Collect. Math. 39 (1988), 201–208

Embeddings of separable Banach star algebras into a Banach star algebra with one generator

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Received 7/APR/89

Abstract

Any separable Banach star algebra A with continuous involution can be embedded isometrically into a Banach star algebra B with one generator (i.e. B is generated by one of its elements and its adjoint).

The embeding of a given structure into a less complicated one is a usual question studied in different situations. In [5] it was proved that any countable ring with identity can be embedded into a ring with two generators so that the embedding preserves the unit. The method of [5] can be adopted easily to prove the similar result for Banach algebras:

If A is a Banach algebra with the unit then there exists a unital Banach algebra B with two generators and an isometric unit preserving isomorphism $f : A \to B$.

Clearly one generator is not sufficiente as any algebra with one generator is necessarily commutative. On the other hand it is an open problem whether any separable commutative Banach algebra can be embeded into a Banach algebra with one generator.

The aim of this paper is to improve the above mentioned result for Banach star algebras. Any Banach star algebra with continuous involution can be embeded

* The research has been partially supported by a grant of the Spanish Ministry of Education and Science.

isometrically into a Banach star algebra with continuous involution generated by one of its elements and its adjoint. The embedding preserves the involution and the unit (if there is any).

Related question concerning the generation of the algebra of all bounded operators in a separable Hilbert space was studied in [2], see also [1,6].

The author wishes to express his gratitude to prof. W. Żelazko. This paper arised essentially in the discussions with him.

All Banach algebras in this paper will be unital, i.e., with the unit element of norm 1. This condition however is not essential and can be avoided easily.

Let A be a unital Banach algebra, $x_1, \ldots, x_n \in A$. We shall denote by $\langle x_1, \ldots, x_n \rangle$ the smallest closed subalgebra of A containing $x_1, \ldots, x_n$ and the unit of A.

By a Banach star algebra we shall mean a Banach algebra with an involution (not necessarily continuous).

Proposition 1

Let A be a unital Banach star algebra with finite number of generators. Then there exist a unital Banach star algebra B, an element $x \in B$ such that $B = \langle x, x^* \rangle$ and an isometric star isomorphism $\varphi : A \to B$ preserving the unit.

Proof. It is possible to find in A a finite number of selfadjoint elements generating A. For this, we may replace any generator g by the pair of selfadjoint elements $g + g^*$, $i(g - g^*)$.

Also it is easy to see that there exist a finite number of selfadjoint generators $g_1, \ldots, g_n \in A$ satisfying

$$\sigma(g_i) \subset \{z \in \mathbb{C} : |z - 1| < 1\}, \qquad (i = 1, \ldots, n).$$

(it is sufficient to replace a generator g by $1_A + g/(2\,\|g\|)$).

By the Ford's square root lemma (see, e.g., [1, p. 65]) there exist invertible square roots h_i such that $h_i^2 = g_i$, $h_i^* = h_i$.

Consider the Banach space

$$X = \underbrace{A \oplus A \oplus \ldots \oplus A}_{4n+1}$$

with the ℓ^1-norm

$$\|(a_1, \ldots, a_{4n+1})\|_X = \sum_{i=1}^{4n+1} \|a_i\|_A \qquad (a_i \in A,\ i = 1, \ldots, 4n + 1).$$

Let $B(X)$ be the Banach algebra of all bounded operators on X with the operator norm

$$\|T\|_{B(X)} = \sup\{\|Tx\|_X : \|x\|_X = 1\}.$$

Let further $M_{4n+1}(A)$ be the star algebra of all $(4n+1)\times(4n+1)$ matrices over A with natural algebraic operations and the involution defined by

$$\left[(t_{ij})_{i,j=1}^{4n+1}\right]^* = (t_{ji}^*)_{i,j=1}^{4n+1}.$$

Clearly a matrix $(t_{ij})_{i,j=1}^{4n+1} \in M_{4n+1}(A)$ defines a bounded operator $T : X \to X$ by

$$T(a_1,\dots,a_{4n+1}) = \left(\sum_{j=1}^{4n+1} t_{1,j}a_j,\dots,\sum_{j=1}^{4n+1} t_{4n+1,j}a_j\right).$$

So we may identify $M_{4n+1}(A)$ with the subalgebra of $B(X)$ of all bounded operators $T : X \to X$ satisfying

$$T(a_1a, a_2a,\dots,a_{4n+1}a) = T(a_1,\dots,a_{4n+1})a.$$

Define the mapping $\phi : A \to M_{4n+1} \subset B(X)$ by $\phi(a) = \operatorname{diag}(a,\dots,a)$, i.e., $\phi(a)(a_1,\dots,a_{4n+1}) = (aa_1,\dots,aa_{4n+1})$. Then ϕ is a star and unit-preserving isometric isomorphism $A \to M_{4n+1}(A)$.

Let $x \in B(X)$ be the operator defined by

$$x(a_1,\dots,a_{4n+1}) = \big(0, h_1^{-1}a_1, h_1a_2, h_1a_3, h_1^{-1}a_4, h_2^{-1}a_5, h_2a_6, h_2a_7,$$
$$h_2^{-1}a_8, h_3^{-1}a_9,\dots,h_n^{-1}a_{4n-3}, h_na_{4n-2}, h_na_{4n-1}, h_n^{-1}a_{4n}\big).$$

Clearly $x \in M_{4n+1}(A)$ (x is the matrix with elements $h_1^{-1}, h_1, h_1, h_1^{-1},\dots, h_n^{-1}, h_n, h_n, h_n^{-1}$ on the first diagonal under the main diagonal). The adjoint element of x is defined by

$$x^*(a_1,\dots,a_{4n+1}) = \big(h_1^{-1}a_2, h_1a_3, h_1a_4, h_1^{-1}a_5,\dots,$$
$$h_n^{-1}a_{4n-2}, h_na_{4n-1}, h_na_{4n}, h_n^{-1}a_{4n+1}\big).$$

It suffices to show that $\phi(A) \subset \langle x, x^*\rangle$. We prove that $\langle x, x^*\rangle$ contains even all diagonal matrices.

For $a \in A$, $1 \le k \le 4n+1$ denote by

$$(a)_k = \operatorname{diag}(\underbrace{0,\dots,0}_{k-1}, a, 0,\dots,0).$$

i.e.,

$$(a)_k(a_1,\ldots,a_{4n+1}) = (\underbrace{0,\ldots,0}_{k-1}, aa_k, 0,\ldots,0).$$

We prove by induction on s that $(a)_s \in \langle x, x^*\rangle$ for all $a \in A$. Suppose that $(a)_i \in \langle x, x^*\rangle$ for all $i \leq s$ and for every $a \in A$. We show that $(a)_{s+1} \in \langle x, x^*\rangle$ for every $a \in A$. We distinguish four cases:

1) Let $s = 4k$, $(k = 0,\ldots,n)$. Then

$$x^{*4n-4k}x^{4n-4k} = \operatorname{diag}(a_1,\ldots,a_{4k}, 1, 0,\ldots,0)$$

for some $a_1,\ldots,a_{4k} \in A$. By the induction hypothesis $(1)_{4k+1} \in \langle x, x^*\rangle$. Further

$$(1)_{4k+1}\, x^{*4j+3}x^{4j+3} = \left(h^2_{k+j+1}\right)_{4k+1} \qquad (j = 0,\ldots,n-k-1),$$

$$(1)_{4k+1}\, x^{4j+3}x^{*4j+3} = \left(h^2_{k-j}\right)_{4k+1} \qquad (j = 0,\ldots,k-1).$$

So $(g_i)_{4k+1} = \left(h_i^2\right)_{4k+1} \in \langle x, x^*\rangle$ for $i = 1,\ldots,n$, i.e., $(a)_{4k+1} \in \langle x, x^*\rangle$ for every $a \in A$.

2) Let $s = 4k+1$, $(k = 0,\ldots,n-1)$. Then

$$x^{*4n-s}x^{4n-s} = \operatorname{diag}(a_1,\ldots,a_s, h^2_{k+1}, 0,\ldots,0)$$

for some $a_1,\ldots,a_s \in A$, so $\left(h^2_{k+1}\right)_{s+1} \in \langle x, x^*\rangle$.

Further

$$\left(h^2_{k+1}\right)_{s+1} xx^*xx^* = \left(h^{-2}_{k+1}\right)_{s+1} \in \langle x, x^*\rangle,$$

$$\left(h^{-2}_{k+1}\right)_{s+1} x^{*4j+6}x^{4j+6} = \left(h^2_{k+j+2}\right)_{s+1} \qquad (j = 0,\ldots,n-k-2),$$

$$\left(h^2_{k+1}\right)_{s+1} x^{4j+4}x^{*4j+4} = \left(h^2_{k-j}\right)_{s+1} \qquad (j = 0,\ldots,k-1),$$

hence $(g_i)_{s+1} \in \langle x, x^*\rangle$ for $i = 1,\ldots,n$ and $(a)_{s+1} \in \langle x, x^*\rangle$ for every $a \in A$.

3) Let $s = 4k+2$, $(k = 0,\ldots,n-1)$. Then

$$x^{*4n-s}x^{4n-s} = \operatorname{diag}(a_1,\ldots,a_s, 1, 0,\ldots,0)$$

for some $a_1,\ldots,a_s \in A$ so $(1)_{s+1} \in \langle x, x^*\rangle$. Further

$$(1)_{s+1}x^{*4j+1}x^{4j+1} = \left(h^2_{k+j+1}\right)_{s+1} \qquad (j = 0,\ldots,n-k-1),$$

$$(1)_{s+1}x^{4j+5}x^{*4j+5} = \left(h^2_{k-j}\right)_{s+1} \qquad (j = 0,\ldots,k-1).$$

So $(a)_{s+1} \in \langle x, x^* \rangle$ for every $a \in A$.

4) Let $s = 4k + 3$, $(k = 0, \ldots, n-1)$. Then

$$x^{*4n-s} x^{4n-s} = \operatorname{diag}(a_1, \ldots, a_s, h_{k+1}^{-2}, 0, \ldots, 0)$$

for some $a_1, \ldots, a_s \in A$, so $\left(h_{k+1}^{-2}\right)_{s+1} \in \langle x, x* \rangle$.

Further

$$\left(h_{k+1}^{-2}\right)_{s+1} x^2 x^{*2} = \left(h_{k+1}^{2}\right)_{s+1} \in \langle x, x^* \rangle.$$

$$\left(h_{k+1}^{2}\right)_{s+1} x^{*4j+4} x^{4j+4} = \left(h_{k+j+2}^{2}\right)_{s+1} \qquad (j = 0, \ldots, n-k-2),$$

$$\left(h_{k+1}^{-2}\right)_{s+1} x^{4j+6} x^{*4j+6} = \left(h_{k-j}^{2}\right)_{s+1} \qquad (j = 0, \ldots, k-1),$$

hence $(a)_{s+1} \in \langle x, x* \rangle$ for every $a \in A$.

We have proved that $\langle x, x^* \rangle$ contains all diagonal matrices, hence $\phi(A) \subset \langle x, x^* \rangle = B$ and this embedding satisfies all required conditions. □

Remark. If the involution in A is continuous then the involution in B is also continuous. If the involution in A is isometric (i.e., $\|a^*\| = \|a\|$ for all $a \in A$) then $\phi : A \to (B, \|\cdot\|')$ is the isometric embedding, where $\|b\|' = \max\{\|b\|, \|b^*\|\}$ for all $b \in B$. Clearly $\|\cdot\|'$ is the isometric involution.

Proposition 2

Let A be a separable unital Banach algebra with an isometric involution. Then there exists a unital Banach star algebra B with an isometric involution, an element $x \in B$ such that $B = \langle x, x^* \rangle$ and an isometric star isomorphism $\phi : A \to B$ preserving the unit.

Proof. We may assume that A has countable many generators $g_1, g_2, \ldots$ satisfying

$$g_i^* = g_i, \quad (i = 1, 2, \ldots),$$

$$\sum_{i=1}^{\infty} \|g_i\| < \infty.$$

Let $K = \{(i,j), i, j = 1, 2, \ldots\}$. Consider the Banach space

$$X = \left\{ \{a_k\}_{k \in K} \in A : \sum_{k \in K} \|a_k\| < \infty \right\}$$

with naturally defined algebraic operations and with the ℓ^1-norm

$$\|\{a_k\}\|_X = \sum_{k\in K} \|a_k\|_A.$$

Denote by $B(X)$ the Banach algebra of all bounded operators on X. Let $M_{K(A)}$ be the set of all matrices $(a_{k,m})_{k,m\in K}$ of elements of A satisfying

$$\sup_{m\in K} \sum_{k\in K} \|a_{k,m}\| < \infty \qquad \text{and} \qquad \sup_{k\in K} \sum_{m\in K} \|a_{k,m}\| < \infty.$$

A matrix $T = (t_{k,m})_{k,m\in K} \in M_K(A)$ defines the operator $T \in B(X)$ by

$$T(\{a_m\}_{m\in K}) = \left\{ \sum_{m\in K} t_{k,m} a_m \right\}_{k\in K}.$$

In this way we may identify $M_K(A)$ with a subalgebra of $B(X)$. Define the involution on $M_K(A)$ by

$$\left[(t_{k,m})_{k,m\in K}\right]^* = \left(t^*_{m,k}\right)_{k,m\in K}$$

and the norm

$$\left\|(t_{k,m})_{k,m\in K}\right\|_{M_K(A)} = \max\left\{\|T\|_{B(X)}, \|T^*\|_{B(X)}\right\}$$
$$= \max\left\{ \sup_{m\in K} \sum_{k\in K} \|a_{k,m}\|, \sup_{k\in K} \sum_{m\in K} \|a_{k,m}\| \right\}$$

where $T \in B(X)$ is the operator corresponding to the matrix $(t_{k,m})$. With the norm and the involution just defined $M_K(A)$ is a Banach star algebra with the isometric involution.

Define the mapping $\phi : A \to M_K(A) \subset B(X)$ by $\phi(a)x = ax$ $(a \in A,\ x \in X)$, i.e., $\phi(a)$ is the diagonal matrix with a in the main diagonal. Clearly ϕ is an isometric star isomorphism preserving the unit. Denote by $K' = \{(i,1),\ i = 1,2,\ldots\} \subset K$, $K'' = K - K'$. Let $u : K' \to K''$ be a bijection.

For $a \in A, k \in K$ denote by $(a)_k$ the element $\{\delta_{k,m}a\}_{m\in K} \in X$ where $\delta_{k,m}$ is the Kronecker's symbol.

Let $V = (v_{k,m})_{k,m\in K}$, $B = (b_{k,m})_{k,m\in K}$ and $C = (c_{k,m})_{k,m\in K}$ be the elements of $M_K(A) \subset B(X)$ defined by

$$v_{k,m} = \begin{cases} 1 & \text{if } k \in K',\ m = u(k) \\ 0 & \text{otherwise,} \end{cases}$$

$$b_{m,k} = \begin{cases} 1 & \text{if } k = (i,j) \in K,\ m = (i, j+1) \\ 0 & \text{otherwise,} \end{cases}$$

$$c_{k,m} = \begin{cases} g_j & \text{if } k = (i,j),\ m = (i,1) \\ 0 & \text{otherwise.} \end{cases}$$

This means that

$$V(a)_k = \begin{cases} (a)_{u(k)} & (k \in K') \\ 0 & (k \in K''). \end{cases}$$

$$V^* = V^{-1},$$

$$B(a)_{(i,j)} = (a)_{(i,j+1)} \qquad (i,j) \in K,$$

$$C(a)_{(i,j)} = (g_j a)_{(i,1)} \qquad (i,j) \in K.$$

We prove that $\phi(A) \subset \langle V, V^*, B, C\rangle$.

Let $t \in \{1,2,\ldots\}$. Denote by

$$Z_t = CB^{t-1}V^*V + VCB^{t-1}V^* \in \langle V, V^*, B, C\rangle.$$

Let $a \in A$, $k = (i,1) \in K'$. Then

$$\begin{aligned} Z_t(a)_k &= CB^{t-1}V^*(a)_{u(k)} = CB^{t-1}(a)_k \\ &= CB^{t-1}(a)_{(i,1)} = C(a)_{(i,t)} = (g_t a)_{(i,1)} = (g_t a)_k. \end{aligned}$$

Let $k \in K''$, $u^{-1}(k) = (i,1)$. Then

$$Z_t(a)_k = VCB^{t-1}(a)_{(i,1)} = VC(a)_{(i,t)} = V(g_t a)_{(i,1)} = (g_t a)_k.$$

Hence $Z_t = \phi(g_t) \in \langle V, V^*, B, C\rangle$ and $\phi(A)$ is a subalgebra of the Banach star algebra $\langle V, V^*, B, B^*, C, C^*\rangle$.

The embedding ϕ is an isometric star isomorphism preserving the unit. □

Theorem 3

Let A be a separable unital Banach star algebra with continuous involution. Then there exists a unital Banach star algebra B with continuous involution, an element $x \in B$ such that $B = \langle x, x^*\rangle$ and an isometric star isometric isomorphism $\phi : A \to B$ preserving the unit.

Proof. Define a new norm in A by $\|a\|'_A = \max\{\|a\|_A, \|a^*\|_A\}$. Then $\|\cdot\|'_A$ is equivalent to the norm $\|\cdot\|_A$ and the involution is isometric with respect to $\|\cdot\|'_A$. By the preceding proposition $(A, \|\cdot\|'_A)$ can be embedded isometrically into a Banach star algebra $(B, \|\cdot\|'_B)$ with an isometric involution. By [3], there exists a norm $\|\cdot\|_B$ on B equivalent to $\|\cdot\|'_B$ such that $\|\cdot\|$ is an extension of the norm $\|\cdot\|_A$. Clearly the involution in $(B, \|\cdot\|_B)$ is continuous and $(A, \|\cdot\|_A)$ is embedded isometrically into $(B, \|\cdot\|)$. □

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