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Lifts and isomorphisms of commutation in bundles of jets

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ABSTRACT

A linear connection or a pseudo-connection on a manifold M can be considered as a generalized connection, i.e., as a (1,1)–tensor field on TM. In this paper we study the relationship between the lifts of ∇ to T_rM when ∇ is considered as a connection (resp. pseudo-connection) and as a tensor field. We prove that certain isomorphims of commutation link both concepts. For this purpose, we study firstly generalized connections, obtaining some new results about them.

In this work we answer the following question: it is known that a linear connection (resp. a linear pseudo-connection) ∇ on M can be understood as a (1,1)-tensor field $F(\nabla)$ on TM. (We shall study carefully this fact in Section 1). If ∇ is a linear connection, we consider its lift ∇^* to the fibre bundle of r-jets, T_rM (defined by Yano and Ishihara [15]). Then, in which way $F(\nabla^*)$ and $(F(\nabla))^{(r)}$ are related? (the second one is the r–lift as a tensor field). The same question is possible if we consider the (α)–lifts of a linear pseudo-connection ∇ to T_rM (defined by us [3]), and we shall study the relationship between $F(\nabla^{(\alpha)})$ and $(F(\nabla))^{(\alpha)}$ when $\alpha \in \{0, 1, \dots, r\}$.

We shall see that the isomorphisms of commutation [9]

$$\alpha_{rs} : T_r(T_sM) \longrightarrow T_s(T_rM)$$

that, from now on, we call Morimoto's isomorphisms, are the key of the answer. Observe that if $r = 2$ and ∇ is a linear connection on M, we have

$$F(\nabla^*) \in T_1^1\big(T(T_2M)\big) \qquad \text{and} \qquad \big(F(\nabla)\big)^{(2)} \in T_1^1\big(T_2(TM)\big).$$

Then, in theorem B, we shall obtain that the local coefficients of $F(\overline{\nabla})$ are the composition of α_{12} and those of $(F(\nabla))^{(2)}$. This kind of answer may be generalized to arbitrary r.

In the case $r = 1$, we shall observe that $\alpha_{11} = S$ is the automorphism $S : TTM \to TTM$ which interchanges both fibred structures of TTM over TM. This automorphism was earlier introduced by Kobayashi [7]. And we obtain an analogous result (theorem A) for the complete and horizontal lifts of linear connections to TM.

For linear pseudo-connections we have the same situation.

Another problem in this work is the following: we develop here a theory of generalized connections, firstly defined by Spesivykh [11-13], in such a way that linear connections and linear pseudo-connections on M may be thought as generalized connections on the tangent bundle of M. So, in Section 1, we take Koszul and Di Comites' definitions of linear connection and pseudo-connection on a manifold M and Vilms' definition of an infinitesimal connection on a vector bundle $\pi : E \to M$. Then, we obtain the generalized connections in a natural way, as a "generalization" of the above concepts. We can express them as in the following diagram:

Manifold		VectorBundle
Linear Pseudo – connection	$\longrightarrow$	Generalized Connection
$\uparrow$		$\uparrow$
Linear Connection	$\longrightarrow$	Infinitesimal Connection

Moreover, in proposition 1 we shall obtain a characterization of infinitesimal connections.

In Section 2 we shall offer a short description of Morimoto's isomorphisms and in Section 3 we shall study the lifts to TM, while in Section 4 to T_2M. Theorems A in Section 3 and B in Section 4 are the main results.

1. Connections

The simplest idea of a linear connection is due to Koszul [8]: A *linear connection on* M is an operator

$$\begin{array}{rccc} \nabla : & T_0^1(M) \times T_0^1(M) & \longrightarrow & T_0^1(M) \\ & (X, Y) & \longrightarrow & \nabla_X Y \end{array}$$

satisfying

(a) $\nabla_{X+Y}Z = \nabla_X Z + \nabla_Y Z$
(b) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$
(c) $\nabla_{fX}Y = f \circledast \nabla_X Y$
(d) $\nabla_X(fY) = f \circledast \nabla_X Y + (Xf) \circledast Y$.

Conditions (b) and (d) prove that ∇_X is a derivative. Remember that in local coordinates,

$$\nabla_X Y = \frac{\partial}{\partial x^i} \circledast \left(\Gamma^i_{kj} Y^j + \frac{\partial Y^i}{\partial x^k} \right) X^k.$$

As it is well known, the set of linear connections on M is not a $T^0_0(M)$–module. We obtain this structure considering the notion of pseudo-connection, which was introduced by Di Comite [1].

A *linear pseudo-connection* on M with fundamental tensor field $F \in T^1_1(M)$ is a $T^0_0(M)$–linear map

$$\nabla : T^1_0(M) \longrightarrow D(M)$$

such that ∇_X is a derivative whose direction vector field is $F(X)$, $D(M)$ being the Lie algebra of directional derivatives on M.

Observe that the set of linear pseudo-connections is a $T^0_0(M)$–module and that a pseudo-connection has the above properties. We obtain

$$\nabla_X Y = \frac{\partial}{\partial x^i} \circledast \left(\Gamma^i_{kj} Y^j + \frac{\partial Y^i}{\partial x^k} F^j_k \right) X^k.$$

This construction is generalized to an arbitrary vector bundle $\pi : E \to M$ in such a way that a linear connection on M is an infinitesimal connection on the tangent bundle $\pi_M : TM \to M$.

The idea is the following [14]: Consider the exact sequence of vector bundles over E

$$0 \longrightarrow VE \xrightarrow{i} TE \xrightarrow{\pi'} \pi^{-1}(TM) \longrightarrow 0 \tag{1}$$

where $VE = \ker \pi_* = \ker \pi'$, $\pi^{-1}(TM)$ is the pull-back of TM and π' the induced morphism. Then, an *infinitesimal connection* on $\pi : E \to M$ is a smooth splitting $V : TE \to VE$, i. e., $Vi = \mathrm{id}_{VE}$.

If we consider the canonical morphism ρ

$$\rho : VE \cong \pi_E^{-1}(E) \subset E \times E \xrightarrow{\mathrm{pr}_2} E,$$

pr_2 being the second projection, we define the map $K = \rho \circ V$ and call the *connection map*.

An infinitesimal connection defines the following maps

$$
\begin{aligned}
&V : TE \longrightarrow VE \\
&K : TE \longrightarrow E \\
&h : TE \longrightarrow TE \\
&H : \pi^{-1}(TM) \longrightarrow TE \\
&F : TE \longrightarrow TE.
\end{aligned} \tag{2}
$$

If we consider local expressions, we have

$$
\pi : E \longrightarrow M \qquad \begin{array}{cccc} \pi : & U^n \times \mathbb{R}^r & \longrightarrow & U^n \\ & (x, a_1) & \longrightarrow & x \end{array}
$$

$$
\rho : VE \to E \qquad \begin{array}{cccc} \rho : & U^n \times \mathbb{R}^r \times \{0\} \times \mathbb{R}^r & \longrightarrow & U^n \times \mathbb{R}^r \\ & (x, a_1, 0, a_2) & \longrightarrow & (x, a_2) \end{array}
$$

and the local expression for (1) is

$$
\begin{array}{cccccccc}
0 \to & U^n \times \mathbb{R}^r \times \{0\} \times \mathbb{R}^r & \to & U^n \times \mathbb{R}^r \times \mathbb{R}^n \times R^r & \to & U^n \times \mathbb{R}^r \times \mathbb{R}^n & \to 0 \\
& (x, a_1, 0, a_2) & \to & (x, a_1, 0, a_2) & & & \\
& & & (x, a_1, c, a_2) & \to & (x, a_1, c) &
\end{array}
$$

The local definitions of the other maps in (2) are:

$$V(x, a_1, c, a_2) = \big(x, a_1, 0, a_2 + \omega(x, a_1)c\big)$$

$$K(x, a_1, c, a_2) = \big(x, a_2 + \omega(x, a_1)c\big)$$

$$h(x, a_1, c, a_2) = \big(x, a_1, c, -\omega(x, a_1)c\big)$$

$$H(x, a_1, c) = \big(x, a_1, c, -\omega(x, a_1)c\big)$$

$$F(x, a_1, c, a_2) = \big(x, a_1, c, -a_2 - 2\omega(x, a_1)c\big)$$

where $\omega : U^n \times \mathbb{R}^r \to L(\mathbb{R}^n, \mathbb{R}^r)$ is the local component of the connection. Then ω is linear on the π_E-fibres.

Remark 1. h is the horizontal projection defined by the connection. H has the same expression and it is a smooth splitting of π'. V is the vertical projection.

Remark 2. A connection is defined giving any of the five maps V, K, h, H or F.

A *linear infinitesimal connection* [14] is an infinitesimal connection such that ω is also linear on the $(\pi_M)_*$-fibres, i. e., such that

$$\big(\omega(x,a_1)c\big)^\alpha = \big(\omega_i(x,a_1)c^i\big)^\alpha = \omega_i^\alpha(x,a_1)c^i = \Gamma_{i\beta}^\alpha(x)\,a_1^\beta c^i .$$

This definition is equivalent to the classical one [2,10]: a linear connection is a smooth decomposition $TE = VE \oplus HE$, HE being invariant by dilations.

Remark 3. An infinitesimal connection on $\pi_M : TM \to M$ is a nonhomogeneous connection on M in the sense of Grifone [6], which is C^∞ on the zero section. So, $F \in T_1^1(TM)$ and

$$J \circ F = J \qquad \text{and} \qquad F \circ J = -J,$$

J being the canonical almost-tangent structure on TM.

This last remark allows us to obtain the notion of generalized connection, which is due to Spesivykh. First of all, observe that $F, h, V \in T_1^1(E)$ and that their local expressions are

$$h = \frac{\partial}{\partial x^i} \otimes dx^i + \frac{\partial}{\partial a_1^\alpha} * \big(-\omega_i^\alpha(x,a_1)\big) \otimes dx^i,$$

$$V = \frac{\partial}{\partial a_1^\alpha} * \omega_i^\alpha(x,a_1) \otimes dx^i + \frac{\partial}{\partial a_1^\alpha} \otimes da_1^\alpha,$$

$$F = \frac{\partial}{\partial x^i} \otimes dx^i + \frac{\partial}{\partial a_1^\alpha} * \big(-2\omega_i^\alpha(x,a_1)\big) \otimes dx^i - \frac{\partial}{\partial a_1^\alpha} \otimes da_1^\alpha .$$

If we consider $F' \in T_1^1(E)$ as

$$F' = \frac{\partial}{\partial x^i} C_j^i \otimes dx^j + \frac{\partial}{\partial x^i} M_\beta^i \otimes da_1^\beta + \frac{\partial}{\partial a_1^\alpha} \Psi_j^\alpha \otimes dx^j + \frac{\partial}{\partial a_1^\alpha} B_\beta^\alpha \otimes da_1^\beta$$

we obtain its matrix expression

$$F' = \begin{bmatrix} C_j^i & M_\beta^i \\ \Psi_j^\alpha & B_\beta^\alpha \end{bmatrix} .$$

Then, we have the matrix expressions for h, V and F:

$$h : \begin{bmatrix} \delta_j^i & 0 \\ -\omega_j^\alpha(x,a_1) & 0 \end{bmatrix} ;$$

$$V : \begin{bmatrix} 0 & 0 \\ \omega_j^\alpha(x,a_1) & \delta_\beta^\alpha \end{bmatrix}$$

and

$$F : \begin{bmatrix} \delta_j^i & 0 \\ -2\omega_j^\alpha(x,a_1) & \delta_\beta^\alpha \end{bmatrix} .$$

A *generalized connection* [13] on a vector bundle $\pi : E \to M$ is a tensor field $F \in T_1^1(E)$.

Remark 4. In the above situation, h, V and F are generalized connections.

A generalized connection is called a pseudo-conection if $VE \subset \ker F$. The local condition is $M^i_\beta = 0 = B^\alpha_\beta$. We justify this name latter: a linear pseudo-connection on M can be understood as a pseudo-connection on $\pi_M : TM \to M$ in this sense. We notice that the original denomination of Spesivykh is different, but we use this one for historical coherence.

Proposition 1

Let $F \in T^1_1(E)$ be a generalized connection on $\pi : E \to M$. Then F is an infinitesimal connection if and only if F is an almost-product structure on TE (i.e., $F^2 = \mathrm{id}$.) and for each $e \in E$, the subspace $S_e(-1)$ of T_eE corresponding to the eigenvalue -1 is the vertical subspace V_eE.

Proof. Let F be an infinitesimal connection and h and V its horizontal and vertical projections. Then, $F = h - V$.

If $\vec{e} \in T_eE$,

$$\begin{aligned} F^2(\vec{e}) &= (h - V)(h - V)(\vec{e}) \\ &= h^2(\vec{e}) - hV(\vec{e}) - Vh(\vec{e}) + V^2(\vec{e}) \\ &= h(\vec{e}) + V(\vec{e}) \\ &= \vec{e}. \end{aligned}$$

And

$$F(\vec{e}) = -\vec{e} \iff h(\vec{e}) - V(\vec{e}) = -h(\vec{e}) - V(\vec{e}) \iff h(\vec{e}) = 0 \iff \vec{e} \in V_eE.$$

Conversely, let F be a generalized connection such that

$$F^2 = \mathrm{id} \qquad \text{and} \qquad S_e(-1) = V_eE.$$

Then, we define

$$h = (1/2)(I + F) \qquad \text{and} \qquad V = (1/2)(I - F)$$

and, immediately, we obtain

$$h^2 = h; \quad V^2 = V; \quad hV = 0; \quad Vh = 0; \quad h + V = I; \quad h - V = F.$$

Using the second hypothesis $F(\vec{e}) = -\vec{e} \iff \vec{e} \in V_eE$, we obtain

$$FV(\vec{e}) = (1/2)F(I - F)(\vec{e}) = (1/2)(F - F^2)(\vec{e}) = (1/2)(F - I)(\vec{e}) = -V(\vec{e}) \implies V(\vec{e}) \in V_eE.$$

Finally, if $\vec{e} \in V_eE$, then

$$Vi(\vec{e}) = V(\vec{e}) = (1/2)(I - F)(\vec{e}) = (1/2)\big(\vec{e} - F(\vec{e})\big) = (1/2)(\vec{e} + \vec{e}) = \vec{e}.$$

We conclude that V is a smooth splitting of i in (1), and it defines a connection whose horizontal projection is h. □

Now we define the notion of covariant derivative of an infinitesimal or generalized connection. As we are interested only in connections defined on tangent bundles, we restrict our construction to this case (for a general theory see [4]).

Let us have an infinitesimal connection Γ on $\pi_M : TM \to M$ given by $V = i^{-1}$ in (3):

$$0 \longrightarrow VTM \xrightarrow{i} TTM \xrightarrow{\tau'} \pi_M^{-1}(TM) \longrightarrow 0 \tag{3}$$

The map

$$\begin{array}{rcl} \nabla_X : T_0^1(M) & \longrightarrow & T_0^1(M) \\ Y & \longrightarrow & \nabla_X Y = K \circ Y_* \circ X \end{array}$$

is called the *covariant derivative* of the given connection [14].

If we have a generalized connection Γ the construction is more difficult. We define

$$\begin{array}{rcl} \nabla_X : T_0^1(M) & \longrightarrow & T_0^1(M) \\ Y & \longrightarrow & \nabla_X Y = (1/2)\rho \circ D \circ Z \end{array}$$

where $D = \left(Y_* \circ (\pi_M)_* \circ F - F\right) \circ Y_* \circ X \circ \pi_M$ and Z is any vector field on M. Its local expression is [4]:

$$\nabla_X Y = \frac{\partial}{\partial x^i} \, \frac{1}{2} \left(\frac{\partial Y^i}{\partial x^l} M^l_k \frac{\partial Y^k}{\partial x^j} + \frac{\partial Y^i}{\partial x^l} C^l_j - B^i_k \frac{\partial Y^k}{\partial x^j} - \Psi^i_j \right) X^j.$$

If Γ is an infinitesimal connection we obtain:

$$\nabla_X Y = \frac{\partial}{\partial x^i} * \left(\frac{\partial Y^i}{\partial x^j} + \omega^i_j \right) X^j.$$

Using local expressions, we have

Proposition 2 [13]

If F and F' are generalized connections such that $F' - F = gI$, for some $g \in T_0^0(TM)$, I being the Kronecker tensor field on TM, then F and F' define the same covariant derivatives.

Proposition 3

Let F be an infinitesimal connection and consider its horizontal and vertical projections h and V as generalized connections. Then, their covariant derivatives $\overset{F}{\nabla}$, $\overset{h}{\nabla}$, $\overset{V}{\nabla}$ are related by

$$\overset{F}{\nabla}_X = \overset{h}{\nabla}_X - \overset{V}{\nabla}_X.$$

Finally, we obtain

Proposition 4

Let ∇ be a linear pseudo-connection on M with fundamental tensor field $G \in T_1^1(M)$ and let F be the pseudo-connection defined by

$$F = \begin{bmatrix} 2G_j^i & 0 \\ -2\Gamma_{jk}^i a_1^k & 0 \end{bmatrix}$$

Then, F as a generalized connection and ∇ as a linear pseudo-connection define the same covariant derivative:

$$\overset{F}{\nabla}_X Y = \nabla_X Y. \qquad \text{for all } X, Y \in T_0^1(M).$$

We introduce the following notation:

(a) Let ∇ be a linear connection on M. When we consider ∇ as a generalized connection on $\pi_M : TM \to M$, we shall write

$$F(\nabla) = \begin{bmatrix} \delta_j^i & 0 \\ -2\Gamma_{jk}^i a_1^k & -\delta_j^i \end{bmatrix} \in T_1^1(TM).$$

(b) Let ∇ be a linear pseudo-connection on M with fundamental tensor field $G \in T_1^1(M)$. When we consider ∇ as a generalized connection on $\pi_M : TM \to M$, we shall write

$$F(\nabla) = \begin{bmatrix} 2G_j^i & 0 \\ -2\Gamma_{jk}^i a_1^k & 0 \end{bmatrix} \in T_1^1(TM).$$

Remark 5. If ∇ is a linear connection, ∇ is also a linear pseudo-connection whose fundamental tensor field is the Kronecker field. Then, we obtain two different expressions for ∇ when it is considered as a generalized connection. By proposition 2, both expressions define the same covariant derivative.

2. Morimoto's isomorphisms

Let $T_r M$ denote the tangent bundle of order r (or the r jet bundle) in the sense of [15,19,3]. Morimoto [9] proved the existence of a diffeomorphism

$$\alpha_{rs} : T_r(T_s M) \longrightarrow T_s(T_r M)$$

in such a way that $\alpha_{rs} \circ \alpha_{sr} = \text{id}$ and $\alpha_{sr} \circ \alpha_{rs} = \text{id}$.

Actually, its construction is more general because it is done for p^r- jets bundles.

The diffeomorphism α_{rs} is characterized by the following relationship, for any $f \in T_0^0(M)$:

$$\left(f^{(\eta)}\right)^{((\epsilon))} = \left(f^{(\epsilon)}\right)^{((\eta))} \circ \alpha_{rs}$$

where $\epsilon \in \{0, 1, \dots, r\}$, $\eta \in \{0, 1, \dots, s\}$ and

$$\begin{aligned} (\epsilon) &: T_0^0(M) \longrightarrow T_0^0(T_r M) \\ (\eta) &: T_0^0(M) \longrightarrow T_0^0(T_s M) \\ ((\epsilon)) &: T_0^0(T_s M) \longrightarrow T_0^0(T_r(T_s M)) \\ ((\eta)) &: T_0^0(T_r M) \longrightarrow T_0^0(T_s(T_r M)) \end{aligned}$$

are the lifts of functions defined in [9] and [15].

Proposition 5

The local expressions of α_{11} and α_{12} are:

$$\begin{array}{ccc} \alpha_{11} : TTM & \longrightarrow & TTM \\ (x^i, x^{i+n}, x^{i+2n}, x^{i+3n}) & \longmapsto & (x^i, x^{i+2n}, x^{i+n}, x^{i+3n}) \end{array}$$

$$\begin{array}{ccc} \alpha_{12} : T(T_2 M) & \longrightarrow & T_2(TM) \\ (x^i, x^{i+n}, x^{i+2n}, x^{i+3n}, x^{i+4n}, x^{i+5n}) & \longmapsto & (x^i, x^{i+3n}, x^{i+n}, x^{i+4n}, x^{i+2n}, x^{i+5n}). \end{array}$$

The proof is obtained by a straightforward and tedious calculation using the characterization above.

Remark 6. α_{11} is the automorphism introduced by Kobayashi [7].

3. Lifts to TM

Using the general theory given in section 1, we have:

(a) A linear connection ∇ on TM has the following matrix expression, when $\bar{\nabla}$ is considered as a generalized connection:

$$F(\nabla) = \begin{bmatrix} \delta^i_j & 0 & 0 & 0 \\ 0 & \delta^i_j & 0 & 0 \\ -2\Omega^i_{00j} & -2\Omega^i_{01j} & -\delta^i_j & 0 \\ -2\Omega^i_{10j} & -2\Omega^i_{01j} & 0 & -\delta^i_j \end{bmatrix} \in T^1_1(TTM)$$

where $\Omega^i_{\alpha\beta j} = \bar{\Gamma}^{i+\alpha n}_{j+\beta n\ k+\gamma n} x^{k+(2+\gamma)n}$, with $\bar{\Gamma}^{i+\alpha n}_{j+\beta n\ k+\gamma n} \in T^0_0(TTM)$, i. e.,

$$\Omega^i_{00j} = \bar{\Gamma}^i_{jk} x^{k+2n} + \bar{\Gamma}^i_{j\ k+n} x^{k+3n}$$

$$\Omega^i_{01j} = \bar{\Gamma}^i_{j+n\ k} x^{k+2n} + \bar{\Gamma}^i_{j+n\ k+n} x^{k+3n}$$

$$\Omega^i_{10j} = \bar{\Gamma}^{i+n}_{jk} x^{k+2n} + \bar{\Gamma}^{i+n}_{j\ k+n} x^{k+3n}$$

$$\Omega^i_{11j} = \bar{\Gamma}^{i+n}_{j+n\ k} x^{k+2n} + \bar{\Gamma}^{i+n}_{j+n\ k+n} x^{k+3n}.$$

(b) A linear pseudo-connection $\bar{\nabla}$ on TM with fundamental tensor field $G \in T^1_1(TM)$ has the following matrix expression:

$$F(\nabla) = \begin{bmatrix} 2G^i_j & 2G^i_{j+n} & 0 & 0 \\ 2G^{i+n}_j & 2G^{i+n}_{j+n} & 0 & 0 \\ -2\Omega^i_{00j} & -2\Omega^i_{01j} & 0 & 0 \\ -2\Omega^i_{10j} & -2\Omega^i_{11j} & 0 & 0 \end{bmatrix} \in T^1_1(TTM)$$

with the same notation.

Let $F \in T^1_1(TM)$. Then, as it is well known [15], its complete and vertical lifts to TTM have the following expressions:

If

$$F = \begin{bmatrix} F^i_j & F^i_{j+n} \\ F^{i+n}_j & F^{i+n}_{j+n} \end{bmatrix},$$

then

$$F^V = \begin{bmatrix} 0 & 0 & 0 & 0 \\ F^i_j & 0 & F^i_{j+n} & 0 \\ 0 & 0 & 0 & 0 \\ F^{i+n}_j & 0 & F^{i+n}_{i+n} & 0 \end{bmatrix}; \qquad F^C = \begin{bmatrix} F^i_j & 0 & F^i_{j+n} & 0 \\ (F^i_j)^C & F^i_j & (F^i_{j+n})^C & F^i_{j+n} \\ F^{i+n}_j & 0 & F^{i+n}_{j+n} & 0 \\ (F^{i+n}_j)^C & F^{i+n}_j & (F^{i+n}_{j+n})^C & F^{i+n}_{j+n} \end{bmatrix}$$

and given a linear connection $\tilde\nabla$ on TM, with symbols $\Gamma^{i+\alpha n}_{j+\beta n\ k+\gamma n}$, $\alpha, \beta, \gamma \in \{0, 1\}$, the horizontal lift on F to TTM (with respect to ∇) is

$$\begin{bmatrix} F^i_j & 0 & F^i_{j+n} & 0 \\ -\Gamma^i_{k+\alpha n} F^{k+\alpha n}_j + F^i_{k+\beta n} \bar\Gamma^{k+\beta n}_j & F^i_j & -\Gamma^i_{k+\alpha n} F^{k+\alpha n}_{j+n} + F^i_{k+\beta n} \bar\Gamma^{k+\beta n}_{j+n} & F^i_{j+n} \\ F^{i+n}_j & 0 & F^{i+n}_{j+n} & 0 \\ -\Gamma^{i+n}_{k+\alpha n} F^{k+\alpha n}_j + F^{i+n}_{k+\beta n} \bar\Gamma^{k+\beta n}_j & F^{i+n}_j & -\Gamma^{i+n}_{k+\alpha n} F^{k+\alpha n}_{j+n} + F^{i+n}_{k+\beta n} \bar\Gamma^{k+\beta n}_{j+n} & F^{i+n}_{j+n} \end{bmatrix}$$

where

$$\bar\Gamma^{i+\alpha n}_{k+\gamma n} = \Gamma^{i+\alpha n}_{j\ k+\gamma n} x^{j+2n} + \Gamma^{i+\alpha n}_{j+n\ k+\gamma n} x^{j+3n}.$$

And now, we introduce the following

Notation. If $F, G \in T^1_1(TTM)$ we say that F and G *coincide modulo* α_{11} if using local coordinates we have

$$F = \frac{\partial}{\partial x^{i+\alpha n}} \otimes f^{i+\alpha n}_{j+\beta n}(x^i, x^{i+n}, x^{i+2n}, x^{i+3n}) dx^{j+\beta n}$$

and

$$G = \frac{\partial}{\partial x^{i+\alpha n}} \otimes (f^{i+\alpha n}_{j+\beta n} \circ \alpha_{11})(x^i, x^{i+n}, x^{i+2n}, x^{i+3n}) dx^{j+\beta n}.$$

i. e., if the local coefficients of F are the composition of α_{11} and those of G. Remember that $\alpha_{11} \circ \alpha_{11} = \mathrm{id}$, and so the definition is symmetric.

Thus we have

Theorem A

(a) Let ∇ be a linear connection on M and let ∇^C be its complete lift (in the sense of [15]) to TM. Then $F(\nabla)^C$ and $F(\nabla^C)$ coincide modulo α_{11}.

(b) Let ∇ be a linear connection on M and let ∇^H be its horizontal lift (in the sense of [15]) to TM. Suppose that the torsion of ∇ vanishes. Then $F(\nabla)^H$ and $F(\nabla^H)$ coincide modulo α_{11}.

(c) Let ∇ be a linear pseudo-connection on M and let ∇^C and ∇^V be the complete and vertical lifts to TM (in the sense of [5]). Then $F(\nabla)^C$ and $F(\nabla^C)$ (resp. $F(\nabla)^V$ and $F(\nabla^V)$) coincide modulo α_{11}.

Proof. (a) The local symbols of ∇^C are [15]:

$$\bar\Gamma^i_{jk} = \Gamma^i_{jk}; \qquad \bar\Gamma^i_{j\;k+n} = 0; \qquad \bar\Gamma^i_{j+n\;k} = 0; \qquad \bar\Gamma^i_{j+n\;k+n} = 0;$$

$$\bar\Gamma^{i+n}_{jk} = \left(\Gamma^i_{jk}\right)^C; \qquad \bar\Gamma^{i+n}_{j\;k+n} = \Gamma^i_{jk}; \qquad \bar\Gamma^{i+n}_{j+n\;k} = \Gamma^i_{jk}; \qquad \bar\Gamma^{i+n}_{j+n\;k+n} = 0.$$

where

$$\left(\Gamma^i_{jk}\right)^C = \frac{\partial \Gamma^i_{jk}}{\partial x^l}\, x^{l+n}.$$

Then $F(\nabla^C)$ is

$$F(\nabla^C) = \begin{bmatrix} \delta^i_j & 0 & 0 & 0 \\ 0 & \delta^i_j & 0 & 0 \\ -2\Gamma^i_{jk}x^{k+2n} & 0 & -\delta^i_j & 0 \\ -2\left(\dfrac{\partial \Gamma^i_{jk}}{\partial x^l}\, x^{l+n}x^{k+2n} + \Gamma^i_{jk}x^{k+3n}\right) & -2\Gamma^i_{jk}x^{k+2n} & 0 & -\delta^i_j \end{bmatrix}$$

On the other hand,

$$F(\nabla) = \begin{bmatrix} \delta^i_j & 0 \\ -2\Gamma^i_{jk}x^{k+n} & -\delta^i_j \end{bmatrix} \in T^1_1(TM)$$

and

$$F(\nabla)^C = \begin{bmatrix} \delta^i_j & 0 & 0 & 0 \\ (\delta^i_j)^C & \delta^i_j & 0 & 0 \\ -2\Gamma^i_{jk}x^{k+n} & 0 & -\delta^i_j & 0 \\ \left(-2\Gamma^i_{jk}x^{k+n}\right)^C & -2\Gamma^i_{jk}x^{k+n} & \left(-\delta^i_j\right)^C & -\delta^i_j \end{bmatrix}$$

$$= \begin{bmatrix} \delta^i_j & 0 & 0 & 0 \\ 0 & \delta^i_j & 0 & 0 \\ -2\Gamma^i_{jk}x^{k+n} & 0 & -\delta^i_j & 0 \\ -2\left(\frac{\partial \Gamma^i_{jk}}{\partial x^l}\, x^{l+2n}x^{k+n} + \Gamma^i_{jk}x^{k+3n}\right) & -2\Gamma^i_{jk}x^{k+n} & 0 & -\delta^i_j \end{bmatrix},$$

as we wanted, beacuse for each $f \in T^0_0(TM)$,

$$f(x^i, x^{i+n})^C = \frac{\partial f}{\partial x^i}\, x^{i+2n} + \frac{\partial f}{\partial x^{i+n}}\, x^{i+3n}.$$

(b) We omit the proof. The only difficulty is that we need $\Gamma^i_{jk} = \Gamma^i_{kj}$ (i. e., $T = 0$) to obtain the equality.

(c) The local symbols of ∇^V and ∇^C are the following [5, 3]:

$$\begin{aligned} \nabla^V: \quad & \bar{\Gamma}^{i+\alpha n}_{j+\beta n\ k+\gamma n} = \left(\Gamma^i_{jk}\right)^{(\alpha-\beta-\gamma-1)} \\ \nabla^C: \quad & \bar{\Gamma}^{i+\alpha n}_{j+\beta n\ k+\gamma n} = \left(\Gamma^i_{jk}\right)^{(\alpha-\beta-\gamma)} \end{aligned}$$

where $\alpha, \beta, \gamma \in \{0, 1\}$ and

$$\left(\Gamma^i_{jk}\right)^{(s)} = 0, \qquad \text{if } s < 0;$$

$$\left(\Gamma^i_{jk}\right)^{(s)} = \left(\Gamma^i_{jk}\right)^V = \Gamma^i_{jk}, \qquad \text{if } s = 0;$$

$$\left(\Gamma^i_{jk}\right)^{(s)} = \left(\Gamma^i_{jk}\right)^C = \frac{\partial \Gamma^i_{jk}}{\partial x^l}\, x^{l+n}, \qquad \text{if } s = 1.$$

Now, we have

$$F(\nabla) = \begin{bmatrix} 2G^i_j & 0 \\ -2\Gamma^i_{jk}x^{k+n} & 0 \end{bmatrix}$$

and then

$$F(\nabla)^V = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2G^i_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2\Gamma^i_{jk}x^{k+n} & 0 & 0 & 0 \end{bmatrix}$$

and, on the other hand,

$$F(\nabla^V) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2G^i_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2\Gamma^i_{jk} x^{k+2n} & 0 & 0 & 0 \end{bmatrix}$$

as we wanted.

In the same way, the rest of the proof. □

4. Lifts to $T_2 M$

First of all, we say that the results of this paragraph can be generalized to $T_r M$, for each $r \in \mathbb{N}$. We choose $r = 2$ for brevity.

Using the general theory, we obtain:

(a) A linear connection $\tilde\nabla$ on $T_2 M$ has the following matrix expression:

$$F(\tilde\nabla) = \begin{bmatrix} \delta^i_j & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta^i_j & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta^i_j & 0 & 0 & 0 \\ -2\Omega^i_{00j} & -2\Omega^i_{01j} & -2\Omega^i_{02j} & -\delta^i_j & 0 & 0 \\ -2\Omega^i_{10j} & -2\Omega^i_{11j} & -2\Omega^i_{12j} & 0 & -\delta^i_j & 0 \\ -2\Omega^i_{20j} & -2\Omega^i_{21j} & -2\Omega^i_{22j} & 0 & 0 & -\delta^i_j \end{bmatrix} \in T^1_1(T(T_2 M))$$

where

$$\Omega^i_{\alpha\beta j} = \tilde\Gamma^{i+\alpha n}_{j+\beta\ k+\gamma n} x^{k+(3+\gamma)n}$$

for each $\alpha, \beta, \gamma \in \{0, 1, 2\}$.

(b) A linear pseudo connection ∇ on $T_2 M$ with fundamental tensor field

$G \in T_1^1(T_2 M)$ has the following matrix expression:

$$F(\nabla) = \begin{bmatrix} 2G_j^i & 2G_{j+n}^i & 2G_{j+2n}^i & 0 & 0 & 0 \\ 2G_j^{i+n} & 2G_{j+n}^{i+n} & 2G_{j+2n}^{i+n} & 0 & 0 & 0 \\ 2G_j^{i+2n} & 2G_{j+n}^{i+2n} & 2G_{j+2n}^{i+2n} & 0 & 0 & 0 \\ -2\Omega_{00j}^i & -2\Omega_{01j}^i & -2\Omega_{02j}^i & 0 & 0 & 0 \\ -2\Omega_{10j}^i & -2\Omega_{11j}^i & -2\Omega_{12j}^i & 0 & 0 & 0 \\ -2\Omega_{20j}^i & -2\Omega_{21j}^i & -2\Omega_{22j}^i & 0 & 0 & 0 \end{bmatrix} \in T_1^1(T(T_2 M))$$

using the same notation that in (a).

Let $F \in T_1^1(TM)$. Then, F admits three lifts $F^{(0)}, F^{(1)}, F^{(2)} \in T_1^1(T_2(TM))$, given by [15]:

$$F = \begin{bmatrix} F_j^i & F_{j+n}^i \\ F_j^{i+n} & F_{j+n}^{i+n} \end{bmatrix}$$

$$F^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ F_j^i & 0 & 0 & F_{j+n}^i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ F_j^{i+n} & 0 & 0 & F_{j+n}^{i+n} & 0 & 0 \end{bmatrix}$$

$$F^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ F_j^i & 0 & 0 & F_{j+n}^i & 0 & 0 \\ (F_j^i)^{(1)} & F_j^i & 0 & (F_{j+n}^i)^{(1)} & F_{j+n}^i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ F_j^{i+n} & 0 & 0 & F_{j+n}^{i+n} & 0 & 0 \\ (F_j^{i+n})^{(1)} & F_j^{i+n} & 0 & (F_{j+n}^{i+n})^{(1)} & F_{j+n}^{i+n} & 0 \end{bmatrix}$$

and

$$F^{(2)} = \begin{bmatrix} F_j^i & 0 & 0 & F_{j+n}^i & 0 & 0 \\ (F_j^i)^{(1)} & F_j^i & 0 & (F_{j+n}^i)^{(1)} & F_{j+n}^i & 0 \\ (F_j^i)^{(2)} & (F_j^i)^{(1)} & F_j^i & (F_{j+n}^i)^{(2)} & (F_{j+n}^i)^{(1)} & F_{j+n}^i \\ F_j^{i+n} & 0 & 0 & F_{j+n}^{i+n} & 0 & 0 \\ (F_j^{i+n})^{(1)} & F_j^{i+n} & 0 & (F_{j+n}^{i+n})^{(1)} & F_{j+n}^{i+n} & 0 \\ (F_j^{i+n})^{(2)} & (F_j^{i+n})^{(1)} & F_j^{i+n} & (F_{j+n}^{i+n})^{(2)} & (F_{j+n}^{i+n})^{(1)} & F_{j+n}^{i+n} \end{bmatrix}$$

where

$$f^{(1)} = \frac{\partial f}{\partial x^i}\, x^{i+2n} + \frac{\partial f}{\partial x^{i+n}}\, x^{i+3n}$$

and

$$f^{(2)} = \frac{\partial f}{\partial x^i}\, x^{i+4n} + \frac{\partial f}{\partial x^{i+n}}\, x^{i+5n} + \frac{\partial^2 f}{\partial x^i \partial x^j}\, x^{i+2n} x^{j+2n}$$
$$+ 2\, \frac{\partial^2 f}{\partial x^i \partial x^{j+n}}\, x^{i+2n} x^{j+3n} + \frac{\partial^2 f}{\partial x^{i+n} \partial x^{j+n}}\, x^{i+3n} x^{j+3n}.$$

Finally, we introduce the following

Notation. If $F \in T_1^1(T(T_2 M))$ and $G \in T_1^1(T_2(TM))$ we say that F and G *coincide modulo* α_{12} if using local coordinates we have the matrix expressions

$$F = \left(f_{j+\beta n}^{i+\alpha n}\right) \qquad \text{and} \qquad G = \left(g_{j+\beta n}^{i+\alpha n}\right)$$

with $\alpha, \beta \in \{0,1,2\}$ and then

$$f_{j+\beta n}^{i+\alpha n} = g_{j+\beta n}^{i+\alpha n} \circ \alpha_{12}.$$

Then, we obtain:

Theorem B

(a) Let ∇ be a linear connection on M and let ∇^* be its lift (in the sense of [15]) to $T_2 M$. Then $F(\nabla)^{(2)}$ and $F(\nabla^*)$ coincide modulo α_{12}.

(b) Let ∇ be a linear pseudo-connection on M and let $\nabla^{(\eta)}$ be the η-lift to $T_2 M$, $\eta \in \{0,1,2\}$ (in the sense of [3]). Then $F(\nabla)^{(\eta)}$ and $F\left(\nabla^{(\eta)}\right)$ coincide modulo α_{12}.

Proof. We omit a complete proof, because this is similar to that of theorem A.

Remember that if ∇ is a linear connection on M, then ∇^* has the following local symbols [15]:

$$\Gamma^{i+\alpha n}_{j+\beta n\ k+\gamma n} = \left(\Gamma^i_{jk}\right)^{(\alpha-\beta-\gamma)}$$

and if ∇ is a linear pseudo-connection on M, then [3] $\nabla^{(\eta)}$ has these symbols

$$\Gamma^{i+\alpha n}_{j+\beta n\ k+\gamma n} = \left(\Gamma^i_{jk}\right)^{(\alpha-\beta-\gamma-(2-\eta))}. \square$$

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