On Pelczynski's property (V*) in vector sequence spaces

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Abstract

Given a sequence (E_n) of Banach spaces, we characterize in this note the (V*)–sets and other related classes of subsets in the space $E = (\sum \oplus E_n)_p$ ($1 \leq p < \infty$ or $p = 0$). As a consequence, we prove that E has the Pelczynski's property (V*) if and only if so does every E_n.

Given a Banach space E, let us recall that a subset $A \subseteq E$ is said to be a *(V*)–set* if for every weakly unconditionally Cauchy (w.u.c. in short) series $\sum x_n^*$ in E^* (i.e., such that $\{\sum_\sigma x_n^* : \sigma \subseteq \mathbb{N} \text{ finite}\}$ is a bounded set of E^*), the following holds:

$$\lim_{n\to\infty} \sup\{x_n^*(x) : x \in A\} = 0.$$

The concept of (V*)–set was introduced by Pelczynski in his important paper [5], as a dual companion of that of (V)–set, which is defined in the same way, interchanching the places of E and E^*.

If we denote by $\mathcal{V}^*(E)$ the class of all (V*)–sets of E, it is easy to see that $\mathcal{V}^*(E)$ is preserved by linear continous images, linear combinations, closed absolutely convex hulls and passing to subsets. Also, every (V*)–set is bounded. All these properties follow from the definition, or can be proved immediately by using the next proposition, which collects the most useful characterizations of (V*)–sets:

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Proposition 1 ([2]; see also [4])

For a bounded subset A of a Banach space E, the following assertions are equivalent:

a) A is a (V^*)–set.

b) Every operator T from E into ℓ_1 maps A into a relatively compact subset.

c) A does not contain a sequence (x_n) equivalent to the unit basis of ℓ_1 and such that the closed span $[x_n]$ is complemented in E.

(The equivalence (a) $\Longleftrightarrow$ (b) follows simply from the bijection between operators $T : E \to \ell_1$ and w.u.c. series in E^*; see e.g., [3, Ch. VII]). With this result at hand, it is obvious that the following relation holds:

$$\mathcal{W}(E) \subseteq \mathcal{WC}(E) \subseteq \mathcal{V}^*(E) \subseteq \mathcal{B}(E), \tag{†}$$

where $\mathcal{W}(E)$, $\mathcal{WC}(E)$ and $\mathcal{B}(E)$ stand for the weakly relatively compact, weakly conditionally compact, and bounded subsets of E, respectively.

In the aforementioned paper, Pelczynski defined the (V*) property by the coincidence of $\mathcal{W}(E)$ and $\mathcal{V}^*(E)$. In view of (†), we see that the (V*) property can be decomposed in the two properties:

a) $\mathcal{W}(E) = \mathcal{WC}(E)$ (i.e., weakly sequential completeness), and

b) $\mathcal{WC}(E) = \mathcal{V}^*(E)$.

We have denoted condition b) as *weak (V*) property*, by obvious reasons. Spaces enjoying this property form a large class. In fact, from Rosenthal's ℓ_1–theorem it follows that whenever E does not contain a copy of ℓ_1, $\mathcal{B}(E) = \mathcal{WC}(E)$, and so E has the weak (V*) property. Less trivial is the following:

Theorem 2 [2]

Every closed subspace of an order continuous Banach lattice has the weak (V) property.*

In particular, $L_1(\mu)$ spaces, being weakly sequentially complete, have the (V*) property (result proved by Pelczynski [5]). This is probably the most important class of non reflexive spaces enjoying the (V*) property. Also the dual of the disk algebra and the space L_1/H_0 have property (V*).

In this note, we are interested in the study of the (V*) and the weak (V*) property in spaces of vector sequences.

Let (E_n) be a sequence of Banach spaces and $1 \leq p < \infty$. We shall denote, as usual, by $(\sum \oplus E_n)_p$ the space of all vector valued sequences $x = (x_n)$ such that $x_n \in E_n$ $(n \in \mathbb{N})$ and

$$\|x\|_p^p = \sum_{n=1}^{\infty} \|x_n\|^p$$

is finite, endowed with the Banach norm $x \longmapsto \|x\|_p$. Similarly, we define the c_0–sum and the ℓ_∞–sum. It is well known that if

$$E = \left(\sum \oplus E_n\right)_p \qquad (1 \leq p < \infty \text{ or } p = 0),$$

then E^*, the topological dual if E, can be identified to $(\sum \oplus E_n^*)_q$, where q is conjugate to p, that is, $(1/p) + (1/q) = 1$ (with the usual conventions). This identification is given by the isometry

$$\left(\sum \oplus E_n^*\right)_q \ni x^* = (x_n^*) \longmapsto T_{x^*} \in \left(\sum \oplus E_n\right)_p^*,$$

where

$$T_{x^*}(x) = \sum_{n=1}^{\infty} \langle x_n, x_n^* \rangle.$$

For every $m \in \mathbb{N}$, I_m will denote the canonical injection

$$E_m \ni y \mapsto (0, \ldots, 0, y, 0, \ldots) \in \left(\sum \oplus E_n\right)_p$$

and π_m the canonical projection

$$\left(\sum \oplus E_n\right)_p \ni x = (x_n) \mapsto x_m \in E_m.$$

We shall need the following result :

Lemma 3 [1, lemma 1.3]

Let (E_n), F be Banach spaces, $1 \leq p < \infty$ and T an operator from $E = (\sum \oplus E_n)_p$ into F. Suppose $A \subseteq E$ satisfies the condition

$$\lim_{n \to \infty} \sup \left\{ \sum_{k=n}^{\infty} \|\pi_k(x)\|^p : x \in A \right\} = 0.$$

Then

$$\lim_{n \to \infty} \sup \left\{ \left\| \sum_{k=1}^{n} T \circ I_k \circ \pi_k(x) - T(x) \right\| : x \in A \right\} = 0.$$

With this result at hand, we can prove the

Proposition 4

Let (E_n) be a sequence of Banach spaces and $A \subset E = (\sum \oplus E_n)_1$ a bounded subset. The following assertions are equivalent:

a) $A \in \mathcal{V}^*(E)$.

b) A satisfies the conditions

i) $\pi_n(A) \in \mathcal{V}^*(E_n)$ for every $n \in \mathbb{N}$.

ii)

$$\lim_{n\to\infty} \sup \left\{ \sum_{k=n}^{\infty} \|x_k\| : x \in A \right\} = 0.$$

Proof. a) $\Longrightarrow$ b): Every π_n is linear and continuous; hence, i) follows. Let $A \in \mathcal{V}^*(E)$ and suppose ii) does not hold. Then, there exists $\varepsilon > 0$, $(x_n) \subseteq A$ and a subsequence of positive integers $p_1 < q_1 < p_2 < q_2 < \cdots$ such that

$$\sum_{k=p_n}^{q_n} \|x_{nk}\| > \varepsilon.$$

For every $p_n \leq k \leq q_n$, let us consider $x_k^* \in E_k^*$ such that $\|x_k^*\| = 1$ and

$$\|x_{nk}\| \leq \langle x_{nk}, x_k^* \rangle + \frac{\varepsilon}{2^k} \cdot$$

Define

$$\varphi_n = \sum_{k=p_n}^{q_n} I_k(x_k^*) \in \left(\sum \oplus E_k^*\right)_\infty \approx E^*.$$

Then, $\sum \varphi_n$ is w.u.c. clearly. However,

$$\varphi_n(x_n) = \sum_{k=p_n}^{q_n} \langle x_{nk}, x_k^* \rangle \geq \sum_{k=p_n}^{q_n} \|x_{nk}\| - \frac{\varepsilon}{2} > \frac{\varepsilon}{2} \, ,$$

contradicting the fact that $A \in \mathcal{V}^*(E)$.

b) $\Longrightarrow$ a): According to proposition 1, in order to show that $A \in \mathcal{V}^*(E)$ it suffices to prove that $T(A)$ is relatively compact for every operator $T : E \to \ell_1$. But if T is such an operator and we write

$$P_m = \sum_{n=1}^{m} T \circ I_n \circ \pi_n : E \longrightarrow \ell_1,$$

we have by ii) and lemma 3:

$$\lim_{n\to\infty} \sup \{\|P_n(x) - T(x)\| : x \in A\} = 0.$$

Then, given $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$T(A) \subseteq P_m(A) + \{a = (a_n) \in \ell_1 : \|a\|_1 \leq \varepsilon\}. \tag{‡}$$

But i) and proposition 1 imply that $P_m(A)$ is relatively compact; hence (‡) proves that $T(A)$ is precompact and so relatively compact. □

The case $p \neq 1$ is treated in the following proposition.

Proposition 5

Let (E_n) be a sequence of Banach spaces and $1 < p < \infty$ or $p = 0$. For a bounded subset $A \subseteq E = (\sum \oplus E_n)_p$, the following assertions are equivalent:

a) $A \in \mathcal{V}^*(E)$.

b) $\pi_k(A) \in \mathcal{V}^*(E_k)$ *for every* $k \in \mathbb{N}$.

Proof. a) $\Longrightarrow$ b) is clear. We shall prove b) $\Longrightarrow$ a) arguing by contradiction. So, suppose b) true, but $A \notin \mathcal{V}^*(E)$. Then there exists a w.u.c. series $\sum x_n^*$ in E^*, a sequence $(x_n) \subseteq A$ and a $\delta > 0$ such that

$$|\langle x_n, x_n^*\rangle| = \left|\sum_{k=1}^{\infty} \langle \pi_k(x_n), \pi_k(x_n^*)\rangle\right| > \delta, \qquad (n = 1, 2, \ldots).$$

As $\sum \pi_k(x_n^*)$ is a w.u.c. series in E_k^*, the hypothesis implies

$$\inf_n |\langle \pi_k(x_n), \pi_k(x_n^*)\rangle| = 0, \qquad (k = 1, 2, \ldots).$$

Hence, we can get by induction two subsequences of positive integers

$$1 = n_1 < n_2 < \cdots, \qquad 0 = k_0 < k_1 < k_2 < \cdots,$$

such that

$$\left|\sum_{j=1}^{k_{i-1}} \langle \pi_j(x_{n_i}), \pi_j(x_{n_i}^*)\rangle\right| < \frac{\delta}{4}$$

and

$$\left|\sum_{j>k_i}\langle \pi_j(x_{n_i}, \pi_j(x^*_{n_i}))\rangle\right| < \frac{\delta}{4}\,.$$

If we put $y_i^* = x_{n_i}^*$, $y_i = (y_{ik})$, where $y_{ik} = \pi_k(x_{n_i})$ for $k_{i-1} < k \leq k_i$ and 0 otherwise, it turns out that $\sum y_i^*$ is still a w.u.c. series in E^*, (y_i) is a bounded sequence in E, $|\langle y_i, y_i^*\rangle \geq \delta/2$, and *the y_i's have pairwise disjoint supports.* In particular, $\{y_i : i \in \mathbb{N}\}$ cannot be a (V*)–set. Normalizing if necessary, we can suppose $\|y_i\| = 1$ for every i. Then, for any finite scalar sequence $a = (a_i)$, we have

$$\left\|\sum a_i y_i\right\|_p^p = \sum |a_i|^p \, \|yi\|^p = \sum |a_i|^p,$$

which proves that (y_i) is equivalent to the unit bases of ℓ_p (c_0 if $p = 0$). In particular, (y_i) is weakly null, hence a (V*)–set, contradicting what we had proved above. □

As a consequence, we get the following corollary that improves [1, proposition 2.5]:

Corollary 6

Let (E_k) be a sequence of Banach spaces and $1 < p < \infty$ or $p = 0$. Suppose (x_n) is basic sequence in $E = (\sum \oplus E_n)_p$, equivalent to the unit basis of ℓ_1 and spanning a complemented subspace. Then there exists $k \in \mathbb{N}$ such that $(\pi_k(x_n))$ contains a subsequence equivalent to the unit vector basis of ℓ_1 that spans a complemented subspace of E_k.

Proof. $A = \{x_n : n \in \mathbb{N}\}$ is not a (V*)–set, by proposition 1, c). Proposition 5 yields a $k \in \mathbb{N}$ such that $\{\pi_k(x_n) : n \in \mathbb{N}\}$ is not a (V*)–set in E_k. The result follows at once from proposition 1, c) again. □

The next lemma is surely well known. We include a proof for the sake of completeness:

Lemma 7

Let (x_n) be a bounded sequence in $E = (\sum \oplus E_n)_p$ ($1 \leq p < \infty$ or $p = 0$). Then the following assertions are equivalent:

a) (x_n) converges weakly to 0 (resp., is weakly Cauchy).

b) We have

i) $(\pi_k(x_n)) = (x_{nk})$ *converges weakly to 0 (resp., is weakly Cauchy) in* E_k*, for every* $k \in \mathbb{N}$.

ii) If $p = 1$*, the following condition holds:*

$$\lim_{n\to\infty} \sup \left\{ \sum_{k=n}^{\infty} \|\pi_k(x_m)\| : m \in \mathbb{N} \right\} = 0.$$

Proof (sketch). That a) implies b) follows from propositions 4 and 5, taking into account that every weakly Cauchy sequence is a (V*)–set. For the converse, suppose $p \neq 1$ and $(\pi_k(x_n))$ weakly null in E_k $(k \in \mathbb{N})$. Let $x^* = (x_k^*) \in \left(\sum \oplus E_k^*\right)_q \approx E^*$ (where $(1/p) + (1/q) = 1$). Then for every $r \in \mathbb{N}$,

$$\begin{aligned} |\langle x_n, x^* \rangle| &\leq \left| \sum_{k=1}^{r} \langle x_{nk}, x_k^* \rangle \right| + \sum_{k>r} |\langle x_{nk}, x_k^* \rangle| \\ &\leq \left| \sum_{k=1}^{r} \langle x_{nk}, x_k^* \rangle \right| + \left(\sum_{k>r} \|x_{nk}\|^q \right)^{1/q} \left(\sum_{k>r} \|x_{nk}\|^p \right)^{1/p}. \end{aligned}$$

(with obvious modifications when $p = 0$).

The fact that (x_n) is bounded enables us to choose $r \in \mathbb{N}$ in such a way that the second term of the last sum is less than a given $\varepsilon > 0$. With this r fixed, the first term can be made arbitrarily small by increasing n. The case $p = 1$ is analogous, using now condition b) ii).

Finally, let us recall that a sequence (x_n) is (weakly) Cauchy if and only if for every pair (n_k) and (m_k) of increasing subsequences of $\mathbb{N}$, the sequence $y_k = x_{n_k} - x_{m_k}$ is (weakly) null. □

The next result is well known and an immediate consequence of the preceding lemma:

Proposition 8

Let (E_n) *be a sequence of Banach spaces and* $1 \leq p < \infty$*. For a bounded subset* $A \subseteq E = \left(\sum \oplus E_n\right)_p$*, the following properties are equivalent :*

a) $A \in \mathcal{W}(E)$ *(resp.,* $\mathcal{WC}(E)$*).*

b) *We have*

i) For every $k \in \mathbb{N}$*,* $\pi_k(A) \in \mathcal{W}(E_k)$ *(resp.,* $\mathcal{WC}(E_k)$*)*

and

ii) If $p = 1$*, the following condition holds:*

$$\lim_{n\to\infty} \sup \left\{ \sum_{k=n}^{\infty} \|x_k\| : x \in a \right\} = 0.$$

Now we can pass to prove the result announced at the beginning:

Theorem 9

Let (E_n) be a sequence of Banach spaces, $1 \leq p < \infty$, and $E = (\sum \oplus E_n)_p$. Then E has the (V) (resp., weak (V*)) property if and only if so does each E_n.*

Proof. As E_n is isomorphic to a complemented subspace of E, the necessity of the condition is obvious. For the sufficiency, let $A \in \mathcal{V}^*(E)$. Then $\pi_k(A) \in \mathcal{V}^*(E_k)$ for every $k \in \mathbb{N}$ and, when $p = 1$, condition b) ii) of proposition 4 is satisfied. The result follows immediately from the hypothesis and proposition 8. □

Remarks.

a) For $p = 1$ and property (V*), theorem 9 appears in [4], with an entirely different proof.

b) We remit the interested reader to [2], where further references can be found. The case of the spaces $L_p(\mu, E)$ of E–valued Bochner p–integrable functions with respect to a finite measure μ is also considered in [2], extending previous results of [6].

References

1. F. Bombal, Operators on vector sequence spaces, in *Geometric Aspects of Banach Spaces* pp. 94–106, London Mathematical Society Lecture Note Series 140, Cambridge University Press, Cambridge, 1989.
2. F. Bombal, On (V*) sets and Pelzynski's property (V*), *Glasgow Math. J.*, to appear.
3. J. Diestel, *Sequences and Series in Banach Spaces*, Springer, New York, 1984.
4. G. Emmanuele, On the Banach spaces with the property (V*) of Pelczynski, *Annali Mat. Pura Appl.* 152 (1988), 171–181.
5. A. Pelczynski, On Banach spaces on which every unconditionally convergent operator is weakly compact, *Bull. Acad. Po. Sci.* 10 91962), 641–648.
6. E. Saab and P. Saab, On Pelczynski's properties (V) and (V*), *Pacif. J. Math.* 125 (1986) 205–210.