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Proper CW-complexes: A category for the study of proper homotopy

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Abstract

The notion of closure finite complexes with weak topology introduced by J. H. C. Whitehead determines an adequate category for the study of homotopy theory. Nevertheless a noncompact space which can be described as a CW-complex always needs an infinite number of cells. In the present paper we develop a new notion that we call proper CW-complex which enables us to describe some noncompact spaces with finitely many cells.

In this new category we give an algorithm which permits to compute the singular homology groups and the proper homology groups associated with a finite regular proper CW-complex. On the other hand, a characterization of the proper homotopy equivalences depending on the Hurewicz and the relative Steenrod groups is obtained. The paper finishes giving some results about proper cellular approximations of proper maps between proper CW-complexes.

1. Introduction

The study of simplicial complexes of dimension one or two can be remounted at least to Euler's time. In 1862, J. B. Listing developed an analysis of simplicial complexes of greater dimension. In 1930, S. Lefschez worked with infinite simplicial complexes. The notion of closure finite complexes with weak topology (CW-complexes) was introduced by J. H. C. Whitehead in 1949. Whitehead worked with infinite CW-complexes to include spaces such as open manifolds or covering complexes. This category of CW-complexes provides a suitable setting in which to study Algebraic Topology. Let us recall some properties of this category: The singular homology and cohomology is determined by the skeletal structure, the maps which are homotopy equivalences can be characterized by the Hurewicz groups, there exist cellular approximations for any continuous map and there is a suitable obstruction theory.

In 1970, Siebenmann suggested that for the study of noncompact spaces the homotopy theory should be developed in the category of proper maps instead of continuous maps. In the category of spaces and proper maps several homotopy functors which are invariant by proper homotopy have been introduced. For instance, the Brown-Grossman homotopy groups [2,10], the Steenrod homotopy groups [1,12]. The authors have also contributed developing some homology and cohomology groups [7].

One of the questions in proper homotopy is to describe a suitable category in which to study algebraic proper topology. The authors think that such a category is the category of proper CW-complexes. A proper CW-complex can be constructed from a discrete space by attaching consecutively compact cells D^n and noncompact cells $D^{n-1} \times J$. One important point is that the maps used for attaching must be proper. The reader can check that there are many simple noncompact spaces which admit a finite decomposition in compact and noncompact cells of the type above, in particular, the interior of a compact PL-manifold with nonempty boundary and the space obtained by removing a subcomplex of a simplicial complex admit such a decomposition. However, to obtain a standard decomposition of these last examples it is necessary to use an infinite number of cells.

In the present paper, we study some properties of these proper CW-complexes In section 3 we give some topological results that will be needed later. Notice that a standard CW-complex is locally compact if and only if it is locally finite, however, a locally compact proper CW-complex need not be locally finite. In section 4, we give an algorithm which computes the singular homology groups and the proper homology groups, whose definition is recalled in section 2, of any finite regular CW-complex. This permits to generalize the proper obstruction theory developed previously by the authors [6,9] to the category of finite regular CW-complexes.

Section 4 contains a Theorem of Whitehead type which characterizes the proper maps between finite proper CW-complexes which are proper homotopy equivalences. A proper map which induces isomorphisms on the Hurewicz and relative Steenrod proper homotopy groups is proved to be a proper homotopy equivalence. Finally, in section 5, several results are given which give sufficient conditions to guarantee the existence of proper cellular approximations of proper maps between finite proper CW-complexes.

2. Preliminaries

DEFINITION 2.1. Let X and Y be topological spaces. A continuous map $f : X \to Y$ is said to be proper if $f^{-1}(K)$ is compact whenever K is a closed-compact subset of Y.

Two proper maps $f, g : X \to Y$ are said to be properly homotopic, $f \simeq_p g$, if there is a homotopy from f to g which is proper. A subspace A of X is said to be proper if the inclusion map of A into X is proper. In this case, we say that (X, A) is a proper pair. The proper maps and homotopies from proper pairs and triplets are also defined in the obvious way. A ray in X is a proper map $\alpha : J \to Y$ where J denotes the semiopen interval $[0, +\infty)$. A proper map between two spaces with base ray $f : (X, \alpha) \to (Y, \beta)$ is a proper map $f : X \to Y$ satisfying $f\alpha = \beta$. For proper pairs and triplets with base ray, that we shall call rayed pairs or rayed triplets, proper maps and homotopies are defined in the expected way.

Let us recall the definition and some properties, which will be used in the present paper, of some proper homotopy invariants:

Let (X, A, α) be a rayed proper pair. In 1980, Čerin [3] defined $\underline{\pi}_n(X, \alpha)$ as the set of proper homotopy classes of proper maps of the form $f : (S^n \times J, * \times J) \to (X, \alpha)$ ($* \in S^n$, the unit n-sphere) and such that $f(*, t) = \alpha(t)$ under the proper homotopy relation relative to $* \times J$. For $n \geq 1$, $\underline{\pi}_n(X, \alpha)$ admits a group structure (abelian, for $n \geq 2$) and $\underline{\pi}_0(X, \alpha)$ is the set of proper homotopy classes of J into X. We shall say that $\underline{\pi}_0(X, A, \alpha)$ is the set of proper ends of X. In the relative case $\underline{\pi}_n(X, A, \alpha)$ is similarly defined by considering proper maps of the form $(D^n \times J, S^{n-1} \times J, * \times J) \to (X, A, \alpha)$ where D^n is the unit n-disk. In a similar way, the second author in 1984 [11] and independently Brin and Thikstum in 1985 [1] defined the proper homotopy groups $\tau_n(X, \alpha)$, $\tau_n(X, A, \alpha)$, changing $S^n \times J$ by $S^n \times J / S^n \times 0$ and $D^n \times J$ by $D^n \times J / D^n \times 0$. In a general way, we shall refer to all these groups as proper Steenrod groups. Alternative definitions of these groups have been given by Hernández and Porter [12]. For a detailed study of its properties using noncompact cubes we refer to the reader to [18]. For each $n \geq 1$ ($n \geq 2$), $\underline{\pi}_n$, τ_n are covariant functors from the category of rayed spaces (proper pairs) and based proper maps to the category of groups and homomorphisms. These functors are invariant of the proper homotopy type and have exact sequences associated with each rayed proper pair or triplet. For a rayed space (X, α) some relations between the Steenrod and Hurewicz groups are given by the exact sequence

$$\cdots \longrightarrow \underline{\pi}_n(X, \alpha) \longrightarrow \pi_n(X, \alpha(0)) \longrightarrow \tau_{n-1}(X, \alpha) \longrightarrow \tau_{n-1}(X, \alpha) \longrightarrow \cdots \qquad (1)$$

An analogous sequence is obtained for the relative case. There are also compatible actions of $\pi_1(X,\alpha)$ ($\pi_1(A,\alpha)$ for the relative case) on the groups and sets in sequence above.

Massey [15] developed the singular homology H_* by using singular n-cubes (proper maps from I^n to X) instead of singular n–simplexes. Similarly, the authors [7] defined the proper homologies J_*, E_*, using proper singular n- cubes, i.e., proper maps from n-cubes $K_1 \times \cdots \times K_n$ to X where for each $i, 1 \leq i \leq n$, K_i is either I or J. Let us recall these proper homologies:

Let $T_n(X)$ denote the free abelian group generated by all proper singular n-cubes of X module degenerate n–cubes. A proper singular n-cube

$$T : K_1 \times \cdots \times K_n \longrightarrow X$$

is said to be degenerate if there exists some i such that

$$T(x_1,\ldots,x_i,\ldots,x_n) = T(x_1,\ldots,x'_i,\ldots,x_n)$$

for every $x_i, x'_i \in K_i$ ($K_i = I$). Given a proper singular n–cube T, the boundary operator is defined by

$$\partial T = \sum_{i=1}^{n}(-1)^i((\alpha_i^0)^*T - (\alpha_i^1)^*T)$$

where $(\alpha_i^l)^*$ is the homomorphism induced by the inclusion α_i^l given by

$$\alpha_i^l(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) = (x_1,\ldots,x_{i-1},l,x_{i+1},\ldots,x_n)$$

where $l = 0,1$ if $K_i = I$ and $l = 0$ if $K_i = J$, in such a case $(\alpha_i^1)^* = 0$. The chain complex obtained is denoted by $T_*(X)$ and its n–th homology group by $J_n(X)$.

Provided $S_*(X)$ denotes the complex of singular cubes of X module degenerate cubes, we consider the quotient chain complex $T_*(X)/S_*(X)$. The n–th homology group of this chain complex is denoted by $E_n(X)$. For a proper pair (X,A) and a group G the proper (co)homology group with coefficients in G is also defined in the usual way.

The homologies J_*, E_*, are covariant functors from the proper homotopy category of proper pairs to the category of abelian groups. As for homotopy groups there is an exact sequence

$$\cdots \longrightarrow E_{n+1}(X) \longrightarrow H_n(X) \longrightarrow J_n(X) \longrightarrow E_n(X) \longrightarrow \cdots \qquad (2)$$

which relates the proper homology groups to the standard homology groups. There is also a similar sequence for the relative case. Moreover, there are theorems of the Hurewicz type for the proper homotopy and homology groups mentioned in this section which relate the absolute and relative exact sequences (1) and (2) in a commutative way [8].

3. Proper CW-complexes

We are going to describe the category of proper CW-complexes that generalizes the category of standard CW-complexes and the category of finite proper cubic complexes [9]. This new category is suitable for the study of proper homotopy as we shall see in the following sections. In the present section, the maximum norm in $\mathbb{R}^n$

$$\|x\| = \max\{|x_i| : i = 1, \dots, n\}, \qquad x = (x_1, \dots, x_n)$$

will be considered, and the following notation will be used:

$$E^n = \{x \in \mathbb{R}^n : \|x\| \le 1\},$$
$$e^n = \{x \in \mathbb{R}^n : \|x\| < 1\},$$
$$S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\},$$
$$\varepsilon^{n+1} = (e^n \times J) - (e^n \times \{0\}) \qquad (n \ge 0),$$
$$e^0 = E^0 = \{0\}.$$

DEFINITION 3.1. A proper CW-complex consists of a Hausdorff space X together with two index sets A_n and B_n for each integer $n \ge 0$ such that $B_0 = \emptyset$, $A_n \cap B_n = \emptyset$, and proper maps

$$\phi^n_\alpha : E^n \to X \qquad \text{for each } n \ge 0 \text{ and } \alpha \in A_n,$$
$$\phi^n_\beta : E^{n-1} \times J \to X \qquad \text{for each } n > 0 \text{ and } \beta \in B_n,$$

satisfying the following properties:

P1) $X = \bigcup_{n,\gamma} \phi^n_\gamma(c^n)$ for each $n \ge 0$ and $\gamma \in A_n \cup B_n$, where $c^n = e^n$ if $\gamma \in A_n$ and $c^n = \varepsilon^n$ if $\gamma \in B_n$.

P2) $\phi^n_\gamma(c^n) \cap \phi^m_\delta(c^m) = \emptyset$ unless $n = m$ and $\gamma = \delta$.

P3) $(\phi^n_\gamma)_{|c^n}$ is a one-to-one map for every $n \ge 0$ and $\gamma \in A_n \cup B_n$.

P4) Let $X^n = \bigcup_{m,\gamma} \phi^m_\gamma(c^m)$ for every $0 \le m \le n$ and every $\gamma \in A_m \cup B_m$. Then,

$$\phi^n_\alpha(S^{n-1}) \subset X^{n-1} \qquad \text{for every } n \ge 1 \text{ and } \alpha \in A_n,$$
$$\phi^n_\beta((E^{n-1} \times \{0\}) \cup (S^{n-2} \times J)) \subset X^{n-1} \qquad \text{for every } n \ge 2 \text{ and } \beta \in B_n,$$
$$\phi^1_\beta(E^0 \times \{0\}) \subset X^0 \qquad \text{for every } \beta \in B_1$$

P5) A subset F of X is closed if and only if for each $n \ge 0$ and each $\gamma \in A_n \cup B_n$, $(\phi^n_\gamma)^{-1}(F)$ is a closed subset in E^n if $\gamma \in A_n$ or in $E^{n-1} \times J$ if $\gamma \in B_n$.

P6) For each $n \ge 0$, $\phi^n_\gamma(\Sigma^n)$ is contained in the union of finitely many subsets of the form $\phi^m_\delta(c^m)$, where $\Sigma^n = E^n$ if $\gamma \in A_n$ and $\Sigma^n = E^{n-1} \times J$ if $\gamma \in B_n$.

The proper maps ϕ_γ^n are called characteristic maps of X, the subspaces $\phi_\gamma^n(E^n)$ compact n-cells of X and the subspaces $\phi_\gamma^n(E^{n-1} \times J)$ noncompact n-cells of X. The subspace X^n is called the n-skeleton of X, and if $X^n = X$ for some n it is said that X has finite dimension, and the least n satisfying $X^n = X$ is called the dimension of X. If there is no n such that $X^n = X$ it is said that X has infinite dimension. If X has only a finite number of cells it is said that X is finite. Of course X has the coherent topology associated to the family of its cells. However, it is interesting to note that X also has the coherent topology associated to the family of compact subsets (X is a k-space). Therefore the cells are closed subsets of X, and hence proper subsets. Moreover each characteristic map $\phi_\gamma^n : \Sigma^n \to \phi_\gamma^n(\Sigma^n)$ is an identification.

On the other hand, as in the standard case a map $f : X \to Y$ between a proper CW-complex and a topological space Y is continuous if and only if $f \circ \phi_\gamma^n$ is continuous for each $n \geq 0$ and each $\gamma \in A_n \cup B_n$. If X is finite, f is proper if and only if $f \circ \phi_\gamma^n$ is proper for each $n \geq 0$ and each $\gamma \in A_n \cup B_n$. Easy examples show that in the last statement the condition of finiteness is necessary.

DEFINITION 3.2. Given a proper CW-complex X, a subspace L of X is said to be a subcomplex if for each $n \geq 0$ there exist subsets A'_n, B'_n, of A_n, B_n, respectively, such that:

a) $L = \bigcup_{n,\gamma} \phi_\gamma^n(e^n)$ for every $n \geq 0$ and $\gamma \in A'_n \cup B'_n$.

b) $\phi_\gamma^n(\Sigma^n) \subset L$ for every $n \geq 0$ and $\gamma \in A_n \cup B_n$.

It is interesting to note that the arbitrary union and intersection of a family of subcomplexes are both subcomplexes.

The following properties of proper CW-complexes will be used later.

Propostion 3.3

i) If L is a subcomplex of a proper CW-complex X, then L is a proper CW-complex, moreover L is a closed (hence, proper) subset of X.

ii) The path-components of a proper CW-complex X are subcomplexes, and if X is connected, it is path-connected too.

Proposition 3.4

Let X be a proper CW-complex, then K is a compact subspace of X if and only if K is contained into a finite subcomplex of X and $K \cap \phi_\gamma^n(\Sigma^n)$ is compact for every $n \geq 0$ and $\gamma \in A_n \cup B_n$.

Proposition 3.5

For a proper CW-complex X, consider the statements:

LFC) The family of cells of X is locally finite; i.e., each point $x \in X$ has a neighbourhood which only meets finitely many cells.

LC) X is locally compact.

LF) X is locally finite; i.e., each cell meets a finite number of cells.

Then LC and LFC are equivalent, and LF implies both LC of LFC.

The following example shows that LC does not implies LF: Let X be the proper CW-complex consisting of a 0-cell $\{0\}$, a non-compact 1-cell $J = [0, \infty)$ and for each integer $i \geq 0$ a compact cell E_i^2 with characteristic attaching map $\phi_i^2 : \partial E_i^2 \to J$ given by $\phi_i^2(x) = i$ for every $x \in \partial E_i^2$. Since J meets an infinite number of cells it follows that X is not locally finite, although it is clear that X is locally compact.

For the standard CW-complexes LC and LF properties are equivalent. This is a remarkable difference between proper and standard CW-complexes.

Proposition 3.6

Let X and Y be proper CW-complexes. Provided that one of them is locally compact, then the product space is also a proper CW-complex.

Proof. Let A_n, B_n be index sets and ϕ_γ^n, $\gamma \in A_n \cup B_n$, characteristic maps for X, and similarly A'_n, B'_n, ψ_δ^n, for Y. Consider for $X \times Y$ the index sets:

$$\bar{A}_n = \bigcup \{ A_i \times A'_j : i + j = n \},$$

$$\bar{B}_n = \bigcup \{ (A_i \times B'_j) \cup (B_i \times A'_j) \cup (B_i \times B'_j) : i + j = n \},$$

and the characteristic maps $\phi_\gamma^i \times \psi_\delta^j$, where it is necessary to take into account the homeomorphisms:

$$E^{i+j} \cong E^i \times E^j;$$

$$E^i \times (E^{j-1} \times J) \cong E^{i+j-1} \times J \cong (E^{i-1} \times J) \times E^j;$$

$$(E^{i-1} \times J) \times (E^{j-1} \times J) \cong E^{i+j-1} \times J.$$

It is easy to check that the product space $X \times Y$ together with the cells just defined above satisfy all the properties of Definition 3.1 except perhaps P5. To check P5, we can see first that P5 follows easily if X and Y are locally compact. For the general case, for example, if X is locally compact, since Y is a k-space (see definition of k-space in [4]) it follows that $X \times Y$ is also a k-space. Finally taking into account Proposition 3.5 and that $X \times Y$ is a k-space it is not difficult to check that P5 is also satisfied. □

Now the absolute proper homotopy extension property (APHEP) can be defined for a proper pair (X,A) as for the standard homotopy, but, of course, taking proper maps and homotopies. For proper CW-complexes we have:

Proposition 3.7

If X is a finite proper CW-complex and L is a subcomplex of X, then (X,L) has the APHEP.

Proof. Let $f : X \to Y$ be a proper map from X into a topological space Y, and let $H : L \times I \to Y$ be a partial proper homotopy of f. Define the proper map $F : X \times 0 \cup L \times I \to Y$ by $F(x,0) = f(x)$ for $x \in X$ and $F(y,t) = H(y,t)$ for each $(y,t) \in L \times I$. Now F can be extended properly over $X \times I$ by using an inductive procedure on the relative skeleton $K^n = X^n \cup L$:

For $n = 0$, for each 0-cell v which is not in L, we can extend properly F by $F(v,t) = v$.

Assume that F is also defined on $X \times 0 \cup K^{n-1} \times I$, then for each n-cell $\phi^n_\gamma(\Sigma^n)$ of X which is not in L consider the proper map

$$\Sigma^n \times 0 \cup \partial\Sigma^n \times I \xrightarrow{\phi^n_\gamma \times id_I} \phi^n_\gamma(\Sigma^n) \times 0 \cup \phi^n_\gamma(\partial\Sigma^n) \times I \subset X \times 0 \cup K^{n-1} \times I \xrightarrow{F} Y$$

where if $\Sigma^n = E^n$, $\partial\Sigma^n = S^{n-1}$ and if $\Sigma^n = E^{n-1} \times J$, $\partial\Sigma^n = E^{n-1} \times 0 \cup S^{n-2} \times J$ for $n > 1$ and $\partial\Sigma^n = E^0 \times 0$ for $n = 1$. Now according to [8, Proposition 2.2] we can choose a proper retraction $r : \Sigma^n \times I \to \Sigma^n \times 0 \cup \partial\Sigma^n \times I$ which induces a new proper map

$$r' : \phi^n_\gamma(\Sigma^n) \times I \longrightarrow \phi^n_\gamma(\Sigma^n) \times 0 \cup \phi^n_\gamma(\partial\Sigma^n) \times I$$

defined by $r'(x,t) = (\phi^n_\gamma \times id_I) \circ r(z,t)$ where $z \in \Sigma^n$ satisfies that $\phi^n_\gamma(z) = x$. Then $F \circ r' : \phi^n_\gamma(\Sigma^n) \times I \to Y$ is a proper extension of F over the n-cell $\phi^n_\gamma(\Sigma^n)$ and the union of all these extensions defines a proper extension over $X \times 0 \cup K^n \times I$. Finally, since X has finite number of cells it follows that the homotopy $F : X \times I \to Y$ obtained by this procedure is proper. $\square$

The following Proposition is concerned with Freudenthal compactifications and we need some previous concepts:

Let X be a space, consider the set $\{K\}$ of closed-compact subsets of X directed by inclusion. The set of Freudenthal's ends of X is defined by

$$\mathcal{F}(X) = \varprojlim \pi_0(X - K).$$

Let $E \subset X$ and $e = \{U_K\} \in \mathcal{F}(X)$, then we denote $e < E$ if there is some closed-compact subset K such that $U_K \subset E$. Denote $E^{\mathcal{F}} = \{e \in \mathcal{F}(X) : e < E\}$ and $E^* = E \cup E^{\mathcal{F}}$. The topology induced on $X \cup \mathcal{F}(X)$ by the base $\{E^* : E$ is an open subset of $X\}$ is called the Freudenthal topology. This topological space extends the topology of X and will be denoted by X^*. Let us recall that if X is a Hausdorff, locally path-connected, locally compact space with a finite number of path-connected components, then X is an open subset of X^* and X^* is a Hausdorff locally connected compact space that will be called Freudenthal's compactification of X.

It is interesting to note that if X is a finite proper CW-complex, from Propositions 3.3 and 3.5 we can assert that X satisfies the conditions above.

Proposition 3.8

Let X be a finite proper CW-complex, then the Freudenthal compactification X^* is a finite standard CW-complex.

Proof. Let A_n, B_n be the index sets and $\{\phi_\gamma^n : n \geq 0,\ \gamma \in A_n \cup B_n\}$ the characteristic maps of X. Since X is finite, it follows easily that $\mathcal{F}(X)$ is also a finite set $\{e_{i_1}, \dots, e_{i_q}\}$. Now, we can give a CW-complex structure to X^* taking as index set $U_0 = A_0 \cup \{i_1, \dots, i_q\}$, $U_n = A_n \cup B_n$ for each $n > 0$, and the characteristic maps are given by $(\phi_\gamma^n)^* = \phi_\gamma^n$ if $\gamma \in A_n$, and for each $\gamma \in B_n$ by the continuous map

$$(\phi_\gamma^n)^* : \left(E^{n-1} \times J\right)^* \cong E^n \longrightarrow X^*$$

induced by the proper characteristic map $\phi_\gamma^n : E^{n-1} \times J \to X$ on the Freudenthal's compactifications. If $n = 0$, we also consider the maps $(\phi_{i_j}^0)^* : E^0 \to \mathcal{F}(X) \subset X^*$ given by $(\phi_{i_j}^0)^*(E^0) = \{e_{i_j}\}$, $j = 1, \dots, q$.

DEFINITION 3.9. A proper CW-complex X is said to be regular if each n–cell admits an injective characteristic map and the boundary $\phi_\gamma^n(\partial\Sigma^n)$ of each n-cell is the union of finitely many $(n-1)$–cells.

Remark 3.10. In Proposition 3.8, it is clear that if X is regular, X^* is regular too. On the other hand, an analogous result is obtained if we take the Alexandroff's compactification $\hat{X}$ of X.

Remark 3.11. As in the standard case, an alternative definition of proper CW-complex can be given as follows [16, 7.3.11]:

Let X be a topological space together with a sequence of subspaces

$$X^0 \subset X^1 \subset X^2 \subset \cdots \subset X$$

satisfying the following properties:

a) $X = \bigcup_{n=0}^{\infty} X^n$.

b) X^0 is a discrete space.

c) For each $n > 0$ there are two index sets A_n, B_n and proper maps

$$\phi_\alpha^n : S^{n-1} \to X^{n-1} \qquad \text{for each } n > 0 \text{ and } \alpha \in A_n,$$

$$\phi_\beta^n : E^{n-1} \times \{0\} \cup S^{n-2} \times J \to X^{n-1} \qquad \text{for each } n \geq 1 \text{ and } \beta \in A_n,$$

such that X^n is obtained from X^{n-1} by attaching n-cells E^n and $E^{n-1} \times J$ through the proper maps ϕ_α^n, ϕ_β^n respectively, i.e., on the disjoint union of X^{n-1} and disjoint copies E_α^n of E^n for each $\alpha \in A_n$ and disjoint copies $(E^{n-1} \times J)_\beta$ of $E^{n-1} \times J$ for each $\beta \in B_n$. We identify, for each $\gamma \in A_n \cup B_n$, x and $\phi_\gamma^n(x)$, where $x \in \partial E^n$ or $\partial(E^{n-1} \times J)$, respectively.

d) For each $\beta \in B_n$, $\phi_\beta^n(\partial(E^{n-1} \times J))$ is contained in a finite union of subsets of the form Σ_δ^m with $m < n$, and Σ_δ^m denotes either E_δ^m or $(E^{m-1} \times J)_\delta$ if $\delta \in B_m$.

e) A subset F is closed in X if and only if for each $n \geq 0$ $F \cap X^n$ is closed in X^n.

To see that this an alternative definition of proper CW-complex we refer to reader to [18].

4. Computing the homologies H_*, J_*, E_* for finite regular proper CW-complexes

Let us consider the following notations:

$S'_*(X)$ is the chain complex of singular compact simplexes of X; i.e., $S'_n(X)$ is the free abelian group generated by maps of the type $\Delta^n \to X$ where Δ^n is the standard n-simplex.

$C'_*(X)$ is the chain complex of singular proper simplexes of X; i.e., $C'_n(X)$ is the free abelian group generated by proper maps of the type $\Delta^n \to X$ or $\Delta^{n-1} \times J \to X$.

$\tilde{C}'_*(X) = C'_*(X)/S'_*(X)$ is the chain complex of singular non-compact simplexes of X; i.e., $\tilde{C}'_n(X)$ is the free abelian group generated by proper maps of the type $\Delta^{n-1} \times J \to X$.

$S_*(X) = \mathcal{S}_*(X)$ is the chain complex of singular compact cubes in X; i.e., $S_n(X)$ is the free abelian group generated by maps of the type $I^n \to X$ module degenerate maps.

$C_*(X)$ is the chain complex of singular proper cubes of X; e.g., $C_n(X)$ is the free abelian group generated by singular proper n-cubes of the type $I^n \to X$ or $I^{n-1} \times J \to X$ module degenerate n-cubes.

$\tilde{C}_*(X) = C_*(X)/S_*(X)$ is the chain complex such that $\tilde{C}_n(X)$ is the free abelian group generated by singular proper n-cubes of the type $I^{n-1} \times J \to X$ module degenerate n cubes.

In all the complexes above the boundary is defined in a natural way, and for non positive integers q, the q-chain corresponding object is always the group zero.

Proposition 4.1

There are homotopy equivalences between the following chain complexes.

(a) $S'_*(X)$ and $S_*(X)$.

(b) $C'_*(X)$, $C_*(X)$ and $\mathcal{T}_*(X)$.

(c) $\tilde{C}'_*(X)$, $\tilde{C}_*(X)$ and $\mathcal{T}_*(X)/\mathcal{S}_*(X)$.

Proof. (a) Since both complexes determine ordinary singular homology theories, the result is a consequence of the homology uniqueness theorems [5].

(c) To see that $\mathcal{T}_*(X)/\mathcal{S}_*(X)$ and $\tilde{C}_*(X)$ are homotopic equivalent we refer to reader to [11]. In order to find a homotopy equivalence between $\tilde{C}_*(X)$ and $\tilde{C}'_*(X)$, let us consider the tangent mapping space

$$T(\hat{X}, \infty) = \{f : I \to \hat{X} : f^{-1}(\infty) = 1\}$$

where $\hat{X}$ is the Alexandroff's compactification of X. Now define

$$\psi' : \tilde{C}'_*(X) \longrightarrow S'_{*-1}(T(\hat{X}, \infty))$$

as follows: given a proper map $k : \Delta^{n-1} \times J \to X$, then for each $x \in \Delta^{n-1}$ define

$$\big(\psi'(k)(x)\big)(t) = k(x,t) \qquad \text{for each } t \in I - \{1\}$$

and

$$\big(\psi(k)(x)\big)(1) = \infty.$$

Similarly, we can define

$$\psi : \tilde{C}_*(X) \longrightarrow S_{*-1}(T(\hat{X}, \infty))$$

just exchanging Δ^{n-1} by I^{n-1}. It is obvious that ψ' and ψ are homotopy equivalences. If we consider the homotopy equivalence

$$\nu : S_{*-1}(T(\hat{X},\infty)) \longrightarrow S'_{*-1}(T(\hat{X},\infty))$$

given by (a), then the composition

$$v = \psi'^{-1} \circ \nu \circ \psi : \tilde{C}_*(X) \longrightarrow \tilde{C}'_*(X)$$

gives the desired homotopy equivalence.

b) From [11] it follows that $T_*(X)$ and $C_*(X)$ are homotopic equivalent. For $C_*(X)$ and $C'_*(X)$, consider the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_*(X) & \longrightarrow & C_*(X) & \longrightarrow & \tilde{C}_*(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle v} & & \\ 0 & \longrightarrow & S'_*(X) & \longrightarrow & C'_*(X) & \longrightarrow & \tilde{C}'_*(X) & \longrightarrow & 0 \end{array}$$

where w is induced by ν and v taking into account that $C_n(X) \cong S_n(X) \oplus \tilde{C}_n(X)$ for each $n \geq 0$. Since in the diagram above the rows are exact and ν and v are homotopy equivalences, it follows that w is also a homotopy equivalence. □

For a topological space X consider the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S'_*(X) & \longrightarrow & C'_*(X) & \longrightarrow & C'_*(X)/S'_*(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \bar{\eta}_X} & & \\ 0 & \longrightarrow & S'_*(X) & \longrightarrow & S'_*(\hat{X},\infty) & \longrightarrow & S'_*(\hat{X})/S'_*\{X,\infty\} & \longrightarrow & 0 \end{array} \qquad (3)$$

where if $A, B \subset X$, $S'_*(X,A)$ denotes the quotient $S'_*(X)/S'_*(A)$, and $S'_*\{A,B\}$ is the chain complex generated by singular simplexes $T : \Delta^q \to X$ such that either $T(\Delta^q) \subset A$ or $T(\Delta^q) \subset B$. The map η_X is induced by the compactification $(\Delta^{n-1} \times J)^\wedge \cong \Delta^n$ and $\bar{\eta}_X$ is induced by η_X in a natural way. Using all these notations we have:

Lemma 4.2

Let X be a topological space of the form $K \times J$, then η_X and $\bar{\eta}_X$ are homotopy equivalences of chain complexes.

Proof. The diagram above induce the following commutative diagram with exact rows

$$\begin{array}{ccccccc} \longrightarrow & H_q(X) & \longrightarrow & J_q(X) & \longrightarrow & E_q(X) & \longrightarrow \\ & \downarrow{\scriptstyle id_*} & & \downarrow{\scriptstyle (\eta_X)_*} & & \downarrow{\scriptstyle (\bar\eta_X)_*} & \\ \longrightarrow & H_q(X) & \longrightarrow & H_q(\hat{X},\infty) & \longrightarrow & H_q(S'_*(\hat{X})/S'_*\{X,\infty\}) & \longrightarrow \end{array}$$

$$\begin{array}{ccccc} \longrightarrow & H_{q-1}(X) & \longrightarrow & J_{q-1}(X) & \longrightarrow \\ & \downarrow{\scriptstyle id_*} & & \downarrow & \\ \longrightarrow & H_{q-1}(X) & \longrightarrow & H_{q-1}(\hat{X},\infty) & \longrightarrow \end{array}$$

Since $X = K \times J$, it follows that $J_q(X) = 0$ for all q (see properties of J_* in [7,8]). Because $\hat{X}$ is contractible we also have that $H_q(\hat{X},\infty) = 0$ for all q. Then id_* and $(\eta_X)_*$ are isomorphisms. Now the five lemma can be applied to obtain that $(\bar\eta_X)_*$ is also an isomorphism. Therefore we can conclude that η_X and $\bar\eta_X$ are homotopy equivalences. □

Proposition 4.3

Let X be a topological space such that $\hat{X}$ is conic at ∞; i.e., there is an open neighbourhood U of ∞ in $\hat{X}$ and a homeomorphism $h : U \to K \times I/K \times \{1\}$ such that $h(\mathrm{fr}\, U) = K \times 0$ and $h(\infty) = *$ where $*$ is the equivalence class $K \times \{1\}$. Then η_X and $\bar\eta_X$ are homotopy equivalences.

Proof. Consider the following commutative diagram where φ and ψ are induced by inclusions.

$$\begin{array}{ccc} C'_*(U-\infty)/S'_*(\dot{U}-\infty) & \xrightarrow{\varphi} & C'_*(X)/S'_*(X) \\ \downarrow{\scriptstyle \bar\eta_{U-\infty}} & & \downarrow{\scriptstyle \bar\eta_X} \\ S'_*(\bar{U})/S'_*(\dot{U}\cap X,\infty) & \xrightarrow{\psi} & S'_*(\hat{X})/S'_*\{X,\infty\} \end{array}$$

First, we are going to see that φ is a homotopy chain equivalence. This is equivalent to prove that $E_q(U-\infty) \to E_q(X)$ is an isomorphism for all q which is equivalent again to see that $E_q(X, U-\infty) \cong 0$ for all q. Taking into account that $\bar{U} - \infty \cong \mathrm{fr}\, U \times [0,1)$, $\mathrm{fr}\, U$ and $X - U$ are compact and $\mathrm{fr}\, U \times [0,1/2]$ has the same proper homotopy type that $\mathrm{fr}\, U$, we can apply the excision property of the homology E_* to obtain $E_q(X, U-\infty) \cong E_q(X-U, \mathrm{fr}\, U) \cong 0$. Recall that E_* vanishes on compact pairs.

In order to prove that ψ is a homotopy chain equivalence, consider the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S'_*\{X,\infty\} & \longrightarrow & S'_*\{X,U\} & \longrightarrow & S'_*(U)/S'_*\{\bar{U}\cap X,\infty\} & \longrightarrow & 0 \\ & & \downarrow id & & \downarrow i & & \downarrow \psi & & \\ 0 & \longrightarrow & S'_*\{X,\infty\} & \longrightarrow & S'_*(\hat{X}) & \longrightarrow & S'_*(\hat{X})/S'_*\{X,\infty\} & \longrightarrow & 0 \end{array}$$

where i is the inclusion map. Note that i is a homotopy chain map since X and U are open and $\hat{X} = X \cup U$ and we can apply [15, Theorem 6.3]. Consequently we also have that ψ is a homotopy chain equivalence.

On the other hand, since $U - \infty \cong K \times J$, from lemma 4.2, $\bar{\eta}_{U-\infty}$ is also a homotopy chain equivalence. Therefore we obtain that $\bar{\eta}_X$ is a homotopy chain equivalence. Finally, by considering diagram (3) the same is obtained for η_X. □

Corollary 4.4

If X is a proper finite regular CW-complex, then η_X and $\bar{\eta}_X$ are homotopy equivalences.

Proof. According to Remark 3.10, $\hat{X}$ is a standard regular CW-complex and ∞ is a 0 cell of $\hat{X}$. Now we can apply [14, Theorem III.1.7] to obtain a subdivision of the cell structure of $\hat{X}$ which is simplicial. Because in a simplicial CW structure a 0-cell always has a conic neighbourhood, we have that $\hat{X}$ is conic at ∞. Now we can apply Proposition 4.3 to obtain the desired result. □

Corollary 4.5

If X is a finite proper CW-complex such that $\hat{X}$ is conic at ∞, then η_X and $\bar{\eta}_X$ are homotopy equivalences.

Let X a finite regular proper CW-complex. Let us choose an orientation for each n-cell of X that will be called positive. Consider the following chain complexes:

$\mathcal{C}_*(X)$ is the free abelian group generated by all compact oriented cells.

$C_*(X)$ is the free abelian group generated by all oriented cells.

$\bar{C}_*(X)$ is the free abelian group generated by all noncompact oriented cells.

Take as boundary operators those induced by the geometric boundary; i.e., if σ is a q-cell and τ is a $(q+1)$-cell, then the incidence number of τ at σ is 1 or -1 if σ is a face of τ and the orientation induced by τ at σ is positive or negative respectively and 0 if σ is not a face of τ. It is interesting to note that the faces of a compact cell are compact, however a noncompact cell can have compact and noncompact faces. Then for the chain complex $\bar{C}_*(X)$ we only consider noncompact faces.

Corolary 4.6

Let X be a finite regular proper CW-complex. Then for each integer q, the q-th homology group of the chain complexes $S_*(X)$, $C_*(X)$, $\bar{C}_*(X)$, are isomorphic to $H_q(X)$, $J_q(X)$, $E_q(X)$, respectively.

Remark 4.7. The obstruction theory for the extension and classification of proper maps from X to Y developed in [9] or [6] can be easily generalized from the results of the present section for the case in which X is a finite regular proper CW-complex.

5. A theorem of Whitehead type for the proper homotopy equivalences

DEFINITION 5.1. A proper map $f : X \to Y$ is said to be a proper homotopy equivalence if there exists a proper map $g : Y \to X$ such that $fg \simeq_p id_Y$ and $gf \simeq_p id_X$. In this case g is said to be the proper homotopic inverse of f.

Proposition 5.2

If f is a proper homotopy equivalence then the induced maps

$$\pi_n(f) : \pi_n(X, x_0) \to \pi_n(Y, f(x_0)),$$

$$\tau_n(f) : \tau_n(X, \alpha) \to \tau_n(Y, f\alpha)$$

are isomorphisms for each $n \geq 0$, $x_0 \in X$ and α ray in X. Moreover, f induces a one-to-one correspondence from the set of proper ends of X into the set of proper ends of Y.

Proof. The result is well known for $\pi_n(f)$. For $\tau_n(f)$, let F be a proper homotopy from id_X to $g \circ f$, then for each ray α in X, F defines a "path" $\mu : J \times I \to X$ between the rays α and $g \circ f \circ \alpha$. According to [18, Theorem I.5.2.] the path μ induces a isomorphism $\mu_n : \tau_n(X, \alpha) \to \tau_n(X, g \circ f \circ \alpha)$ for each n. It is easy to check that $\tau_n(g) \circ \tau_n(f) = \mu_n$. Analogously, the composition map $\tau_n(f) \circ \tau_n(g)$ is proved to be an isomorphism. Consequently $\tau_n(f)$ is an isomorphism. The last part of this Proposition follows from the functorial properties of π_0, see section 2. □

DEFINITION 5.3. A proper map $f : X \to Y$ such that

1) f induces a bijection from the set of proper ends of X to the set of proper ends of Y,

and

2) $\pi_n(f)$ and $\tau_n(f)$ are 1-1 maps for each $n \geq 0$, $x_0 \in X$, and α ray in X,

will be called weak proper homotopy equivalence.

To prove the converse of Proposition 5.2 in the category of finite proper CW-complexes. let us recall briefly the cylinder of a given proper map $f : X \to Y$. Denote by W the disjoint union $(X \times I) \cup Y$, then the cilinder M_f of f is defined as the quotient of W induced by the identification given by $(x, 1) \sim f(x)$ for each $x \in X$. Recall that the identification map $p : W \to M_f$ satisfies:

a) $p_{|Y} : Y \to p(Y)$ is a homeomorphism and $p(Y)$ is a closed subset of M_f.

b) $p_{|(X \times I - X \times 1)} : (X \times I - X \times 1) \to p(X \times I - X \times 1)$ is a homeomorphism and $p(X \times I - X \times 1)$ is an open subset of M_f.

From these properties above it follows that $X \cong X \times 0$ and Y can be considered as closed disjoint subspaces of M_f.

Proposition 5.5

If $f : X \to Y$ is proper, then $p : W \to M_f$ is proper.

Proof. Since $W = (X \times I) \cup Y$ and the inclusion $i = p_{|Y} : Y \to M_f$ is proper, it will suffice to prove that $q = p_{|X \times I} : X \times I \to M_f$ is proper. For this consider a closed-compact subset K of M_f and let $\{A_\alpha\}_{\alpha \in \mathcal{A}}$ be an open covering of $q^{-1}(K)$ in $X \times I$. Then $\{A_\alpha \cap (X \times 1)\}_{\alpha \in \mathcal{A}}$ is an open covering of $q^{-1}(K) \cap (X \times 1)$ in $X \times 1$, and because $q^{-1}(K) \cap (X \times 1)$ is homeomorphic to $(i \circ f)^{-1}(K)$ which is compact, it follows that a finite subcovering $\{A_{\alpha_i} \cap (X \times 1)\}_{i=1}^{n}$ can be chosen. Now we have that $q^{-1}(K) \cap \big((X \times I) - (A_{\alpha_1} \cup \cdots \cup A_{\alpha_n})\big)$ is a closed subset of $X \times I$ and it is contained in $q^{-1}(K) \cap \big((X \times I) - (X \times 1)\big)$. Then $p\big(q^{-1}(K) \cap \big((X \times I) - (A_{\alpha_1} \cup \cdots \cup A_{\alpha_n})\big)\big)$ is a closed subset of M_f which is contained in K, hence it is compact. Now taking into account property b) above we can conclude that $q^{-1}(K) \cap \big((X \times I) - (A_{\alpha_1} \cup \cdots \cup A_{\alpha_n})\big)$ is also compact. Therefore we can complete the initial family $\{A_{\alpha_1}, \ldots, A_{\alpha_n}\}$ to form a finite subcovering of $q^{-1}(K)$. □

Proposition 5.6

If $f : X \to Y$ is proper, then Y is a strong proper deformation retract of M_f.

Proof. Let $H : M_f \times I \to M_f$ the homotopy given by:

$$H\big(p(x, s), t\big) = p(x, s + t - st). \qquad (x, s) \in X \times I,\ t \in I.$$

$$H(y, t) = y, \qquad y \in Y,\ t \in I.$$

Since I is a locally compact Hausdorff space the product space $M_f \times I$ can also be considered as a quotient of $W \times I$ given by the identification $\big((x, 1), t\big) \sim \big(f(x), t\big)$.

Then to see that H is proper it suffices to prove that $g_1 = q \circ l_1$, $g_2 = i \circ l_2$ are proper, where q, i are the maps defined in the proof of the Proposition 5.5 and

$$l_1 : (X \times I) \times I \to X \times I, \qquad l_2 : Y \times I \to Y$$

are given by

$$l_1\big((x,s),t\big) = \big(x, s+t-st\big), \qquad l_2(y,t) = y,$$

respectively. Because l_1, q, i, l_2 are proper it follows that g_1, g_2 are proper too. Hence H is proper.

Now define $r : M_f \to Y$ by $r = H_1$, and then $r \circ i = id_Y$ and $H : i \circ r \simeq_p id_{M_f}$ (relative to Y). □

Given two topological spaces X and Y, the set of proper homotopy classes will be denoted by $[X,Y]_p$. Note that a proper map $f : X \to Y$ induces the map $f_* : [Z,X]_p \to [Z,Y]_p$. With this notation we have:

Theorem 5.7

Let (X,Y) be a proper pair such that the inclusion map is a weak proper homotopy equivalence. Then if K is a finite proper CW-complex the induced map $i_* : [K,Y]_p \to [K,X]_p$ is a one-to-one correspondence.

Proof. First let us prove that i_* is surjective. Given a proper map $g : K \to X$ we shall construct inductively a proper homotopy $F : K \times I \to X$ such that $F_0 = g$ and $F_1(K) \subset Y$. Then F_1 will be proper and $i_*[F_1] = [g]$.

To define F on the 0 skeleton K^0, since the correspondence $\pi_0(i) : \pi_0(Y) \to \pi_0(X)$ is one-to-one for each 0–cell v of K, we can choose a path $\gamma : I \to X$ from $g(v)$ to some $y \in Y$. Define $F(v,t) = \gamma(t)$.

Suppose that F has been defined on $(K \times 0) \cup (K^{n-1} \times I) \to X$, $n \geq 1$. Now F can be properly extended on the n–skeleton as follows: For each compact n–cell we can proceed as in standard homotopy [16]. If c is a noncompact n–cell of K with characteristic map $h_c : E^{n-1} \times J \to K$, we can consider the map

$$\kappa : \big(E^{n-1} \times J \times 0\big) \cup \big(\partial(E^{n-1} \times J)\big) \times I\big) \xrightarrow{h_c \times id_I} \big(K \times 0\big) \cup \big(K^{n-1} \times I\big) \xrightarrow{F} X$$

and a homeomorphism L from $E^{n-1} \times J \times I$ to $I^{n-1} \times J \times I$ which maps

$$\big(E^{n-1} \times J \times 0\big) \cup \big(\partial(E^{n-1} \times J) \times I\big)$$

onto $I^{n-1} \times J \times 0$. Let us denote $\beta = \kappa \circ L^{-1}(0 \times J \times 0)$. Then from the fact that

$$\kappa \circ (L^{-1})_{|I^{n-1}\times J\times 0} : \left(I^{n-1} \times J, \partial(I^{n-1} \times J), 0 \times J\right) \longrightarrow (X, Y, \beta)$$

represents the zero element of the group $\pi_{n-1}(X, Y, \beta) = 0$, it follows easily that F extends to $e \times I$ in the required way. Of course we can extend F over the other noncompact n-cells in the same way. Note that in case of a noncompact 1-cell we must take into account that $\pi_0(Y,\beta) \to \pi_0(X,\beta)$ is surjective, which is the right interpretation for the condition $\pi_0(X, Y, \beta) = 0$. Because K is finite the inductive procedure finally produces the desired proper homotopy F.

i_* injective: Let $f, g : K \to Y$ be proper maps such that $F : i \circ f \simeq_p i \circ g$. Let L be the subcomplex $(K \times 0) \cup (K \times 1)$ of the proper CW-complex $K \times I$. Define $H(r,t) = F(r)$ for each $(r,t) \in L \times I$. Since $\pi_n(X, Y, y) = 0 = \pi_n(X, Y, \beta)$ for each n, $y \in Y$, and β ray in Y, we can use an argument similar to the one given above to extend it to a proper homotopy $H : (K \times I) \times I \to X$ such that $H_0 = F$ and $H_1(K \times I) \subset Y$. Then H_1 is a proper homotopy between f and g. □

Theorem 5.8

Let X, Y be spaces and let $f : X \to Y$ be a weak proper homotopy equivalence. If K is a finite proper CW-complex, then $f_* : [K, X]_p \to [K, Y]_p$ is a one-to-one correspondence.

Proof. Let M_f be the cylinder of f, and consider the inclusion $g : X \to M_f$ given by $g(x) = p(x, 0)$ and the retraction $r : M_f \to Y$ defined in Proposition 5.5, which is a proper homotopy equivalence. From Proposition 5.2, the hypothesis and the fact that $f = r \circ g$, it follows that g is also a weak proper homotopy equivalence. Applying Proposition 5.7 we obtain that $g_* : [K, X]_p \to [K, M_f]_p$ is one-to-one. Since r is a proper homotopy equivalence we also have that $r_* : [K, M_f]_p \to [K, Y]_p$ is one-to-one. Therefore we can conclude that $f_* = r_* \circ g_*$ is also one-to-one □

The following result is the converse of Proposition 5.2 and it can be considered as the main Theorem of this section.

Theorem 5.9

Let $f : X \to Y$ be a proper map between finite proper CW-complexes. Then f is a proper homotopy equivalence if and only if f is a weak proper homotopy equivalence.

Proof. Suppose that f is a weak proper homotopy equivalence. From Theorem 5.8, it follows that $f_* : [Y, X]_p \to [Y, Y]_p$ is a one-to-one correspondence, hence there is a proper map $h : Y \to X$ such that $f \circ h \simeq_p id_Y$. Because f and id_Y are weak proper homotopy equivalences we also have that h is a weak proper homotopy equivalence. Applying again Theorem 5.8 we obtain that $h_* : [X, Y]_p \to [X, X]_p$ is one-to-one. Therefore there is $f' : X \to Y$ such that $h \circ f' \simeq_p id_X$. From the associative property, it follows that

$$f' \simeq_p (f \circ h) \circ f' \simeq_p f \circ (h \circ f') \simeq_p f.$$

Hence h is the proper homotopic inverse of f. □

Remarks 5.10.

1. There is also a "pointed" version of Theorem 5.9. In this case we must consider proper maps and homotopies between rayed spaces of the form (X, α). You can give a proof of this rayed version following the process of this section. Note that it is convenient to define the rayed cylinder of a proper map preserving base rays.

2. For the standard Whitehead Theorem there are several examples which show that two CW-complexes X, Y, have isomorphic Hurewicz homotopy groups, but X, Y, do not have the same homotopy type. The following modification of [10, Example 1, p. 183] proves that the same occurs for proper homotopy.

Let m, n be even and odd positive integers respectively, such that $m > n > 1$. Let S^n be the n–sphere and $\mathbb{P}^m$ the real projective m–space. Consider the finite proper CW-complexes

$$X = S^n \times \mathbb{P}^m \times J / S^n \times \mathbb{P} \times 0, \qquad Y = S^m \times \mathbb{P}^n \times J / S^m \times \mathbb{P}^n \times 0.$$

It is interesting to note that X and Y are contractible path-connected spaces with one proper end, and it is easy to check that $\pi_q(X) \cong \pi_q(Y)$ for all $q \geq 0$. Nevertheless for $q = m + n + 1$ the proper homology theory J_* satisfies

$$J_{n+m+1}(X) \cong H_m(\mathbb{P}^m) = 0$$

$$J_{n+m+1}(Y) \cong H_n(\mathbb{P}^n) \neq 0.$$

Therefore X and Y have different proper homotopy type.

6. Proper cellular approximation theorems

DEFINITION 6.1. Let X, Y, be proper CW-complexes. A map $g : X \to Y$ is said to be cellular if $g(X^n) \subset Y^n$ for all $n \geq 0$.

We are trying to establish in this section when a proper map $f : X \to Y$ between CW-complexes admits a proper cellular approximation; that is, when there is a proper cellular map $g : X \to Y$ such that $f \simeq_p g$. For a continuous map between CW-complexes it is always possible to find a cellular approximation, for instance see [16, 7.4]. Nevertheless, for proper maps between CW-complexes do not necessarily admit proper cellular approximations as the following example shows:

On the semiopen interval J consider the proper CW-complex structure X given by one 0-cell for each nonnegative integer n, and one compact 1-cell $[n, n+1]$ for each consecutive nonnegative integers, and the structure Y consisting of one 0-cell, 0, and one non-compact 1 cell J. The identity map $id_J : X \to Y$ has a continuous cellular approximation $g : X \to Y$ given by $g(x) = 0$ for all $x \in X$, but id_J does not admit any proper cellular approximation because any cellular map $f : X \to Y$ satisfies that $\mathbb{Z}_+ \subset f^{-1}(0)$.

In this section, several theorems on the homotopy groups π, τ, $\check{\pi}$ of proper CW-complexes are developed to prove finally a proper cellular approximation theorem.

PROPOSITION 6.2

Let X be a proper m dimensional CW-complex. Then $\pi_n(X, X^{m-1}, x_0) = 0$ for each $x_0 \in X^{m-1}$ and each integer n satisfying $1 \leq n < m$.

Proof. Let $\{h_\gamma^m(E_\gamma^m)\}_{\gamma \in A_m \cup B_m}$ the family of m-cells of X. If $y \in A_m$, denote

$$U_\gamma = h_\gamma^m\{x \in E^m : \|x\| < 2/3\},$$

$$V_\gamma = h_\gamma^m\{x \in E^m : \|x\| \leq 1/3\}.$$

And if $y \in B_m$, denote

$$U_\gamma = h_\gamma^m\{(x,t) \in E^{m-1} \times J : \|x\| < 2/3,\ t > 1/3\},$$

$$V_\gamma = h_\gamma^m\{(x,t) \in E^{m-1} \times J : \|x\| \leq 1/3,\ t \geq 2/3\}.$$

Let L be the closed subset of X given by $L = \bigcup_\gamma V_\gamma$ and let W_γ be the open subset of X defined by $W_\gamma = U_\gamma \cap (X - L)$ for each $\gamma \in A_m \cup B_m$.

Given a representative map $f : (I^n, \partial I^n, *) \to (X, X^{m-1}, x_0)$ of an element of $\pi_n(X, X^{m-1}, x_0)$ we can consider the open covering

$$\{f^{-1}(U_\gamma), f^{-1}(X - L) : \gamma \in A_m \cup B_m\}$$

of I^n. Now we can subdivide I^n to obtain a new finite regular CW-complex structure M such that each cell of M is mapped by f either in $X - L$ or in some U_γ. Since, for each $\gamma \in A_m$, W_γ retracts to S^{m-1} by strong deformation, and, if $\gamma \in B_m$, W_γ retracts also to $\mathbb{R}^{m-1}$, a homotopy F from f to $g : M \to X$ can be constructed such that g maps the cells contained in $f^{-1}(U_\gamma)$ into W_γ. If $f(x) \in U_\gamma$ then $F(x,t) \in U_\gamma$ for every $t \in I$ and F is stationary in $f^{-1}(X - L)$. The construction of this F can be done inductively on the dimension of the skeletons of M in a similar way to the standard case [16]. Consequently $[f] = [g] \in \pi_n(X, X^{m-1}, x_0)$, but $[g] = i_*[g]$ where $i_* : \pi_n(X - L, X^{m-1}) \to \pi_n(X, X^{m-1})$ and $\pi_n(X - L, X^{m-1}) = 0$ because X^{m-1} is a strong deformation retract of $X - L$. Hence $[f] = 0$. □

Note 6.3. Attaching cells of dimension greater than 1 does not increase the number of path-components. Then, for each $m \geq 1$, $i_* : \pi_0(X^{m-1}) \to \pi_0(X)$ is a surjective map. This fact will be denoted by $\pi_0(X, X^{m-1}) = 0$.

Corollary 6.4

Let X a proper CW-complex. Then $\pi_r(X, X^n, x_0) = 0$ for each $r \leq n$, $x_0 \in X^n$.

Corollary 6.5

Let X be a proper CW-complex such that X^n admits a ray α. Then
a) $\tau_n(X, X^n, \alpha) \to \pi_n(X, X^n, \alpha)$ is surjective for $n \geq 1$.
b) $\tau_{q-1}(X, X^n, \alpha) \to \tau_{q-1}(X, X^n, \alpha)$ is a isomorphic for $2 \leq q \leq n$.

Proposition 6.6

Let X be an m-dimensional proper CW-complex such that every cell of X^{m-1} meets only a finite number of m-cells. Then for each ray α in X^{m-1},

$$\tau_{n-1}(X, X^{m-1}, \alpha) = 0$$

for every n satisfying $2 \leq n \leq m - 1$.

Proof. The notations h_γ^m, U_γ, V_γ, L, of the proof of Proposition 6.2 will be considered again.

Let

$$f : (I^{n-1} \times J, \partial(I^{n-1} \times J), * \times J) \longrightarrow (X, X^{m-1}, \alpha)$$

be a representative proper map of an element of $\pi_{n-1}(X, X^{m-1}, \alpha)$. Then

$$\{f^{-1}(U_\gamma), f^{-1}(X - L) : \gamma \in A_m \cup B_m\}$$

is an open covering of

$$I^{n-1} \times J = \bigcup_{k=0}^{\infty} (I^{n-1} \times [k, k+1]).$$

For each k, there exists a subdivision of $I^{n-1} \times [k, k+1]$ such that every cell is contained in an open subset of the covering above. All these subdivisions together give a structure of n dimensional regular CW-complex M in $I^{n-1} \times J$ satisfying that each cell of M is contained in some open subset of the covering.

For each noncompact cell of X with characteristic map $h_\gamma^m : E^{m-1} \times J \to X$ let us consider the increasing sequence of compact subsets of X,

$$\{h_\gamma^m(E^{m-1} \times [0, N])\}_{N=0}^{\infty},$$

whose union is $h_\gamma^m(E^{m-1} \times J)$. Since f is a proper map there exists an increasing sequence of compact subsets

$$\{I^{n-1} \times [0, d_\gamma^N]\}_{N=0}^{\infty},$$

where $d_\gamma^N \in \mathbb{R}$, such that

$$f^{-1}(h_\gamma^m(E^{m-1} \times [0, N])) \subset I^{n-1} \times [0, d_\gamma^N].$$

In the following paragraph, we are going to construct a proper map $g : M \to X$ satisfying for each r cell $\phi_\beta^r(E^r)$ of M that, if $\phi_\beta^r(E^r) \subset f^{-1}(X - L)$, then $g_{|\phi_\beta^r(E^r)} = f_{|\phi_\beta^r(E^r)}$, and, if $\phi_\beta^r(E^r) \subset f^{-1}(U_\gamma)$, then $g \circ \phi_\beta^r(E^r) \subset W_\gamma$. Moreover, there is a proper homotopy F from f to g relative to $\partial(I^{n-1} \times J)$ which has the following properties: if $f(x) \in U_\gamma$, then $F(x,t) \in U_\gamma$ for all t, and, if $f(x) \in X - L$, then F is stationary at x. The construction is made inductively on the skeletons of M as follows:

Given a 0-cell v of M, if $f(v) \in X - L$ we define $g(v) = f(v)$ and the homotopy, which is denoted by F, by $F(v,t) = f(v)$ for each $t \in I$. If $f(v) \in U_\gamma$ with $\gamma \in A_m$, because U_γ is path-connected, there exists a path from $f(v)$ to a point $g(v)$ in W_γ and the homotopy F is defined by the path chosen. If $f(v) \in U_\gamma$ with $\gamma \in B_m$, U_γ is also path-connected, however, to obtain a proper homotopy F, we must choose an adequate path. Whether $v \notin I^{n-1} \times [0, d_\gamma^N]$, then $f(v) \notin h_\gamma^m(E^{m-1} \times [0,N])$, and, because $U_\gamma \cap h_\gamma^m\left(E^{m-1} \times (N,+\infty)\right)$ is path-connected, there exists a path from $f(v)$ to a point $g(v)$ of $W_\gamma \cap h_\gamma^m\left(E^{m-1} \times (N,+\infty)\right)$ and the homotopy is defined again by the path. If $v \in I^{n-1} \times [0,d_\gamma^0]$ since $f(v) \in U_\gamma \cap h_\gamma^m\left(E^{m-1} \times (0,+\infty)\right)$ we can proceed as above for $N = 0$.

Suppose that F has been defined on M^{r-1}, $r \leq n$, satisfying the required conditions. Then for an r-cell $\phi_\beta^r(E^r)$ of M, if $\phi_\beta^r(E^r) \subset f^{-1}(X - L)$, then $g = f$ on $\phi_\beta^r(E^r) \subset f^{-1}(U_\gamma)$, with $\gamma \in A_m$. $g_{|\phi_\beta^r(S^{r-1})}$ can be extended to a map g of $\phi_\beta^r(E^r)$ into W_γ in the same way that in Proposition 6.2, and the same occurs for the homotopy from f to g on $\phi_\beta^r(S^{r-1})$; if

$$\phi_\beta^r(E^r) \subset (I^{n-1} \times J) - (I^{n-1} \times [0,d_\gamma^N]),$$

then

$$f \circ \phi_\beta^r(E^r) \subset U_\gamma \cap \left(h_\gamma^m(E^{m-1} \times J) - h_\gamma^m(E^{m-1} \times [0,N])\right),$$

which will be denoted by $U'_{\gamma N}$ and

$$g \circ \phi_\beta^r(S^{r-1}) \subset U'_{\gamma N} \cap W_\gamma,$$

which will be denoted by $W'_{\gamma N}$. Note that, if $N = 0$, $W'_{\gamma N}$ is contractible, and, for $N \neq 0$, $W'_{\gamma N}$ retracts to S^{m-2} but not in a proper way. Since $r \leq n < m - 1$, it follows that $\pi_{r-1}(W'_{\gamma N}) = 0$ for every N, therefore g can be extended to $\phi_\beta^r(E^r)$. On the other hand, since $\pi_r(U'_{\gamma N}) = 0$, F can be extended to a homotopy in $U'_{\gamma N}$ from $f_{|\phi_\beta^r(E^r)}$ to $g_{|\phi_\beta^r(E^r)}$. In the case that $\phi_\beta^r(E^r) \cap (I^{n-1} \times [0,d_\gamma^0]) \neq \emptyset$ we have that $f \circ \phi_\beta^r(E^r) \subset U'_{\gamma 0}$ and we can proceed as above for $N = 0$.

The homotopy $F : I^n \times J \times I \to X$ just constructed is continuous and satisfies the required conditions, but it is necessary to prove that F is proper. For this let us consider the notations:

$$\tilde{V}_\eta = \begin{cases} h_\eta^m\{x \in E^m : \|x\| < 1/2\} & \text{if } \eta \in A_m, \\ h_\eta^m\{(x,t) \in E^{m-1} \times J : \|x\| < 1/2, t > 1/2\} & \text{if } \eta \in B_m, \end{cases}$$

$$\tilde{L} = \bigcup_{\eta \in A_m \cup B_m} \tilde{V}_\eta .$$

Note that $X - \tilde{L}$ is a closed subset of X contained in $X - L$.

Now let K be a closed-compact subset of X, then by Proposition 3.4 it follows that

$$K = \bigcup_{\text{finite}} (K \cap h_\gamma^m(\Sigma^m)) \cup (K \cap (X - \tilde{L})).$$

Because the subset $K \cap (X - \tilde{L})$ is closed compact in X and by the construction of F it follows that the closed subset $F^{-1}(K \cap (X - \tilde{L}))$ is contained in $f^{-1}(K \cap (X - \tilde{L})) \times I$ which is compact because f is proper. Hence $F^{-1}(K \cap (X - \tilde{L}))$ is compact. For $\gamma \in A_m$,

$$F^{-1}\left(K \cap h_\gamma^m(\Sigma^m)\right) \subset f^{-1}\left(h_\gamma^m(\Sigma^m)\right) \times I,$$

which is compact because f is proper. For $\gamma \in B'_m$, by Proposition 3.4, $K \cap h_\gamma^m(\Sigma^m)$ is compact in $h_\gamma^m(\Sigma^m)$, then there exists $N \in \mathbb{N}$ such that

$$K \cap h_\gamma^m(\Sigma^m) \subset h_\gamma^m\left(E^{m-1} \times [0, N]\right).$$

Hence

$$f^{-1}\left(K \cap h_\gamma^m(\Sigma^m)\right) \subset I^{n-1} \times [0, d_\gamma^N].$$

Now taking into account the construction of F, this implies that

$$F^{-1}\left(K \cap h_\gamma^m(\Sigma^m)\right) \subset I^{n-1} \times [0, d_\gamma^N] \times I.$$

Then $F^{-1}\left(K \cap h_\gamma^m(\Sigma^m)\right)$ is compact and consequently F is proper.

Finally, since each cell of X^{m-1} meets only a finite number of m-cells it follows that X^{m-1} is a strong proper deformation retract of $X - \text{int}\, L$. Therefore $[f] = [g]$ represents the zero element of $\tau_{n-1}(X, X^{m-1}, \alpha)$. □

Note 6.7. If all m-cells of X are compact, following the proof of Proposition 6.6, it follows that $\tau_{n-1}(X, X^{m-1}, \alpha) = 0$ for each n satisfying $2 \leq n \leq m - 1$.

Remark 6.8. Let X be a proper CW-complex of finite dimension greater than 1. If we attach r-cells to X^1, $r \geq 2$, then the number of proper ends of X does not increase, therefore the map $\pi_0(X^1, \alpha) \to \pi_0(X, \alpha)$ is surjective. Consequently for each $N \geq 1$ the map $\pi_0(X^n, \alpha) \to \pi_0(X, \alpha)$ is also surjective and the same occurs for $\tau_0(X^n, \alpha) \to \tau_0(X, \alpha)$. These facts will be briefly denoted by $\pi_0(X, X^n, \alpha) = 0 = \tau_0(X, X^n \alpha)$.

Corollary 6.9

Let X be a proper CW-complex with finite dimension greater than n. If each k-cell $(k \geq n)$ of X meets only a finite number of $(k+1)$ cells of X then for each ray α in X^n, $\tau_{r-1}(X, X^n, \alpha) = 0$ for each r sastisfying $1 \leq r < n$.

Lemma 6.10

Let X be a finite path-connected proper CW-complex with only one proper end satisfying that the Alexandroff compactification $\hat{X}$ is conic at ∞ (see Proposition 3.4). Then for each $n \geq 3$, given a proper map $f : \left(I^{n-1} \times J, \partial(I^{n-1} \times J)\right) \to (X, X^n)$ there exists a proper map $g : I^{n-1} \times J \to X^n$ such that $f \simeq_p g$ through a proper homotopy H satisfying $H\left(\partial(I^{n-1} \times J) \times I\right) \subset X^n$.

Proof. First note that because $\pi_1(X, X^n, \alpha) = 0$ for each ray α in X^n and $\pi_0(X) = 0$ it follows that $\pi_0(X^n) = 0$. According to Corollaries 6.4 and 6.9 we have that $\pi_i(X, X^n, \alpha(0)) = 0$ for each i satisfying $0 \leq i \leq n$ and $\tau_{i-1}(X, X^n, \alpha) = 0$ for each i satisfying $0 \leq i < n$. Applying the exact sequences of Section 2, we also have that $\pi_{i-1}(X, X^n, \alpha) = 0$ for each i satisfying $0 \leq i < n$. Now we can apply the Theorem of Hurewicz type given at [8] to obtain the epimorphism

$$\rho_\pi : \pi_{n-1}(X, X^n, \alpha) \longrightarrow E_n(X, X^n)$$

whose kernel is the subgroup generated by the elements of the form $\xi - u * \xi$ where $\xi \in \pi_{n-1}(X, X^n, \alpha)$, $u \in \pi_1(X^n, \alpha)$ and $u * \xi$ denotes the action of u on ξ.

On the other hand, from Corollary 4.5 we have that $H_i(X, X^n) = J_i(X, X^n) = 0$ for all $i \leq n$. Now let us consider the following commutative diagram:

$$\begin{array}{ccccccccc}
\longrightarrow & 0 = \pi_n(X, X^n, \alpha(0)) & \longrightarrow & \tau_{n-1}(X, X^n, \alpha) & \longrightarrow & \pi_{n-1}(X, X^n, \alpha) & \longrightarrow \\
 & \big\downarrow \rho_\pi & & \big\downarrow \rho_\tau & & \big\downarrow \rho_\pi & \\
\longrightarrow & 0 = H_n(X, X^n) & \longrightarrow & 0 = J_n(X, X^n) & \longrightarrow & 0 = E_n(X, X^n) & \longrightarrow
\end{array}$$

$$\begin{array}{ccc}
\longrightarrow & \pi_{n-1}(X, X^n, \alpha(0)) = 0 & \longrightarrow \\
 & \big\downarrow \rho_\pi & \\
\longrightarrow & H_{n-1}(X, X^n) = 0 & \longrightarrow
\end{array}$$

Then we obtain that $\tau_{n-1}(X, X^n, \alpha) = \Omega_\tau^{n-1}(X, X^n, \alpha)$ is generated by the elements of the form $\xi - u * \xi$ where $\xi \in \tau_{n-1}(X, X^n, \alpha)$, $u \in \pi_1(X^n, \alpha)$ and $u * \xi$ denotes the action of u on ξ. Therefore the quotient $\pi_{n-1}(X, X^n, \alpha)/\Omega_\tau^{n-1}(X, X^n, \alpha)$ is trivial. However this quotient is isomorphic to the set of free proper homotopy classes of proper maps of pairs of the form $f : \left(I^{n-1} \times J, \partial(I^{n-1} \times J)\right) \to (X, X^n)$ by [18, III.3]. Now the result holds trivially. □

Theorem 6.11

Let X be a finite regular proper CW-complex. Then for $n \geq 3$, given a proper map

$$f : (I^{n-1} \times J, \partial(I^{n-1} \times J)) \longrightarrow (X, X^n),$$

there exist a proper map

$$g : I^{n-1} \times J \longrightarrow X^n$$

such that $f \simeq_p g$ through a proper homotopy T satisfying that

$$T(\partial(I^{n-1} \times J) \times I) \subset X^n.$$

Proof. The proper map f induces the continuous map $f^* : (I^{n-1} \times J)^* \to X^*$ between the respective Freudenthal's compactifications. The unique Freudenthal's end of $I^{n-1} \times J$ is transformed by f^* into a vertex v of the finite regular CW-complex X^* (see Prop. 3.8). Consider the subcomplex $Y = \overline{\mathrm{st}}\, v$ (start of v) of X^*. Then there is $p \in \mathbb{N}$ such that $f(I^{n-1} \times [p, +\infty)) \subset Y - \{v\}$ which is a path-connected subcomplex of X with one proper end and will be denoted by M. Note that M is under conditions of Lemma 6.10.

Now the restriction

$$f : (I^{n-1} \times \{p\}, \partial I^{n-1} \times \{p\}) \longrightarrow (M, M^n)$$

represents an element of $\pi_{n-1}(M, M^n) = 0$. Because $\pi_{n-1}(M, M^n) = 0$, there is a homotopy defined on $I^{n-1} \times \{p\}$ which can be extended by the absolute proper homotopy extension property to obtain a homotopy

$$F : I^{n-1} \times]p, +\infty) \times I \to M$$

such that

$$F_0 = f_{|I^{n-1} \times [p,+\infty)},$$

$$F(\partial I^{n-1} \times [p, +\infty) \times I) \subset M^n$$

and

$$F(\partial(I^{n-1} \times [p, +\infty)) \times \{1\}) \subset M^n.$$

Now from Lemma 6.10 there exists

$$G : (I^{n-1} \times [p, +\infty) \times I, \partial(I^{n-1} \times [p, +\infty)) \times I) \to (M, M^n)$$

such that $F_1 = G_0$ and $G_1(I^{n-1} \times [p, +\infty)) \subset M^n$. Let H be the proper homotopy on $I^{n-1} \times [p, +\infty)$ defined by the composition of F and G. We can extend H to $\partial(I^{n-1} \times [0,p])$ by the APHEP in such a way that

$$H(((I^{n-1} \times \{0\}) \cup (\partial I^{n-1} \times [0,p])) \times I) \subset X^n.$$

Finally because $\pi_n(X, X^n) = 0$, see Corollary 6.4, it follows that H can be extended on $I^{n-1} \times [0,p]$. The proper homotopy H satisfies that $H_0 = f$,

$$H(\partial(I^{n-1} \times J) \times I) \subset X^n$$

and

$$H_1(I^{n-1} \times J) \subset X^n. \square$$

Let X be a finite proper CW-complex. If each noncompact cell τ can be subdivided by a compact cell σ_τ in two new cells τ_c, τ_n, which are compact and noncompact respectively, and the new cells satisfy that $L = \bigcup \sigma_\tau$, $K = \bigcup \tau_n$ are proper CW-complexes such that the cellular structures of K and the product $L \times J$ are isomorphic, X is said to be of cylinder type at infinity.

Proposition 6.12

If X is a finite proper CW-complex of cylinder type at infinity with only one proper end, then for each ray α in X^2, $\tau_1(X, X^2, \alpha) = 0$.

Proof. Since X is of cylinder type at infinite, then we can subdivide the noncompact cells of X in the way described above. Using the same notation that [18] and [13, Corollary 14.6], we have

$$\pi_1(X, \alpha) \simeq \lambda_1(\hat{X}, \infty) \simeq \pi_1(L),$$

$$\pi_1(X^2, \alpha) \simeq \lambda_1(\hat{X}^2, \infty) \simeq \pi_1(L^1).$$

Since $\pi_1(L, L^1) = 0$, it follows that $\pi_1(L^1) \to \pi_1(L)$ is surjective. Therefore $i_* : \pi_1(X^2, \alpha) \to \pi_1(X, \alpha)$ is also surjective. Now from the exact sequence

$$\cdots \to \pi_1(X^2, \alpha) \to \pi_1(X, \alpha) \to \pi_1(X, X^2, \alpha) \to \pi_0(X^2, \alpha) \to \cdots$$

because $\pi_0(X^2, \alpha) = \{*\}$, we obtain that $\pi_1(X, X^2, \alpha) = 0$.

Finally in the exact sequence

$$\cdots \to \pi_2(X, X^2, \alpha(0)) \to \tau_1(X, X^2, \alpha) \to \pi_1(X, X^2, \alpha) \to \cdots$$

by Corollary 6.4, we have that $\pi_2(X, X^2, \alpha(0)) = 0$. Hence $\tau_1(X, X^2, \alpha) = 0$. $\square$

Proposition 6.13

If X is a finite regular proper CW-complex with only one proper end, then for each ray α in X^2, $\pi_1(X, X^2, \alpha) = 0$.

Proof. First we can suppose that the dimension of X is less than or equal to 3. In this case from the Schönflies Theorem [17, p. 26] we can argue to conclude that X is of cylinder type at infinite. Then Proposition 6.12 is applied to obtain that $\pi_1(X, X^2, \alpha) = 0$.

In the general case, we can consider the exact sequence associated with the triplet (X, X^3, X^2, α)

$$\cdots \to \pi_1(X^3, X^2, \alpha) \to \pi_1(X, X^2, \alpha) \to \pi_1(X, X^3, \alpha) \to \cdots$$

By Corollary 6.9 we know that $\pi_1(X, X^3, \alpha) = 0$. Therefore $\pi_1(X, X^2, \alpha) = 0$. □

Theorem 6.14

Let X and Y be finite CW-complexes such that either Y is regular or Y is of cylinder type at infinity and let $f : X \to Y$ be a proper map such that $f_{|M}$ is cellular for some subcomplex M of X. Then there is a cellular proper map $g : X \to Y$ such that $g_{|M} = f_{|M}$ and g is properly homotopic to f relative to M. (Proper cellular approximation theorem).

Proof. Consider the proper map $F : (X \times 0) \cup (M \times I) \to Y$ given by $F(x, 0) = f(x)$ for each $x \in X$ and $F(m, t) = f(m)$ for each $m \in M$ and $t \in I$. The map F will be properly extended to $X \times I$ by induction and the top level map F_1 will be cellular.

For each 0-cell a of X which is not contained in M there exists a path α in Y from $f(a)$ to some vertex v of Y, see Note 6.3. Define $g(a) = v$ and the homotopy F is defined at a by the path α. Now suppose that F has been extended to $X^{n-1} \times I$ and $F(X^{n-1} \times 1) \subset Y^{n-1}$ $(n \geq 1)$. Given an n-cell $h^n_\gamma(\Sigma^n)$ of X which is not contained in M, we define $k : (\Sigma^n \times 0) \cup (\partial\Sigma^n \times I) \to Y$ by $k = F \circ (h^n_\gamma \times id_I)$. If Σ^n is compact k represents an element of $\pi_n(Y, Y^n)$ which is trivial by Corollary 6.4, therefore k extends to a continuous map $G : E^n \times I \to Y$ such that $G(E^n \times 1) \subset Y^n$, and an extension $F : h^n_\gamma(E^n) \times I \to Y$ can be defined such that $F \circ (h^n_\gamma \times id_I) = G$.

If Σ^n is not compact we can consider a homeomorphism

$$l : E^{n-1} \times J \times I \longrightarrow I^{n-1} \times J \times I$$

which maps

$$(E^{n-1} \times J \times 0) \cup (\partial(E^{n-1} \times J) \times I)$$

onto $I^{n-1} \times J \times 0$ and $E^{n-1} \times J \times 1$ onto $(I^{n-1} \times J \times 1) \cup (\partial(I^{n-1} \times J) \times I)$. Then,

$$k \circ l^{-1}_{|I^{n-1} \times J \times 0} : (I^{n-1} \times J \times 0, \partial(I^{n-1} \times J) \times 0) \to (Y, Y^n)$$

is a proper map. For $n = 1$, $\beta(t) = k \circ l^{-1}(0, t, 0)$ represents a proper end of Y. Since $\pi_0(Y^1) \to \pi_0(Y)$ is a surjective map, then there is a proper homotopy $H : J \times I \to I$ such that $H_0 = \beta$ and $H(J \times 1) \subset Y^1$. Then $H_{|0 \times I}$ represents an element of $\pi_0(Y, Y^1, \beta(0)) = 0$. Therefore a new proper homotopy $G : I^0 \times J \times I \to Y$ such that

$$G_{|I^0 \times J \times 0} = (k \circ l^{-1})_{|I^0 \times J \times 0}$$

and

$$G((I^0 \times J \times 1) \cup (\partial(I^0 \times J) \times I)) \subset Y^1$$

can be constructed. For $n = 2$, we apply either Proposition 6.13 or Proposition 6.12 depending if Y is either regular or cylinder type at infinity respectively, and for $n \geq 3$ the Theorem 6.11 can be used. In all the cases there is a proper map $G : I^n \times J \times I \to Y$ satisfying

$$G_{|I^{n-1} \times J \times 0} = k \circ l^{-1}_{|I^{n-1} \times J \times 0}$$

and

$$G((I^{n-1} \times J \times 1) \cup (\partial(I^{n-1} \times J) \times I)) \subset Y^n$$

is obtained.

Therefore, for $n \geq 1$, $(G \circ l) : E^{n-1} \times J \times I \to Y$ is a proper extension of k and satisfies that $G \circ l(E^{n-1} \times J \times I) \subset Y^n$. This map induces a proper extension of F to $h^n_\gamma(E^n \times J) \times I$ such that $F \circ (h^n_\gamma \times id_I) = G$ and $F(h^n_\gamma(E^n \times J) \times 1) \subset Y^n$. Finally we can assert that the extension $F : X \times I \to Y$ obtained by this inductive process is proper because X is a finite proper CW-complex. □

Remarks 6.15.

1. Let X and Y be standard CW-complexes and suppose that X is finite, then since X is compact there is a proper cellular approximation for any proper map $f : X \to Y$.

2. Let X be a locally finite standard CW-complex of finite dimension and Y a locally finite standard CW-complex. In this case there is no proper cellular approximation theorem. For example, let $X = [0, +\infty)$ be the CW-complex whith a 0–cell for each nonnegative integer n, and a 1–cell $[n, n+1]$ for each pair of consecutive nonnegative integers, and let Y be the CW-complex defined as follows. First a 1-cell

α^1 is attached to a single 0 cell α^0, let X^1 be the CW-complex obtained. Secondly a 2-cell α^2 is attached to X^1 by identifying all the boundary to a point in the interior of α^1. Afterwards we proceed by induction attaching an $(n+1)$-cell to a point in the interior of the corresponding n-cell. The resulting space Y satisfies that there are proper maps of the form $f : X \to Y$ and there are no proper cellular maps of the form $g : X \to Y$.

3. If X is a finite proper CW-complex and Y is a locally finite standard CW-complex of finite dimension, then there is proper cellular approximation for any proper map $f : X \to Y$. This can be proved similarly to the proof of Theorem 6.14 and by using that $\tau_{n-1}(Y, Y^{m-1}, \alpha) = 0$ for each n satisfying $2 \leq n \leq m-1$, see Note 6.7.

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