \\
&= t^{n_0} \max_{j \in J''} |\lambda_j| \\
&\geq t^{n_0} \max_{j \in J''} |\lambda_j| \, \|e_j\|.
\end{aligned}
$$

2) Put $D = \overline{[\{e_i | i \in J\}]}$. Then $\{e_i | i \in J\}$ is a t^{n_0}-orthogonal base of $(D, \|\cdot\|)$. See 1.7 for a proof.

3) Define $q : D \to \mathbb{R}^+$ by

$$x = \sum_{i \in J} \lambda_i e_i \ (\lambda_i \longrightarrow 0) \longmapsto t^{n_0} \max_{i \in J} |\lambda_i|.$$

Then q is a norm on D, equivalent to $\|\cdot\|$.

It is easy to see that q is a norm on D.

Now, put $x = \sum_{i \in J} \lambda_i e_i$ with $\lambda_i \longrightarrow 0$ and put

$$J_1 = \left\{ i \in J : |\lambda_i| = \max_{j \in J} |\lambda_j| \right\}.$$

We will assume that $x \neq 0$. Choose $\lambda \in \mathbb{K}$ such that $|\lambda| = \max_{j \in J} |\lambda_j|$.

Then

$$
\begin{aligned}
\|x\| &= \left\| \sum_{i \in J \setminus J_1} \lambda_i e_i + \sum_{i \in J_1} \lambda_i e_i \right\| \\
&\geq |\lambda| p\left(\sum_{i \in J \setminus J_1} \overline{\left(\frac{\lambda_i}{\lambda}\right)} s_i + \sum_{i \in J_1} \overline{\left(\frac{\lambda_i}{\lambda}\right)} s_i \right) \\
&= |\lambda| p\left(\sum_{i \in J_1} \overline{\left(\frac{\lambda_i}{\lambda}\right)} s_i \right) \\
&= |\lambda| \max_{i \in J_1} p(s_i) \\
&\geq t^{n_0} \max_{i \in J} |\lambda_i| \\
&= q(x) \\
&\geq t^{n_0} \max_{i \in J} |\lambda_i| \, \|e_i\| \\
&\geq t^{n_0} \|x\|.
\end{aligned}
$$

4) We can extend q to a norm $\bar{q}$ on E, equivalent to $\|\cdot\|$ and such that $\bar{q}_{|D} = q$. For $y \in E$, put

$$\bar{q}(y) = \inf_{d \in D} \max\{\|y - d\|, q(d)\}.$$

See also [5, lemma 6.14].

5) Now, let $(i_k)_{k\in\mathbb{N}_0}$ be a subset of I_{n_0}. Choose $(\eta_k)_{k\in\mathbb{N}_0}$ in $\mathbb{K}$ such that

$$0 < |\eta_1| < |\eta_2| < \cdots < 1, \qquad \lim_{k\to\infty} |\eta_k| = 1.$$

We define $f : D \to \mathbb{K}$ by

$$\sum_{i\in J} \lambda_i e_i \longmapsto \sum_{k\in\mathbb{N}_0} \lambda_{i_k} \eta_k.$$

Clearly, f is linear and well-defined. f is also continuous.

For $x = \sum_{i\in J} \lambda_i e_i$ with $\lambda_i \in \mathbb{K}$ and $\lambda_i \longrightarrow 0$, we have that:

$$\begin{aligned}
\left| f\left(\sum_{i\in J} \lambda_i e_i\right)\right| &= \left| f\left(\sum_{k\in\mathbb{N}_0} \lambda_{i_k} e_{i_k}\right)\right| \\
&= \left|\sum_{k\in\mathbb{N}_0} \lambda_{i_k} \eta_{i_k}\right| \\
&\leq \max_{k\in\mathbb{N}_0} |\lambda_{i_k}|\,|\eta_k| \\
&\leq \max_{k\in\mathbb{N}_0} |\eta_k| \frac{1}{\|e_{i_k}\|} |\lambda_{i_k}|\,\|e_{i_k}\| \\
&\leq \frac{1}{t^{2n_0}} \|x\|.
\end{aligned}$$

6) For $x \in D$, $|f(x)| \leq t^{-n_0} q(x)$. Hence, $|f(x)| \leq t^{-n_0} q(x)$ for $x \in D$.

We can extend $f \in D'$ to $\bar{f} \in E'$ such that $\bar{f}_{|D} = f$ and $|\bar{f}(x)| \leq t^{-n_0} q(x)$ for $x \in D$.

7) Finally, we arrive at our contradiction:

$$\begin{aligned}
\bar{f}(e_i) = f(e_i) = 0 \qquad &\forall i \in J \setminus \{i_k : k \in \mathbb{N}_0\},\\
|\bar{f}(e_{i_k})| = |f(e_{i_k})| = |\eta_k| < 1 \qquad &\forall i \in \{i_k : k \in \mathbb{N}_0\}\\
|\bar{f}(e_i)| \leq \frac{1}{t^{n_0}} q(e_i) \leq \frac{1}{t^{n_0}} \|e_i\| < \frac{1}{t^{n_0+1}} p(s_i) \leq 1 \qquad &\forall i \in I \setminus J.
\end{aligned}$$

Hence, $|\bar{f}(e_i)| < 1$ for $i \in I$, but

$$\sup_{x\in A} |\bar{f}(x)| = \sup_{i\in I} |\bar{f}(e_i)| = 1,$$

which is a contradiction. □

5.2.9. Proposition

$\lim_{i\in I} \|e_i\| = 0$.

Proof. $\lim_{i\in I} p(s_i) = 0$. Indeed, for every $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $t^n < \epsilon$ for all $n > N_\epsilon$. Then

$$p(s_i) < \epsilon, \qquad i \in I \setminus \bigcup_{k=1}^{N_\epsilon} I_k,$$

hence

$$\|e_i\| < \epsilon, \qquad i \in I \setminus \bigcup_{k=1}^{N_\epsilon} I_k,$$

and therefore

$$\lim_{i\in I} \|e_i\| = 0. \ \square$$

5.2.10. Corollary

$$\inf_{i\in I} p(s_i) = \inf_{i\in I} \|s_i\|.$$

5.2.11. Corollary

The topology induced by $\|\cdot\|$ on V is not discrete.

5.2.12. Definition. $B \subset E$ a.c. is called a (pure) compactoid if for each zero neighbourhood U of E, there exists a finite set $F \subset E$ ($F \subset B$) such that $B \subset U + \operatorname{co} F$.

5.2.13. Theorem

A is a pure compactoid.

Proof. $A = \overline{\operatorname{co}}\{e_i : i \in I\}$ and $\lim \|e_i\| = 0$. $\square$

5.2.14. Remark. This theorem answers a question asked by W.H. Schikhof [4], namely the question if each weakly c'-compact, a.c. subset of a B.S. over a spherically complete field $\mathbb{K}$ is a pure compactoid.

5.2.15. Corollary

If E is a locally convex space over a spherically complete field $\mathbb{K}$ with dense valuation, and if B is an a.c., weakly c'-compact subset of E, then B is a pure compactoid.

Proof. Let $U \subset E$ be a neighbourhood of 0. There is a continuous seminorm q such that $\{x \in E : q(x) \leq 1\} \subset U$. Let $\hat{E}_q$ be the completion of E_q. The canonical map $\pi_q : E \to E_q \subset \hat{E}_q$ is continuous, hence $\pi_q(B)$ is weakly c'-compact in $\hat{E}_q$. From 5.2.13 we deduce that $\pi_q(B)$ is a pure compactoid in $\hat{E}_q$, hence in E_q. Since $\pi_q(U)$ is open in $\hat{E}_q$, there exists a finite set $F \subset B$, such that:

$$\pi_q(B) \subset \pi_q(U) + \pi_q(F),$$

so we have:

$$B \subset U + \operatorname{co} F + \operatorname{Ker} q \subset U + \operatorname{co} F. \square$$

5.3. A is finite-dimensional.

5.3.1. Remark. Throughout 5.3 A is finite-dimensional and there is no assumption on the valuation of K.

5.3.2. Proposition

Without any assumption on the valuation of K, the following are equivalent:
i) A is n-dimensional;
ii) V is n-dimensional.

Proof. Choose $t \in (0,1)$ and let p be a norm on V such that for all $v \in V \setminus \{0\}$ $t\,\|v\| < p(v) \leq \|v\|$ and

$$p(V) \subset \{t^n : n \in \mathbb{Z}\} \cup \{0\}.$$

Let $\{s_i : i \in I\}$ be an orthogonal base of (V,p). For $i \in I$, choose $e_i \in A$ such that $\pi(e_i) = s_i$. Then

$$A = \overline{\operatorname{co}}\,\{e_i : i \in I\} = \operatorname{co}\{e_i : i \in I\}$$

and $(e_i)_{i\in I}$ is a linearly independent family in $[A]$.

i) $\Rightarrow$ ii) If $\#I \neq n$, it follows that A is not n-dimensional.

ii) $\Rightarrow$ i) If $\#I = n$, it follows that A is n dimensional.

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