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A note on the Köthe dual of Banach-valued echelon spaces

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Abstract

Several different ways of defining the Köthe α-dual of echelon spaces of Banach-valued functions are shown to be equivalent.

Let (E, Σ, μ) be a measure space where E is a locally compact topological Hausdorff space and μ a regular non-negative σ-finite measure defined on a σ-algebra Σ containing all Borel sets in E. Let $g_1, g_2, \ldots$, be an increasing sequence of non-negative measurable functions such that

$$\mu\{x \in E : g_k(x) = 0 \text{ for all } k = 1, 2, \ldots\} = 0.$$

For $p \geq 1$ the echelon Köthe space of order p associated to $(g_k)_k$ is defined as the space $\Lambda^p = \Lambda^p(E, \Sigma, \mu, (g_k)_k)$ of all measurable functions $f : E \to \mathbb{R}$ such that

$$p_k(f) := \left(\int_E |f|^p g_k \, d\mu \right)^{1/p} < +\infty \qquad \text{for all } k = 1, 2, \ldots.$$

With the system of seminorms $p_1, p_2, \ldots$, Λ^p is a Fréchet space and its topological dual is the same as its Köthe α-dual $(\Lambda^p)^\alpha$ defined by:

$$(\Lambda^p)^\alpha := \left\{ g : E \to \mathbb{R} : g \text{ is measurable and } \int |fg| \, d\mu < +\infty \text{ for all } f \in \Lambda^p \right\}.$$

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The theory of echelon Köthe spaces has been widely studied by J. A. López Molina [8–11], J. C. Díaz Alcaide [1–3] and K. Reiher [13,14]. This theory can be extended to functions with values in a Banach space: Let X be a Banach space with dual X', a function $f : E \to X$ is said to be μ measurable if it is the μ-a.e. limit of a sequence of simple functions [4. II.1]. We define the corresponding Banach–valued echelon Köthe space as follows:

$$\Lambda^p(X) := \{f : E \to X : f \text{ is } \mu - \text{measurable and } \|f\| \in \Lambda^p\}.$$

When endowed with the topology defined by the system of seminorms $q_k(f) := p_k(\|f\|)$, $k = 1, 2, \ldots$, $\Lambda^p(X)$ is a Fréchet space.

To extend the Köthe α duality to this setting, one can follow several approaches, see, e. g., [7], [12] or [15]. Our purpose here is to show that these approaches are essentially the same, namely:

Theorem

For a μ-measurable function $g : E \to X'$, the following are equivalent:

(i)

$$\int_E \|f\| \, \|g\| \, d\mu < +\infty \qquad \text{for all } f \in \Lambda^p(X).$$

(ii) $\|g\| \in (\Lambda^p)^\alpha$.

(iii)

$$\int_E \left|\langle f(x), g(x)\rangle\right| d\mu(x) < +\infty \qquad \text{for all } f \in \Lambda^p(X).$$

To prove this, we need the following slight extension of a lemma which was stated in [6] and may be of independent interest. We include its proof for the sake of completeness. We shall make use of of the following form of Luzin's theorem: "*If X is a Banach space, $A \in \Sigma$ has finite measure, $f : A \to X$ is a μ measurable function and $\varepsilon > 0$, then a compact set $K \subset A$ there exists such that $\mu(A \setminus K) < \varepsilon$ and f is continuous on K*" [5, 9.1 and 10.2].

Lemma

Let $g : E \to X'$ and $\varepsilon : E \to \Re$ be μ measurable functions, ε in addition strictly positive. Then there exists a μ–measurable function $n : E \to X$ such that

(1) n is countably valued with values in the unit ball of X, and

(2) $\|g(x)\| \leq \langle g(x), n(x)\rangle + \varepsilon(x)$ $\quad$ *μ-a.e. in E.*

Proof. Since (E,Σ,μ) is σ finite, E can be covered by a sequence of pairwise disjoint sets all of them having finite measure and, by using Luzin's theorem repeatedly, we can find a sequence $(A_m)_m$ of compact, pairwise disjoint subsets of E such that $\mu(E \setminus \bigcup_m A_m) = 0$, and g and ε are continuous on each A_m. For every m we shall construct a simple function n_m satisfying conditions (1) and (2) on A_m. Then $n := \sum_m n_m$ will be the required n.

Fix $K = A_m$. For $x \in K$ there exists a vector $e(x) \in X$, with $\|e(x)\| \leq 1$ and such that

$$\|g(x)\| < \langle g(x), e(x)\rangle + \varepsilon(x).$$

Now, for $x \in K$ the function

$$t \in K \longmapsto \langle g(t), e(x)\rangle + \varepsilon(t) - \|g(t)\|$$

is continuous on K and strictly positive on x, therefore we can find an open neighbourhood of x, $U(x)$, such that for $t \in K \cap U(x)$ we have:

$$\|g(t)\| \leq \langle g(t), e(x)\rangle + \varepsilon(t).$$

Now $\{U(x) : x \in K\}$ is an open covering of the compact set K and therefore we may take a finite covering from it: $K \subset U(x_1) \cup \cdots \cup U(x_r)$. Take $B_1 = K \cap U(x_1)$ and in general

$$B_j = (K \cap U(x_j)) \setminus \bigcup_{i=1}^{j-1} B_i \qquad \text{for } j = 2,3,\ldots,r.$$

Then $K = \bigcup_i B_i$ and, for $e_i = e(x_i)$, we have for all $t \in K$

$$\|g(t)\| \leq \left\langle g(t), \sum_{i=1}^{r} e_i \chi_{B_i}(t) \right\rangle + \varepsilon(t).$$

Finally,

$$n_K(x) := \sum_{i=1}^{r} e_i \chi_{B_i}(x)$$

is the desired function on K. □

Proof of the theorem. Bearing in mind that for $f \in \Lambda^p$ and $u \in X$ we have $fu \in \Lambda^p(X)$, a straightforward computation proves (i) $\Longleftrightarrow$ (ii) $\Longrightarrow$ (iii). To prove (iii) $\Longrightarrow$ (ii), take a μ-measurable function $g : E \to X'$ such that

$$\int_E \left|\langle f(x), g(x)\rangle\right| d\mu(x) < +\infty$$

whenever $f \in \Lambda^p(X)$. Let $(E_n)_{n=1}^{\infty}$ be a sequence of pairwise disjoint, measurable sets, all of them having finite measure, that covers E and take

$$\varepsilon(x) := \sum_{n=1}^{\infty} \frac{\chi_{E_n}(x)}{2^n\left(\mu(E_n)+1\right)} .$$

Then $\varepsilon(x) > 0$ for all $x \in E$, ε is μ-measurable and

$$\int_E \varepsilon(x)\, d\mu(x) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Take $h \in \Lambda^p$ arbitrary, and apply the lemma above to ε and hg: there exists a μ measurable function n from E into the unit ball of X such that:

$$\|h(x)g(x)\| \leq \langle h(x)g(x), n(x)\rangle + \varepsilon(x) \qquad \mu - \text{a.e.}$$

Now, since $\|h(x)n(x)\| \leq |h(x)|$, we have that $hn \in \Lambda^p(X)$ and therefore:

$$\begin{aligned}
\int_E |h(x)|\,\|g(x)\|\, d\mu(x) &= \int_E \|h(x)g(x)\|\, d\mu(x) \\
&\leq \int_E \langle h(x)g(x), n(x)\rangle\, d\mu(x) + \int_E \varepsilon(x)\, d\mu(x) \\
&\leq \int_E \langle g(x), h(x)n(x)\rangle\, d\mu(x) + 1 \\
&< +\infty.
\end{aligned}$$

Since h was arbitrary, we have that $\|g\| \in (\Lambda^p)^\alpha$. □

DEFINITION. According to our theorem the Köthe α-dual of $\Lambda^p(X)$ is defined as the space of all μ-measurable funtions from E into X' satisfying either (i), (ii) or (iii).

Remark. For the case of echelon Köthe spaces, our result extends [12, Prop. 12] where the Banach space was assumed to be separable and reflexive. Also, (iii) provides a new characterization of the topological dual of $\Lambda^p(X)$ when X' has the Radon-Nikodým Property, see [4, IV.1] or, more generally, [7, Thm. 5].

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