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Equilibrium points for a wide class of games

EZIO MARCHI AND PABLO TARAZAGA

Universidad Nacional de San Luis, San Luis, Argentina

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ABSTRACT

In this paper we present the existence of equilibrium points for extension of finite games having sum of quotients of expectation functions as payoff function.

Introduction

It is interesting to see that the minimax theorem was generalized by von Neumann [6] for games with payoff functions which are rates of expectations. On the other hand Nash's equilibrium points might be suitable for such a generalization which was obtained by Marchi [3].

Here we will consider two existence theorems with different techniques for non-cooperative n person games with payoff function given by sum of quotients of expectations.

1. General games

Let us consider an extension of the finite n-person game with payoff

$$\sum_{k=1}^{r_i} \frac{G_k^i(w_1,\ldots,w_n)}{H_k^i(w_1,\ldots,w_n)}$$

with the set of players N. The player $i \in N$ has the finite set of strategies W_i. Then the extension is defined on the set

$$X_{i=1}^{n} \overline{W_i} = X$$

where $n = |N|$ is the number of players and $\overline{W_i} = X_i$ the set of mixed strategies.

The payoff functions are given by

$$
\begin{aligned}
f^i(x_1,\ldots,x_n) &= \sum_{k=1}^{r_i} \frac{E_k^i(x_1,\ldots,x_n)}{F_k^i(x_1,\ldots,x_n)} \\
&= \frac{\sum_{j=1}^{r_i} \prod_{k\neq j}^{r_i} F_k^i(x_1,\ldots,x_n)\, E_j^i(x_1,\ldots,x_n)}{\prod_{k=1}^{r_i} F_k^i(x_1,\ldots,x_n)}
\end{aligned}
$$

where without loss of generality we assume all the functions to be strictly positive. Introducing the notation

$$(x_i|y) = (y_1,\ldots,y_{i-1},x_i,y_{i+1},\ldots,y_n)$$

for $x_i \in X_i$ and $y = (y_1,\ldots,y_n) \in X$ we define the following functions

$$L_j^i(x_i|x) = \prod_{k\neq j}^{r_i} F_k^i(x)\, E_k^i(x) \doteq \prod_{k\neq j}^{r_i} F_k^i(x_i|x)\, E_k^i(x_i|x)$$

and

$$M^i(x_i|x) = \prod_{k=1}^{r_i} F_k^i(x) = \prod_{k=1}^{r_i} F_k^i(x_i|x).$$

Therefore

$$f^i(x) = \sum_{k=1}^{r_i} \frac{E_k^i(x)}{F_k^i(x)} = \frac{\sum_{j=1}^{r_i} L_j^i(x_i|x)}{M^i(x_i|x)}\,.$$

We have that the functions L_j^i are concave in $x_i \in X_i$ for fixed $x \in X$ if and only if the condition a)

$$(1)\qquad \nabla_i\, L_j^i(x_i^0|x)\cdot(x_i - x_i^0) \geq L_j^i(x_i|x) - L_j^i(x_i^0|x)$$

is satisfied for each x_i^0, x_i in X_i. Here ∇_i indicates the gradient and $\cdot$ the inner product. On the other hand, the payoff M^i is convex in $x_i \in X_i$ for fixed $x \in X$ if and only if

$$(2)\qquad \nabla_i\, M^i(x_i^0|x)\cdot(x_i - x_i^0) \leq M^i(x_i|x) - M^i(x_i^0|x)$$

for each x_i^0, $x_i \in X_i$. This condition is b).

As a first result, we have the following

Theorem 1

Under conditions a) and b) the extension game has a Nash equilibrium point.

Proof. For a fixed $y \in X$ and any $x_i \in X_i$, let

$$\Theta_i(x_i, y) = M^i(y) \sum_{j=1}^{r_i} L_j^i(x_i|y) - \sum_{j=1}^{r_i} L_j^i(y)\, M^i(x_i|y)$$

which is concave in $x_i \in X_i$.

Therefore the function

$$\Theta(x, y) = \sum_{i=1}^{n} \Theta_i(x_i, y)$$

is also concave with respect to the variable $x \in X$. Moreover, it is continuous. Defining the non-empty convex correspondence $K(y) \subset X$ of those points $x \in X$ such that

$$\Theta(x, y) = \max_{z \in X} \Theta(z, y)$$

then, by virtue of Kakutani's fixed point theorem, we have the existence of a point $\bar{x} \in X$ such that $\bar{x} \in K(\bar{x})$, or

$$\Theta(x, \bar{x}) \geq \Theta(x, \bar{x})$$

for any $x \in X$. This is equivalent to

$$0 \geq M^i(\bar{x}) \sum_{j=1}^{r_i} L_j^i(x_i|x) - \sum_{j=1}^{r_i} L_j^i(\bar{x})\, M^i(x_i|x)$$

for each player $i \in N$ and each strategy $x_i \in X_i$.

Dividing this inequality by $M^i(\bar{x})\, M^i(x_i|x) > 0$, we have the existence of a Nash equilibrium point.

Now we would like to show the gradient formulas which are given by

$$\nabla_i L_j^i(x_i^0|x) = \nabla_i E_j^i(x_i^0|x) \prod_{s \neq j}^{r_i} F_s^i(x_i^0|x) + \sum_{k \neq j}^{r_i} \nabla_i F_k^i(x_i^0|x) \prod_{s \neq j,k}^{r_i} F_s^i(x_i^0|x)\, E_j^i(x_i^0|x)$$

and

$$\nabla_i M^i(x_i^0|x) = \sum_{k=1}^{r_i} \nabla_i F_k^i(x_i^0|x) \prod_{j\neq k}^{r_i} F_k^i(x_i^0|x).$$

As an example, we have that when the payoff function are given by

$$E_k^i(x) = E_k^i(x_k) = \sum_{w_k} a_k^i(w_k)\, x_k(w_k)$$

$$F_k^i(x) = F_k^i(x_k) = \sum_{w_k} b_k^i(w_k)\, x_k(w_k)$$

then, it holds true

$$\nabla_i E_j^i(x_i^0|x) = \delta_{ij}\, \nabla_i E_j^i(x_i^0) = \delta_{ij}\big(a_j^i(w_j^1), \ldots, a_j^i(w_j^n j)\big)$$

where δ_{ij} indicates Kronecker's delta. On the other hand

$$\nabla_i F_k^i(x_i^0|x) = \delta_{ik}\, \nabla_i F_k^i(x_i^0) = \delta_{ik}\big(b_k^i(w_k^i), \ldots, b_k^i(w_k^n k)\big).$$

Therefore, it is easy to see that (1) and (2) hold true. In order to verify (1) it is convenient to split the cases when $j = i$ and $j \neq i$.

As a second example, we introduce the case when $r_i = 1$ and $L_j^i = \delta_{ij}\, E_i^i$, then

$$\nabla_i L_i^i = \nabla_i E_i^i \qquad \text{and} \qquad \nabla_i M^i = \nabla_i F_i^i$$

Then b) and c) are assured. Moreover if a) is assumed, we are generalizing von Neumann's case with n–players. Besides it, when F_i^i is constant for each $i \in N$, the existence of Nash equilibrium point is gotten. □

2. A further approach

As we have pointed out in the introduction, we are now going to study the existence of equilibrium point for the extension already introduced. Here we will do this using a different technique which is based on a generalization of Knaster-Kuratowski-Mazurkiewicz lemma due to Peleg [5]. Using it we will present a different proof of Theorem 1, only in the case for $r_i = 1$. This second technique will be as powerful as that using Kakutani's fixed point theorem for such a case.

Theorem 2 (Peleg)

Let $C^i_{w_i} \subset X$ be a closed subset for any $i \in N$ and any $w_i \in W_i$, such that

$$\bigcup_{w_i \in Q_i} C^i_{w_i} \{x \in X : x_i(w_i) = 0 \qquad \text{for all } w_i \in W_i - Q_i\}$$

Then

$$\bigcap_{i \in N} \bigcap_{w_i \in W_i} C^i_{w_i} \neq \emptyset.$$

Here $\emptyset$ denotes the empty set.

Using this important Peleg's result, we now have our second proof of Theorem 1.

Let us consider the set $D^i_{w_i}$ of the $x \in X$ such that

$$F_i(x)\big[E_i(w_i|x) - E_i(w_i|x)\big] - E_i(x)\big[F_i(\bar{w}_i|x) - F_i(w_i|x)\big] \geq 0$$

for all $\bar{w}_i \in S(x_i)$ where $E_i = E^i_1$, $F_i = F^i_1$, and $S(x_i)$ is the support of $x_i \in X_i$. First, we want to see that such a set is closed. Indeed, consider a converging sequence $x(n) \longrightarrow x$ such that for each natural n, $x(n) \in D^i_{w_i}$, then for each $\bar{w}_i \in S(x_i(n))$:

$$\begin{aligned} H_i(w_i, \bar{w}_i, x(n)) = {} & F_i(x(n))\big[E_i(\dot{w}_i|x(n)) - E_i(w_i|x(n))\big] \\ & - E_i(x(n))\big[F_i(\bar{w}_i|x(n)) - F_i(w_i|x(n))\big] \\ \geq {} & 0 \end{aligned}$$

but since each term is continuous, in the limit because there is an n_0 such that, for all $n \geq n_0$, $S(x(n)) \supset S(x)$, we have that the just written inequality holds true also for $x \in X$. Thus, $x \in D^i_{w_i}$ and $D^i_{w_i}$ is closed. On the other hand, given any subset $Q_i = \{\tilde{w}^1_i, \ldots, \tilde{w}^s_i\}$, let $\tilde{x}_i$ with $S(\tilde{x}_i) \subset Q_i$ and $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$. Therefore $\tilde{x}_i(w_i) = 0$ for $w_i \in W_i - Q_i$.

Consider a $\hat{w}_i$ such that

$$\min_{\bar{w}_i \in Q_i} H_i(\hat{w}_i, w_i, \tilde{x}) = \min_{w_i \in Q_i} \min_{w_i \in Q_i} H_i(w_i, w_i, \tilde{x})$$

which by finiteness always exist. For such a $\hat{w}_i$ it holds true $H_i(\hat{w}_i, w_i, \tilde{x}) \geq 0$ for each $w_i \in Q_i$. Thus $\tilde{x} \in D^i_{w_i}$, and Peleg's condition in the Theorem 2 for the sets $D^i_{w_i}$ is satisfied. Thus

$$\begin{aligned} \emptyset \neq & \bigcap_{i \in N} \bigcap_{w_i \in W} D^i_{w_i} \\ = & \bigcap_{i \in N} \bigcap_{w_i \in W_i} \{x \in X : H_i(w_i, \dot{w}_i, x) \geq 0 \quad \text{for all } w_i \in S(x_i)\} \end{aligned}$$

or equivalently there exist a point $x \in X$ such that for each $i \in N$ and each $w_i \in W_i$.

$$w_i \in S(x_i) : H_i(w_i, w_i, \bar{x}) \geq 0 :$$
$$F_i(x)\, E_i(w_i|x) - E_i(x)\, F_i(w_i|\bar{x}) \geq F_i(x)\, E_i(w_i|x) - E_i(x)\, F_i(w_i|x)\,. \qquad (3)$$

The right hand amount in this inequality has to be constant for each $\bar{w}_i \in S(x_i)$. therefore by convex combination of $x_i \in X_i$ since the function are linear:

$$0 = F_i(x)\, E_i(\bar{x}_i|\bar{x}) - E_i(x)\, F_i(x_i|\bar{x}) \geq F_i(x)\, E_i(w_i|x) - E_i(\bar{x})\, F_i(w_i|x)$$

for all $w_i \in W_i$. By combining in a convex combination for $x_i \in X_i$:

$$E_i(\bar{x})\, F_i(x_i|x) \geq F_i(x)\, E_i(x_i|\bar{x})$$

for each $x_i \in X_i$. Dividing by $F_i(\bar{x})\, F_i(x_i|x) > 0$ by condition a) we have that $x \in X$ is an equilibrium point. □

We would like to mention that the above results can be extended in a natural way for games defined on suitable linear topological spaces by using the Fan-Glicksberg generalization of Kakutani's fixed point theorem [1,2]. This can be done using the same technique mentioned in the proof of Theorem 1.

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