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Prime nondegenerate Jordan algebras with nonzero socle and the symmetric Martindale algebra of quotients

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ABSTRACT

In this note we compute the symmetric Martindale algebra of quotients of a prime associative algebra with nonzero socle. Then we use this result to get, via Zelmanov's theorem for prime nondegenerate Jordan algebras, a shorter proof of the structure theorem of Osborn and Racine for prime Jordan algebras with nonzero socle.

1. Computing the left algebra of quotients and the symmetric algebra of quotien

Let A be a prime associative algebra (possibly without 1) over a commutative a sociative ring Φ with 1. Consider the set of all (f, U), $f : {}_AU \to {}_AA$ being a le A–module Φ–linear function and where U ranges over all nonzero (two-sided) idea of A. Two such functions (f, U) and (g, V) are said to be equivalent if they agr on their common domain $U \cap V$ which is nonzero since A is prime. It is easy to s that it is an equivalence relation [6, Lemma 1.1]. Let $[f, U]$ denote the equivalen class of (f, U) and let $Q_1 = Q_1(A)$ be the set of all such equivalence classes. Und the usual operations:

$$[f,U] + [g,V] = [f+g, U \cap V], \qquad \alpha[f,U] = [\alpha f, U], \qquad [f,U][g,V] = [fg, VU$$

Q_1 becomes an associative Φ–algebra with 1 called the *left Martindale algebra quotients* of A. The mapping $a \mapsto [R_a, A]$ ($bR_a = ba$, $b \in A$) is an embedding of into Q_1. Moreover, under this identification, we have

Proposition

Let A be a prime associative algebra and let Q_1 be the left Martindale algebra of quotients of A.

(i) Given $q \in Q_1$ there exists a nonzero ideal U of A such that $Uq \subset A$.

(ii) If $q \in Q_1$, U is a nonzero ideal of A and $Uq = 0$, then $q = 0$.

(iii) Every subalgebra B of Q_1 containing A is prime.

Proof. (i) and (ii) follow from [6, Prop. 1.2]. To show (iii) suppose that $q_1 B q_2 = 0$ for $q_1, q_2 \in B$. By (i), there exists a nonzero ideal U of A such that $Uq_i \subset A$, $i = 1,2$. Then $Uq_1 A U q_2 = 0$ implies $Uq_1 = 0$ or $Uq_2 = 0$ since A is prime. Hence, by (ii), $q_1 = 0$ or $q_2 = 0$. □

Following [6], the *symmetric Martindale algebra of quotients* of A is the subalgebra $Q_s(A)$ of $Q_1(A)$ given by

$$Q_s(A) = \{q \in Q_1(A) : Uq + qU \subset A \text{ for a nonzero ideal } U \text{ of } A\}.$$

Since A is contained in $Q_s(A)$, we have by Proposition 1 (iii) that $Q_s(A)$ is prime. Suppose now that A has an involution $* : A \to A$. Then $*$ can be extended to an involution of $Q_s(A)$ as follows:

For $q \in Q_s(A)$ choose a nonzero *–ideal U of A such that $Uq + qU \subset A$ and define $q^* = [g, U]$, where $xg = (qx^*)^*$, $x \in U$. Since

$$(ax)g = \big(q(ax)^*\big)^* = \big((qx^*)a^*\big)^* = a(qx^*)^* = a(xg) \qquad (a \in A,\ x \in U),$$

$q \mapsto q^*$ defines an involution on $Q_s(A)$. This extension is unique. Let $^\# : Q_s(A) \to Q_s(A)$ be a new involution extending $* : A \to A$. Given $q \in Q_s(A)$ choose a nonzero *–ideal U of A such that $qU + Uq \subset A$. Then for $u \in U$,

$$q^\# u^* = (uq)^\# = (uq)^* = q^* u^* \Longrightarrow (q^\# - q^*)U = 0 \Longrightarrow \text{ by Proposition (ii) } q^\# = q^*.$$

Let (X, Y, φ) be a pair of dual vector spaces over a division associative Φ–algebra Δ, i. e., X is a left vector space over Δ, Y is a right vector space and $\varphi : X \times Y \to \Delta$ is a nondegenerate bilinear form. A linear operator $a : X \to X$ is said to be continuous (relative to the dual pair (X, Y, φ)) if there exists a (necessarily unique) linear operator $a^\# : Y \to Y$ such that

$$\varphi(xa, y) = \varphi(x, a^\# y), \qquad x \in X,\ y \in Y.$$

The set of all continuous linear operators is a prime associative algebra $L_Y(X)$ whose *socle* (the sum of all minimal right (left) ideals) coincides with the ideal of all finite-rank continuous linear operators $F_Y(X)$. Conversely, every prime associative algebra A with nonzero socle is isomorphic to a subalgebra of $L_Y(X)$ containing $F_Y(X)$. If A has an involution $* : A \to A$, then the dual pair comes from a self-dual vector space X relative to a hermitian or alternate inner product $\langle \cdot, \cdot \rangle$, and $*$ is the canonical adjoint involution: $\langle xa, x' \rangle = \langle x, x' a^* \rangle$ [3, p. 13].

Theorem 1

(i) Let A be a prime associative algebra with nonzero socle. Then there exists a pair of dual vector spaces (X, Y, φ) over a division associative algebra Δ such that $Q_1(A) \cong \mathrm{End}_\Delta(X)$ and $Q_s(A) \cong L_Y(X)$.

(ii) Suppose now that A has an involution $* : A \to A$. Then there exists a hermitian or alternate self-dual vector space $(X, \langle\cdot,\cdot\rangle)$ such that $(Q_s(A), *)$ is isomorphic to $L_Y(X)$ with the adjoint involution.

Proof. (i) By above we may assume that A is a subalgebra of $L_Y(X)$ containing $F_Y(X)$. Write $M = F_Y(X)$. Then M is a simple ideal of A and a right ideal of $\mathrm{End}_\Delta(X)$. The latter follows since M is the linear span of all continuous linear operators $y \otimes x$ ($x \in X$, $y \in Y$) defined by $x'(y \otimes x) = \varphi(x', y)x$, $x' \in X$. Hence we have that the mapping $a \mapsto [R_a, M]$ is a monomorphism of $\mathrm{End}_\Delta(X)$ into $Q_1(A)$. We must prove that this is onto.

Let $[f, M] \in Q_1(A)$ and fix $0 \neq y_1 \in Y$, $0 \neq x_1 \in X$ such that $\varphi(x_1, y_1) = 1$. Then $e_1 = y_1 \otimes x_1$ is a nonzero idempotent in M. For every $x \in X$ we have that $e_1(y_1 \otimes x) = y_1 \otimes x$ and hence

$$(y_1 \otimes x)f = \big(e_1(y_1 \otimes x)\big)f = e_1\big((y_1 \otimes x)f\big) = y_1 \otimes u$$

for a unique $u \in X$. Write $u = xa$. We show that $a : X \to X$ is a linear operator For $x, x' \in X$ we have

$$\begin{aligned} y_1 \otimes (x + x')a &= \big(y_1 \otimes (x + x')\big)f \\ &= (y_1 \otimes x + y_1 \otimes x')f \\ &= (y_1 \otimes x)f + (y_1 \otimes x')f \\ &= y_1 \otimes xa + y_1 \otimes x'a \\ &= y_1 \otimes (xa + x'a). \end{aligned}$$

Thus $(x + x')a = xa + x'a$.

For $x \in X$, $\alpha \in \Delta$ we have

$$y_1 \otimes \alpha x = (y_1 \otimes \alpha x_1)(y_1 \otimes x)$$

and hence

$$\begin{aligned} y_1 \otimes (\alpha x)a &= (y_1 \otimes \alpha x)f \\ &= \big((y_1 \otimes \alpha x_1)(y_1 \otimes x)\big)f \\ &= (y_1 \otimes \alpha x_1)\big((y_1 \otimes x)f\big) \\ &= (y_1 \otimes \alpha x_1)(y_1 \otimes xa) \\ &= y_1 \otimes \alpha(xa). \end{aligned}$$

Thus $(\alpha x)a = \alpha(xa)$.

Finally, we prove that $[f, M] = [R_a, M]$ by showing that $a \in \mathrm{End}_\Delta(X)$ does not depend on the choice of y_1.

Let $0 \neq y_2 \in Y$. There exists $c \in M$ such that $c^\#(y_1) = y_2$, so

$$y_2 \otimes x = c^\#(y_1) \otimes x = c(y_1 \otimes x_1).$$

Then

$$(y_2 \otimes x)f = \big(c(y_1 \otimes x)\big)f = c\big((y_1 \otimes x)f\big) = c(y_1 \otimes xa) = y_2 \otimes xa$$

which proves that $a \in \mathrm{End}_\Delta(X)$ does not depend on the choice of y_1. Hence

$$\left(\sum y_i \otimes x_i\right) f = \sum y_i \otimes x_i a = \left(\sum y_i \otimes x_i\right) R_a.$$

Suppose now that $q = [R_a, M] \in Q_s(A)$ with $a \in \mathrm{End}_\Delta(X)$. For all $x \in X$, $y \in Y$ we have that $a(y \otimes x) \in M$, so there exists $w \in Y$ such that

$$a(y \otimes x) = w \otimes x.$$

Hence $\varphi(x'a, y) = \varphi(x', w)$ for all $x' \in X$. This proves that $a \in L_Y(X)$.

(ii). It follows from (i) and from the uniqueness of the extension of the involution to $Q_s(A)$. □

Remark. The symmetric Martindale ring of quotients of a semiprime associative ring with essential socle has been also considered by P. Ara [1].

2. Prime nondegenerate Jordan algebras with nonzero socle

All the algebras we consider in this section are over a field F of characteristic different from 2. A (nonassociative) algebra J satisfying:
(i) $xy = yx$
(ii) $(x^2y)x = x^2(yx)$
for all $x, y \in J$ is called a *Jordan algebra* (our standard references for Jordan algebras are [4], [8]). Every associative algebra A (with product xy) gives rise to Jordan algebra A^+ under the new multiplication defined by

$$x.y = 1/2(xy + yx).$$

Jordan algebras which are subalgebras of a Jordan algebra A^+ are called *special Jordan algebras*. For every associative algebra A with an involution $* : A \to A$, the set of all hermitian elements $H(A,*) = \{a \in A : a^* = a\}$ is a subalgebra of A^+, and therefore special. Another important class of special Jordan algebras is obtained as follows. Let V be a vector space over a field F with a symmetric bilinear form $\langle\cdot,\cdot\rangle : V \times V \to F$. Consider the vector space direct sum $J = F \oplus V$ and define

$$(\alpha,x)(\beta,y) = \big(\alpha\beta + \langle x,y\rangle, \alpha y + \beta x\big).$$

Then J with this product is a special Jordan algebra (J is a Jordan subalgebra of the Clifford algebra $C(V,\langle\cdot,\cdot\rangle)$). If $\langle\cdot,\cdot\rangle$ is nondegenerate and $\dim_F V > 1$, then J is a simple Jordan algebra.

Every Jordan algebra wich is not special is called an *exceptional Jordan algebra*. Let C be a Cayley-Dickson algebra over F (C is an 8-dimensional alternative algebra obtained by doubling a quaternion algebra Q by the Cayley-Dickson process). Then the set

$$H_3(C,\gamma) = H\big(M_3(C),*\big)$$

of all 3×3 matrices with entries in C which are hermitian under the involution $X^* = \gamma^{-1}\bar{X}^t\gamma$ ($\gamma = \operatorname{diag}(\gamma_1,\gamma_2,\gamma_3)$ for $\gamma_i \neq 0$ in F) is a simple 27-dimensional exceptional Jordan algebra.

Let J be a Jordan algebra. For any element $a \in J$, U_a denotes the linear operator $U_a : J \to J$ defined by

$$U_a x = 2a(ax) - a^2 x$$

for all $x \in J$. The Jordan algebra J is said to be *nondegenerate* (respectively, *prime*) if $U_a J = 0$ implies $a = 0$ (respectively, $U_T S = 0$ implies $T = 0$ or $S = 0$, T, S ideals of J). For an associative algebra A, the Jordan algebra A^+ is nondegenerate if and only if A is semiprime. Moreover, if A is prime then A^+ and $H(A,*)$ are prime for every involution $* : A \to A$.

The *centre* of a Jordan algebra J is defined by the set

$$Z(J) = \{z \in J : (zx)y = z(xy) \text{ for all } x,y \in J\}.$$

It is easy to see thast $Z(J)$ is a subalgebra of J, that can be zero. Suppose now that J is prime and that $Z(J) \neq 0$. Then J has no zero divisor in $Z(J)$, so we can consider the *central localization* $\big(Z(J) - \{0\}\big)^{-1}J$ which is a Jordan algebra over the field $\big(Z(J) - \{0\}\big)^{-1}Z(J)$ [8, p. 185].

A Jordan algebra J with 1 is called a *division Jordan algebra* if U_x is invertible for every $0 \neq x \in J$. In this case there exists a unique $y \in J$, the inverse of x, such that $U_x^{-1} = U_y$. By [4, p. 179], every simple exceptional Jordan algebra which is finite-dimensional over its centre Z is either a division Jordan algebra or a Jordan algebra $H_3(C,\gamma)$, where C is a Cayley-Dickson algebra over Z. In both cases J is 27-dimensional over its centre. This follows since the scalar extension $J_K = K \otimes_Z J$, K being the algebraic closure of Z, is a finite-dimensional simple exceptional Jordan algebra over the algebraically closed field K. By [3, p. 204], $J_K \cong H_3(C,*)$, where C is the (unique) Cayley-Dickson algebra over K. Hence $\dim_Z J = \dim_K J_K = 27$.

The classification of all prime nondegenerate Jordan Jordan algebras was achieved by Zelmanov [7].

Theorem 2 (Zelmanov)

For every prime nondegenerate Jordan algebra J one of the following conditions holds:

(i) $Z(J) \neq 0$ and the central localization $\left(Z(J) - \{0\}\right)^{-1} J$ is the simple Jordan algebra of a nondegenerate symmetric bilinear form on a vector space Y over the field $\left(Z(J) - \{0\}\right)^{-1} Z(J)$.

(ii) $Z(J) \neq 0$ and the central localization $\left(Z(J) - \{0\}\right)^{-1} J$ is a simple 27-dimensional exceptional Jordan algebra over the field $\left(Z(J) - \{0\}\right)^{-1} Z(J)$.

(iii) J contains an ideal I isomorphic to the Jordan algebra A^+, where A is a prime associative algebra such that

$$A^+ \triangleleft J \subseteq Q_s(A)^+.$$

(iv) J contains an ideal I isomorphic to the Jordan algebra $H(A,*)$, where A is a prime associative algebra with an involution $* : A \to A$ such that

$$H(A,*) \triangleleft J \subseteq H\left(Q_s(A),*\right).$$

An *inner* ideal of Jordan algebra J is subspace K of J such that $U_x J \subseteq K$ for all $x \in K$. For a nondegenerate Jordan algebra J, the sum of all minimal inner ideals K of J is an ideal $\mathrm{Soc}(J)$ called the *socle* of J. If J contains minimal inner ideals then $\mathrm{Soc}(J)$ is a direct sum of simple ideals each of which contains a minimal inner ideal [5]. If A is a semiprime associative algebra then the socle of the nondegenerate Jordan algebra A^+ coincides with the socle of the associative algebra A. Moreover, if A has an involution $* : A \to A$ then $\mathrm{Soc}\left(H(A,*)\right) = H\left(\mathrm{Soc}(A),*\right)$ [2, Prop. 2.6].

Prime nondegenerate Jordan algebras with nonzero socle were classified by Osborn and Racine [5]. To finish we derive the following result from Theorems 1 and 2.

Theorem 3 (Osborn-Racine)

Let J be a prime nondegenerate Jordan algebra with nonzero socle. Then one of the following conditions holds:

(a) J is a simple 27-dimensional exceptional Jordan algebra over its centre.

(b) J is the simple Jordan algebra of a nondegenerate symmetric bilinear form.

(c) There exists a pair of dual vector spaces (X, Y, φ) over a division associative algebra Δ such that

$$F_Y(X)^+ \triangleleft J \subseteq L_Y(X)^+.$$

(d) There exists a hermitian or alternate self-dual vector space $\big(X, \langle \cdot, \cdot \rangle\big)$ over Δ such that

$$H\big(F_X(X), *\big) \triangleleft J \subseteq H\big(L_X(X), *\big)$$

where $* : L_X(X) \to L_X(X)$ denotes the adjoint involution.

Proof. Suppose (Zelmanov's theorem, cases (i), (ii)) that $Z(J) \neq 0$ and that the central localization $\big(Z(J) - \{0\}\big)^{-1} J$ is a simple Jordan algebra. Set $M = \mathrm{Soc}(J)$. Since J is prime, M is a simple ideal. Then for every $0 \neq z \in Z(J)$ the mapping $x \to zx$ $(x \in M)$ is a bijection from M onto M. Hence M remains ideal in the central localization $\big(Z(J) - \{0\}\big)^{-1} J$ which is a simple algebra. Thus

$$J = M = \big(Z(J) - \{0\}\big)^{-1} J$$

is either a simple 27-dimensional exceptional Jordan algebra or the simple Jordan algebra of a nondegenerate symmetric bilinear form on a vector space Y over the field $Z(J)$,$\dim_{Z(J)}(Y) > 1$.

Suppose now (Zelmanov's theorem case (iii)) that J contains an ideal I isomorphic to the Jordan algebra A^+, where A is a prime Jordan algebra such that

$$A^+ \triangleleft J \subseteq Q_s(A)^+.$$

Since $\mathrm{Soc}(A)^+ = \mathrm{Soc}(J)$ is a simple ideal of J [2, Prop. 2.5 and 2.6], we have

$$Soc(A)^+ \triangleleft J \subseteq Q_s(A)^+.$$

Then, by Theorem 1, there exists a pair of dual vector spaces (X, Y, φ) such that

$$F_Y(X)^+ \triangleleft J \subseteq L_Y(X)^+.$$

Suppose finally (Zelmanov's theorem case (iv)) that J contains an ideal I isomorphic to the Jordan algebra $H(A, *)$, where A is a prime associative algebra with an involution $* : A \to A$. By [2, Prop. 2.5 and 2.6] again, $\mathrm{Soc}\big(H(A, *)\big) = \mathrm{Soc}(J)$ is a simple ideal of J . Then $H\big(\mathrm{Soc}(A), *\big) \triangleleft J \subseteq H\big(Q_s(A), *\big)$.

Hence, by Theorem 1, there exists a hermitian or alternate self-dual inner space $(X, \langle \cdot, \cdot \rangle)$ such that $H\big(F_X(X), *\big) \triangleleft J \subseteq H\big(L_X(X), *\big)$, $* : L_X(X) \to L_X(X)$ being the adjoint involution.

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