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Some asymptotic formulæ for spectra of a general convex domain

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Abstract

The object of this paper is to derive some interesting asymptotic formulæ for spectra of a general convex domain in two or three dimensions, linked with variation of two positive distinct functions entering the boundary conditions on two "parts" of the smooth boundary of this domain. Further results may be obtained.

1. Introduction

The underlying problem (1.1)–(1.3) can be considered as a generalisation of recent work of Zayed [4], which determine the precise shape of a vibrating membrane from the complete knowledge of the eigenvalues $\mu_k(\sigma_1, \sigma_2)$ for the Laplace operator Δ_n in $\mathbb{R}^n$, $n = 2$ or 3.

Let Ω be a simply connected bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ in the case $n = 2$, or a smooth bounding surface S in the case $n = 3$. Consider the impedance problem

$$(\Delta_n + \lambda) u = 0 \qquad \text{in } \Omega, \tag{1.1}$$

$$\left(\frac{\partial}{\partial n_1} + \sigma_1\right) u = 0 \qquad \text{on } \partial_1\Omega \qquad (\text{or } S_1), \tag{1.2}$$

$$\left(\frac{\partial}{\partial n_2} + \sigma_2\right) u = 0 \qquad \text{on } \partial_2\Omega \qquad (\text{or } S_2), \tag{1.3}$$

where $\partial/\partial n_1$ and $\partial/\partial n_2$ denote differentiations along the inward pointing normals to $\partial_1\Omega$ (orS_1) and $\partial_2\Omega$ (orS_2) respectively, in which $\partial_1\Omega$ is a part of the boundary $\partial\Omega$ and $\partial_2\Omega$ is the remaining part of $\partial\Omega$ while S_1 is a part of the bounding surface S and S_2 is the remaining part of S, σ_1 and σ_2 are positive functions. Denote its eigenvalues, counted according to multiplicity, by

$$0 < \mu_1(\sigma_1,\sigma_2) \le \mu_2(\sigma_1,\sigma_2) \le \cdots \le \mu_k(\sigma_1,\sigma_2) \le \ldots \to \infty \tag{1.4}$$

as $k \to \infty$.

At the beginning of this century the principal problem was that of investigating the asymptotic distribution of the eigenvalues (1.4). It is well known [1] that, in the case $n = 2$,

$$\mu_k(\sigma_1,\sigma_2) \sim \left(\frac{4\pi}{|\Omega|}\right) k \tag{1.5}$$

as $k \to \infty$, while, in the case $n = 3$,

$$\mu_k(\sigma_1,\sigma_2) \sim \left(\frac{6\pi^2}{V} k\right)^{2/3} \tag{1.6}$$

as $k \to \infty$, where $|\Omega|$ and V respectively are the area and the volume of the domain Ω.

The problem of determining further information about the geometry of Ω as well as the impedances σ_1 and σ_2 from a complete knowledge of the eigenvalues $\mu_k(\sigma_1,\sigma_2)$ has been discussed recently in [5] when $n = 2$ and in [6] when $n = 3$, in the case σ_1 and σ_2 are positive constants, through the asymptotic expansions of the spectral function

$$\theta(t) = \sum_{k=1}^{\infty} \exp\{-t\mu_k(\sigma_1,\sigma_2)\}, \tag{1.7}$$

for small positive t.

Thus in the case $n = 2$, $0 < \sigma_1,\sigma_2 << 1$

$$\theta(t) = \frac{|\Omega|}{4\pi t} + \frac{|\partial\Omega|}{8(\pi t)^{1/2}} + \frac{1}{6}\left[1 - \frac{3}{\pi}\left(\sigma_1|\partial_1\Omega| + \sigma_2|\partial_2\Omega|\right)\right] + O\left(t^{1/2}\right) \tag{1.8}$$

as $t \to 0$, while in the case $n = 3$, $0 < \sigma_1,\sigma_2 << 1$

$$\theta(t) = \frac{V}{(4\pi t)^{3/2}} + \frac{S}{16\pi t} + \frac{1}{12\pi^{3/2}t^{1/2}}\left\{\int_{S_1}(H - 3\sigma_1)\,dS_1 + \int_{S_2}(H - 3\sigma_2)\,dS_2\right\} + O\left(t^{1/2}\right) \tag{1.9}$$

as $t \to 0$, where

$$H = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right),$$

R_1 and R_2 are the principal radii of curvature.

With reference to [5, Section 2], (1.8) may be interpreted as:

(i) $\Omega \subseteq \mathbb{R}^2$ is a convex domain and we have the impedance boundary conditions (1.2), (1.3) with small impedances σ_1 and σ_2, or

(ii) $\Omega \subseteq \mathbb{R}^2$ is a convex domain with

$$\frac{3}{\pi}(\sigma_1|\partial_1\Omega| + \sigma_2|\partial_2\Omega|)$$

holes and has area $|\Omega|$ and its boundary has length $|\partial\Omega| = |\partial_1\Omega| + |\partial_2\Omega|$ together with Neumann boundary condition, provided $3/\pi\,(\sigma_1|\partial_1\Omega| + \sigma_2|\partial_2\Omega|)$ is an integer, where $|\partial_1\Omega|$ and $|\partial_2\Omega|$ are the lengths of $\partial_1\Omega$ and $\partial_2\Omega$ respectively.

Similarly, with reference to [6, Section 2], (1.9) may be interpreted as:

(i) $\Omega \subseteq \mathbb{R}^3$ is a convex domain and we have the impedance boundary conditions (1.2), (1.3) with small impedances σ_1 and σ_2, or

(ii) $\Omega \subseteq \mathbb{R}^3$ is a convex domain, has volume V and its surface has area $S = S_1 + S_2$ and a part of the surface has area S_1 and mean curvature $(H - 3\sigma_1)$, while the other part has area S_2 and mean curvature $(H - 3\sigma_2)$ together with Neumann boundary condition.

In Theorem 1, we generalise the results (1.8) and (1.9) to the case when σ_1 and σ_2 are positive functions satisfying the Lipschitz condition, by using the expression

$$\sum_{k=1}^{\infty}\{\mu_k(\sigma_1,\sigma_2) + P\}^{-2}, \tag{1.10}$$

where P is a positive constant.

In Theorem 2, we show that this generalisation plays an important role in establishing a method to study the asymptotic behaviour of the difference

$$\sum_{\mu_k(\sigma_1,\sigma_2)\leq\lambda}\{\mu_k(\alpha_2,\beta_2) - \mu_k(\alpha_1,\beta_1)\}, \tag{1.11}$$

for large values of λ, where the three pairs of functions (σ_1,σ_2), (α_1,β_1) and (α_2,β_2) are distinct and satisfy the Lipschitz condition and the summation is taken over all values of k for which $\mu_k(\sigma_1,\sigma_2) \leq \lambda$. The method uses an interesting and important Tauberian theorem due to Hardy and Littlewood and developed by Titchmarsh [3].

Theorems 3 and 4 contain further results wich can be considered as a generalisation of the results of Theorem 2.

2. Statement and proofs of results

Theorem 1

If the functions $\sigma_1(Q)$, $\sigma_2(Q)$ satisfy the Lipschitz condition and if P is a positive constant, then in the case $n = 2$

$$\sum_{k=1}^{\infty}\{\mu_k(\sigma_1,\sigma_2)+P\}^{-2} = \frac{|\Omega|}{4\pi P} + \frac{|\partial\Omega|}{16P^{3/2}} + \frac{1}{6P^2}\left\{1-\frac{3}{\pi}\left[\int_{\partial_1\Omega}\sigma_1(Q)\,dQ + \int_{\partial_2\Omega}\sigma_2(Q)\,dQ\right]\right\} + O\left(\frac{1}{P^{5/2}}\right) \tag{2.1}$$

as $P \to \infty$, while in the case $n = 3$

$$\sum_{k=1}^{\infty}\{\mu_k(\sigma_1,\sigma_2)+P\}^{-2} = \frac{V}{8\pi P^{1/2}} + \frac{S}{16\pi P} + \frac{1}{24\pi P^{3/2}}\left\{\int_{S_1}[H(Q)-3\sigma_1(Q)]\,dQ + \int_{S_2}[H(Q)-3\sigma_2(Q)]\,dQ\right\} + O\left(\frac{1}{P^2}\right) \tag{2.2}$$

as $P \to \infty$.

Remark 1. The expression (1.10) is just the Laplace transform of the function $t\theta(t)$ with respect to t, and $P > 0$ is the Laplace transform parameter. Using this we deduce that the formulæ (2.1) and (2.2) can be considered as a generalisation of the formulæ (1.8) and (1.9) respectively.

Theorem 2

If the three pairs of functions $(\sigma_1(Q),\sigma_2(Q))$, $(\alpha_1(Q),\beta_1(Q))$, $(\alpha_2(Q),\beta_2(Q))$ are distinct and satisfying the Lipschitz condition, then in the case $n = 2$,

$$\sum_{\mu_k(\sigma_1,\sigma_2)\le\lambda}\{\mu_k(\alpha_2,\beta_2)-\mu_k(\alpha_1,\beta_1)\} = \frac{\lambda}{2\pi}\left\{\int_{\partial_1\Omega}[\alpha_2(Q)-\alpha_1(Q)]\,dQ + \int_{\partial_2\Omega}[\beta_2(Q)-\beta_1(Q)]\,dQ\right\} + o(\lambda) \tag{2.3}$$

as $\lambda \to \infty$, while in the case $n = 3$

$$\sum_{\mu_k(\sigma_1,\sigma_2)\le\lambda}\{\mu_k(\alpha_2,\beta_2)-\mu_k(\alpha_1,\beta_1)\} = \frac{\lambda^{3/2}}{3\pi^2}\left\{\int_{S_1}[\alpha_2(Q)-\alpha_1(Q)]\,dQ + \int_{S_2}[\beta_2(Q)-\beta_1(Q)]\,dQ\right\} + o(\lambda^{3/2}) \tag{2.4}$$

as $\lambda \to \infty$.

Formulæ (2.1)–(2.4) can be considered as a generalisation of the famillIiar formulæ of Gel'fand and Levitan [2] for the difference of traces of two Sturm-Liouville operators. These formulæ also can be considered as a useful extension of recent results of Zayed [4, Section 2].

Let us now give the proofs of Theorems 1, 2, using the Laplace transform of Green's function of the heat equation

$$\left(\Delta_n - \frac{\partial}{\partial t}\right) u = 0, \qquad n = 2 \text{ or } 3,$$

with respect to the time t, and use s^2 as the Laplace transform parameter [4].

Proof of Theorem 1. With reference to [4, Section 2], let

$$G(M, M_1; -s^2) = \frac{1}{2\pi} K_0(s\,|M - M_1|) - \bar{g}(M, M_1; -s^2), \tag{2.5}$$

be the Green's function of the expression $(\Delta_2 - s^2)u$ in the domain $\Omega \subseteq \mathbb{R}^2$ for the impedance boundary conditions (1.2), (1.3) on $\partial_1\Omega$ and $\partial_2\Omega$ respectively, where s is a sufficiently large positive constant while M and M_1 are points belonging to the domain Ω. As $M_1 \to M$ the equality

$$G(M, M_1; -s_1^2) - \bar{G}(M, M_1; -s^2) =$$

$$= (s^2 - s_1^2) \sum_{k=1}^{\infty} \frac{\phi_k(M_1)\,\phi_k(M)}{\{\mu_k(\sigma_1, \sigma_2) + s_1^2\}\{\mu_k(\sigma_1, \sigma_2) + s^2\}}\,, \tag{2.6}$$

where $\{\phi_k(M)\}$ are normalized eigenfunctions and $s \neq s_1$, implies

$$\frac{1}{2\pi} \log\left(\frac{s_1}{s}\right) + g(M, M; -s_1^2) - g(M, M; -s^2) =$$

$$= (s_1^2 - s^2) \sum_{k=1}^{\infty} \frac{\phi_k^2(M)}{\{\mu_k(\sigma_1, \sigma_2) + s_1^2\}\,\{\mu_k(\sigma_1, \sigma_2) + s^2\}}\,. \tag{2.7}$$

Thus we get the formula

$$\sum_{k=1}^{\infty} \{\mu_k(\sigma_1, \sigma_2) + s^2\}^{-2} = \frac{|\Omega|}{4\pi s^2} + \frac{1}{2s} \int\!\!\int_{\Omega} g_s'(M, M; -s^2)\, dM. \tag{2.8}$$

Using methods similar to those used in [5] we can show that

$$\int\int_{\Omega} \bar{g}'_s(M,M;-s^2)\,dM = \frac{|\partial\Omega|}{8s^2} + \frac{1}{3s^3}\left\{1 - \frac{3}{\pi}\left[\int_{\partial_1\Omega}\sigma_1(Q)\,dQ + \int_{\partial_2\Omega}\sigma_2(Q)dQ\right]\right\} + O\left(\frac{1}{s^4}\right) \tag{2.9}$$

as $s \to \infty$. On inserting (2.9) into (2.8) and letting $s^2 = P$ we arrive at the formula (2.1).

Similarly, let

$$G(M,M_1;-s^2) = \frac{\exp(-s\,|M-M_1|)}{4\pi\,|M-M_1|} - \bar{g}(M,M_1;-s^2), \tag{2.10}$$

be the Green's function of the expression $(\Delta_3 - s^2)u$ in the domain $\Omega \subseteq \mathbf{R}^3$ for the impedance boundary conditions (1.2), (1.3) on S_1 and S_2 respectively. As $M_1 \to M$, the equality (2.6) implies

$$\frac{(s_1 - s)}{4\pi} + \bar{g}(M,M;-s_1^2) - g(M,M;-s^2) =$$

$$= (s_1^2 - s^2)\sum_{k=1}^{\infty}\frac{\phi_k^2(M)}{\{\mu_k(\sigma_1,\sigma_2)+s_1^2\}\,\{\mu_k(\sigma_1,\sigma_2)+s^2\}}\,. \tag{2.11}$$

Thus we get the formula

$$\sum_{k=1}^{\infty}\{\mu_k(\sigma_1,\sigma_2)+s^2\}^{-2} = \frac{V}{8\pi s} + \frac{1}{2s}\int\int\int_{\Omega} g'_s(M,M;-s^2)\,dM. \tag{2.12}$$

Using methods similar to those used in [6] we can show that

$$\int\int\int_{\Omega} g'_s(M,M;-s^2)\,dM = \frac{S}{8\pi s} + \frac{1}{12\pi s^2}\left\{\int_{S_1}[H(Q) - 3\sigma_1(Q)]\,dQ + \int_{S_2}[H(Q) - 3\sigma_2(Q)]\,dQ\right\} + O\left(\frac{1}{s^3}\right) \tag{2.13}$$

as $s \to \infty$. On inserting (2.13) into (2.12) and letting $s^2 = P$, we arrive at the formula (2.2). □

Note that the proof of either (2.9) or (2.13) is omitted here since it is very similar to those obtained in [5] or [6] respectively.

Proof of Theorem 2. With reference to [4, Section 2], let us assume that $\alpha_2(Q) \geq \alpha_1(Q)$, $(Q \in \partial_1\Omega)$ and $\beta_2(Q) \geq \beta_1(Q)$, $(Q \in \partial_2\Omega)$ and introduce the non-negative and non-decreasing function

$$\Phi(\lambda) = \sum_{\mu_k(\alpha_2,\beta_2)\leq\lambda} \{\mu_k(\alpha_2,\beta_2) - \mu_k(\alpha_1,\beta_1)\}; \tag{2.14}$$

moreover we let

$$\Psi(P) = \sum_{k=1}^{\infty} \frac{\{\mu_k(\alpha_2,\beta_2) - \mu_k(\alpha_1,\beta_1)\}^2 \{\mu_k(\alpha_2,\beta_2) + 2\mu_k(\alpha_1,\beta_1) + 3P\}}{\{\mu_k(\alpha_2,\beta_2) + P\}^3 \{\mu_k(\alpha_1,\beta_1) + P\}} . \tag{2.15}$$

Using formula (2.1) first for the functions $(\alpha_1(Q),\beta_1(Q))$, then for the functions $(\alpha_2(Q),\beta_2(Q))$ and subtracting the second one from the first, we find after some reduction that

$$2\sum_{k=1}^{\infty} \frac{\{\mu_k(\alpha_2,\beta_2) - \mu_k(\alpha_1,\beta_1)\}}{\{\mu_k(\alpha_2,\beta_2) + P\}^3} + \Psi(P) = \frac{1}{2\pi P^2}\left\{\int_{\partial_1\Omega} [\alpha_2(Q) - \alpha_1(Q)]\, dQ + \int_{\partial_2\Omega} [\beta_2(Q) - \beta_1(Q)]\, dQ\right\} + O\left(\frac{1}{P^{5/2}}\right) \tag{2.16}$$

as $P \to \infty$, wich can be rewritten for any $a < \mu_1(\alpha_2,\beta_2)$ in the equivalent form

$$2\int_a^{+\infty} \frac{d\Phi(\lambda)}{(\lambda + P)^3} + \Psi(P) = \frac{1}{2\pi P^2}\left\{\int_{\partial_1\Omega} [\alpha_2(Q) - \alpha_1(Q)]\, dQ + \int_{\partial_2\Omega} [\beta_2(Q) - \beta_1(Q)]\, dQ\right\} + O\left(\frac{1}{P^{5/2}}\right) \tag{2.17}$$

as $P \to \infty$.

Further, noting that

$$\Psi(P) = o\left\{\int_a^{+\infty} (\lambda + P)^{-3}\, d\Phi(\lambda)\right\}$$

as $P \to \infty$, we get

$$\int_a^{+\infty} \frac{d\Phi(\lambda)}{(\lambda+P)^3} \sim \frac{1}{4\pi P^2}\left\{\int_{\partial_1\Omega}[\alpha_2(Q)-\alpha_1(Q)]\,dQ + \int_{\partial_2\Omega}[\beta_2(Q)-\beta_1(Q)]\,dQ\right\} \tag{2.18}$$

as $P \to \infty$.

Applying a Tauberian Theorem of Hardy and Littlewood (see, for example, [3, p. 364]), we find that

$$\Phi(\lambda) \sim \frac{\lambda}{2\pi}\left\{\int_{\partial_1\Omega}[\alpha_2(Q)-\alpha_1(Q)]\,dQ + \int_{\partial_2\Omega}[\beta_2(Q)-\beta_1(Q)]\,dQ\right\} \tag{2.19}$$

as $\lambda \to \infty$. Analogously, one establishes the asymptotic formula

$$\sum_{\mu_k(\alpha_1,\beta_1)\le\lambda}\{\mu_k(\alpha_2,\beta_2)-\mu_k(\alpha_1,\beta_1)\} \sim \frac{\lambda}{2\pi}\left\{\int_{\partial_1\Omega}[\alpha_2(Q)-\alpha_1(Q)]\,dQ + \int_{\partial_2\Omega}[\beta_2(Q)-\beta_1(Q)]\,dQ\right\} \tag{2.20}$$

as $\lambda \to \infty$.

Further, noting that

$$\sum_{\mu_k(\alpha_2^*,\beta_2^*)\le\lambda}\{\mu_k(\alpha_2,\beta_2)-\mu_k(\alpha_1,\beta_1)\} \le \sum_{\mu_k(\sigma_1,\sigma_2)\le\lambda}\{\mu_k(\alpha_2,\beta_2)-\mu_k(\alpha_1,\beta_1)\} \le \sum_{\mu_k(\alpha_1^*,\beta_1^*)\le\lambda}\{\mu_k(\alpha_2,\beta_2)-\mu_k(\alpha_1,\beta_1)\} \tag{2.21}$$

where

$$\alpha_2^*(Q) = \max\{\sigma_1(Q),\alpha_2(Q)\}, \qquad \beta_2^*(Q) = \max\{\sigma_2(Q),\beta_2(Q)\},$$
$$\alpha_1^*(Q) = \min\{\sigma_1(Q),\alpha_1(Q)\}, \qquad \beta_1^*(Q) = \min\{\sigma_2(Q),\beta_1(Q)\},$$

and the fact that as $\lambda \to \infty$ the functions:

$$\sum_{\mu_k(\alpha_2^*,\beta_2^*)\le\lambda}\{\mu_k(\alpha_2,\beta_2)-\mu_k(\alpha_1,\beta_1)\} = \sum_{\mu_k(\alpha_2^*,\beta_2^*)\le\lambda}\{\mu_k(\alpha_2^*,\beta_2^*)-\mu_k(\alpha_1,\beta_1)\} - \sum_{\mu_k(\alpha_2^*,\beta_2^*)\le\lambda}\{\mu_k(\alpha_2^*,\beta_2^*)-\mu_k(\alpha_2,\beta_2)\}, \tag{2.22}$$

and likewise for $(\alpha*_1, \beta_1^*)$ are asymptotically equal to

$$\frac{\lambda}{2\pi}\left\{\int_{\partial_1\Omega}[\alpha_2(Q)-\alpha_1(Q)]\,dQ+\int_{\partial_2\Omega}[\beta_2(Q)-\beta_1(Q)]\,dQ\right\},$$

we obtain (2.3) for the special case $\alpha_2(Q) \geq \alpha_1(Q)$ and $\beta_2(Q) \geq \beta_1(Q)$. Similarly, we derive (2.4) for the special case $\alpha_2(Q) \geq \alpha_1(Q)$, $(Q \in S_1)$ and $\beta_2(Q) \geq \beta_1(Q)$, $(Q \in S_2)$ as follows: Using the formula (2.2) first for the functions $(\alpha_1(Q), \beta_1(Q))$, then for the functions $(\alpha_2(Q), \beta_2(Q))$ and subtracting the second one from the first, we find for any $a < \mu_1(\alpha_2, \beta_2)$ that

$$2\int_a^{+\infty}\frac{d\Phi(\lambda)}{(\lambda+P)^3}+\Psi(P)=\frac{1}{8\pi P^{3/2}}\left\{\int_{S_1}[\alpha_2(Q)-\alpha_1(Q)]\,dQ\right.$$
$$\left.+\int_{S_2}[\beta_2(Q)-\beta_1(Q)]\,dQ\right\}+O\left(\frac{1}{P^2}\right) \quad (2.23)$$

as $P \to \infty$, and consequently

$$\int_a^{+\infty}\frac{d\Phi(\lambda)}{(\lambda+P)^3}\sim\frac{1}{16\pi P^{3/2}}\left\{\int_{S_1}[\alpha_2(Q)-\alpha_1(Q)]\,dQ\right.$$
$$\left.+\int_{S_2}[\beta_2(Q)-\beta_1(Q)]\,dQ\right\} \quad (2.24)$$

as $P \to \infty$.

As we have done before, we see that

$$\Phi(\lambda)\sim\frac{\lambda^{3/2}}{3\pi^2}\left\{\int_{S_1}[\alpha_2(Q)-\alpha_1(Q)]\,dQ+\int_{S_2}[\beta_2(Q)-\beta_1(Q)]dQ\right\} \quad (2.25)$$

as $\lambda \to \infty$. Analogously, one establishes the asymptotic formula

$$\sum_{\mu_k(\alpha_1,\beta_1)\leq\lambda}\{\mu_k(\alpha_2,\beta_2)-\mu_k(\alpha_1,\beta_1)\}\sim\frac{\lambda^{3/2}}{3\pi^2}\left\{\int_{S_1}[\alpha_2(Q)-\alpha_1(Q)]\,dQ\right.$$
$$\left.+\int_{S_2}[\beta_2(Q)-\beta_1(Q)]\,dQ\right\} \quad (2.26)$$

as $\lambda \to \infty$.

On using (2.21) and the fact that as $\lambda \to \infty$ the functions (2.22) for (α_2^*, β_2^*) and likewise for (α_1^*, β_1^*) are asymptotically equal to

$$\frac{\lambda^{3/2}}{3\pi^2}\left\{\int_{S_1}[\alpha_2(Q)-\alpha_1(Q)]\,dQ+\int_{S_2}[\beta_2(Q)-\beta_1(Q)]\,dQ\right\},$$

we obtain (2.4) for the special case $\alpha_2(Q) \geq \alpha_1(Q)$ and $\beta_2(Q) \geq \beta_1(Q)$.

In order to prove the theorem in the general case it is sufficient to apply the equality

$$\sum_{\mu_k(\sigma_1,\sigma_2)\leq\lambda} \{\mu_k(\alpha_2,\beta_2) - \mu_k(\alpha_1,\beta_1)\} = \sum_{\mu_k(\sigma_1,\sigma_2)\leq\lambda} \{\mu_k(\sigma_0^*,\sigma_0^{**}) - \mu_k(\alpha_1,\beta_1)\} - \sum_{\mu_k(\sigma_1,\sigma_2)\leq\lambda} \{\mu_k(\sigma_0^*,\sigma_0^{**}) - \mu_k(\alpha_2,\beta_2)\},$$

where

$$\sigma_0^*(Q) = \max\{\alpha_1(Q), \alpha_2(Q)\},$$

$$\sigma_0^{**}(Q) = \max\{\beta_1(Q), \beta_2(Q)\}$$

and apply the special case of the theorem which we just proved. □

3. Further results

Using formulæ (1.5) and (1.6) we obtain:

Corollary 1

As $N \to \infty$ we easily show that in the case $n = 2$

$$\sum_{k=1}^{N}\{\mu_k(\alpha_2,\beta_2) - \mu_k(\alpha_1,\beta_1)\} = \left(\frac{2I}{|\Omega|}\right) N + o(N), \tag{3.1}$$

while in the case $n = 3$

$$\sum_{k=1}^{N}\{\mu_k(\alpha_2,\beta_2) - \mu_k(\alpha_1,\beta_1)\} = \left(\frac{2J}{V}\right) N + o(N), \tag{3.2}$$

where

$$I = \int_{\partial_1\Omega} [\alpha_2(Q) - \alpha_1(Q)]\, dQ + \int_{\partial_2\Omega} [\beta_2(Q) - \beta_1(Q)]\, dQ, \tag{3.3}$$

and

$$J = \int_{S_1} [\alpha_2(Q) - \alpha_1(Q)]\, dQ + \int_{S_2} [\beta_2(Q) - \beta_1(Q)]\, dQ. \tag{3.4}$$

Remark 2. Theorem 2 (or Corollary 1) implies that if the quantities $I \neq 0$ and $J \neq 0$, then for an infinite set of values of the number k the difference

$$\{\mu_k(\alpha_2,\beta_2) - \mu_k(\alpha_1,\beta_1)\}$$

has the same sign as the quantities I and J.

Using Theorem 2 we easily prove the following Theorems:

Theorem 3

Let the three pairs of functions $(\sigma_1(Q), \sigma_2(Q))$, $(\alpha_1(Q),\beta_1(Q))$, $(\alpha_2(Q),\beta_1(Q))$ and the quantity $I \neq 0$ be the same as in (3.3). Furthermore, on the half-axis $[a,+\infty)$ let a function $f(\lambda)$ of constant sign be given which is absolutely continuous on each interval $[a,b]$, $b < \infty$; further we assume that the expression $\lambda f'(\lambda)/f(\lambda)$ is bounded almost everywhere and

$$\int_a^{+\infty} f(\lambda)d\lambda = \infty.$$

Then as $\lambda \to \infty$

$$\sum_{0<\mu_k(\sigma_1,\sigma_2)\leq\lambda} f\big(\mu_k(\sigma_1,\sigma_2)\big)\{\mu_k(\alpha_2,\beta_2) - \mu_k(\alpha_1,\beta_1)\} = \left\{\frac{I}{2\pi} + o(1)\right\} \int_a^\lambda f(t)\,dt. \tag{3.5}$$

Theorem 4

Let the three pairs of fuctions $(\sigma_1(Q),\sigma_2(Q))$, $(\alpha_1(Q),\beta_1(Q))$, $(\alpha_2(Q),\beta_2(Q))$ and the quantity $J \neq 0$ be the same as in (3.4). Furthermore, on the half-axis $[a,+\infty)$ let a function $f(\lambda)$ of constant sign be given which is absolutely continuous on each interval $[a,b]$, $b < \infty$; further we assume that the expression $\lambda f'(\lambda)/f(\lambda)$ is bounded almost everywhere and

$$\int_a^{+\infty} \lambda^{1/2} f(\lambda)\,d\lambda = \infty.$$

Then as $\lambda \to \infty$

$$\sum_{0<\mu_k(\sigma_1,\sigma_2)\leq\lambda} f\big(\mu_k(\sigma_1,\sigma_2)\big)\{\mu_k(\alpha_2,\beta_2) - \mu_k(\alpha_1,\beta_1)\} =$$

$$= \left\{\frac{J}{2\pi^2} + o(1)\right\} \int_a^\lambda |t|^{1/2} f(t)\,dt. \tag{3.6}$$

Proof. On setting

$$Z(\lambda) = \sum_{0<\mu_k(\sigma_1,\sigma_2)\leq\lambda} \{\mu_k(\alpha_2,\beta_2) - \mu_k(\alpha_1,\beta_1)\},$$

we deduce for any $a < \mu_1(\sigma_1,\sigma_2)$ that

$$\sum_{0<\mu_k(\sigma_1,\sigma_2)\leq\lambda} f(\mu_k(\sigma_1,\sigma_2))\{\mu_k(\alpha_2,\beta_2) - \mu_k(\alpha_1,\beta_1)\} = \int_a^{+\infty} f(\lambda)\, dZ(\lambda). \quad (3.7)$$

On inserting (2.3) and (2.4) into (3.7), we get easily the formulæ (3.5) and (3.6) respectively. □

Using Theorems 3 and 4 we obtain :

Corollary 2

Assuming that the function $f(\lambda)$ of Theorem 3 has the form

$$f(\lambda) = \lambda^m, \qquad m \geq -1$$

we find as $\lambda \to \infty$ that

$$\sum_{0<\mu_k(\sigma_1,\sigma_2)\leq\lambda} \mu_k^m(\sigma_1,\sigma_2)\{\mu_k(\alpha_2,\beta_2) - \mu_k(\alpha_1,\beta_1)\} =$$

$$= \begin{cases} \dfrac{I}{2\pi(m+1)}\, \lambda^{m+1} + o(\lambda^{m+1}) & \text{if } m > -1, \\ \dfrac{I}{2\pi} \ln\lambda + o(\ln\lambda) & \text{if } m = -1. \end{cases} \quad (3.8)$$

Corollary 3

Assuming that the function $f(\lambda)$ of Theorem 4 has the form

$$f(\lambda) = \lambda^m, \qquad m \geq -\frac{3}{2}$$

we find as $\lambda \to \infty$ that

$$\sum_{0<\mu_k(\sigma_1,\sigma_2)\leq\lambda} \mu_k^m(\sigma_1,\sigma_2)\{\mu_k(\alpha_2,\beta_2) - \mu_k(\alpha_1,\beta_1)\} =$$

$$= \begin{cases} \dfrac{J}{2\pi^2(m+3/2)}\, \lambda^{m+3/2} + o(\lambda^{m+3/2}) & \text{if } m > -\frac{3}{2} \\ \dfrac{J}{2\pi^2} \ln\lambda + o(\ln\lambda) & \text{if } m = -\frac{3}{2}. \end{cases} \quad (3.9)$$

Remark 3. Formulæ (3.5) (3.9) can be considered as a generalisation of those obtained in [4, Section 3].

From Corollaries 2 and 3 we easily derive the following propositions:

Proposition 1

If $\mu_k(\alpha_1,\beta_1) \neq 0$, then in the case $n = 2$

$$\sum_{k=1}^{N} \frac{\mu_k(\alpha_2,\beta_2)}{\mu_k(\alpha_1,\beta_1)} = N + \frac{I}{2\pi} \ln\left(\frac{4\pi}{|\Omega|}N\right) + o\left(\ln\left(\frac{4\pi}{|\Omega|}N\right)\right) \tag{3.10}$$

as $N \to \infty$, while in the case $n = 3$

$$\sum_{k=1}^{N} \frac{\mu_k(\alpha_2,\beta_2)}{\mu_k(\alpha_1,\beta_1)} = N + J\left(\frac{6}{\pi^4 V}\right)^{1/3} N^{1/3} + o(N^{1/3}) \tag{3.11}$$

as $N \to \infty$.

Proposition 2

If $\mu_1(\alpha_1,\beta_1) > 0$ and $\mu_1(\alpha_2,\beta_2) > 0$ then as $N \to \infty$, we deduce that in the case $n = 2$

$$\sum_{k=1}^{N} \{\mu_k^m(\alpha_2,\beta_2) - \mu_k^m(\alpha_1,\beta_1)\} =$$

$$= \begin{cases} \dfrac{I}{2\pi m}\left(\dfrac{4\pi}{|\Omega|}\right)^m N^m + o(N^m) & \text{if } m > 0, \\ \dfrac{I}{2\pi} \ln\left(\dfrac{4\pi}{|\Omega|}N\right) + o\left(\ln\left(\dfrac{4\pi}{|\Omega|}N\right)\right) & \text{if } m = 0, \end{cases} \tag{3.12}$$

while in the case $n = 3$

$$\sum_{k=1}^{N} \{\mu_k^m(\alpha_2,\beta_2) - \mu_k^m(\alpha_1,\beta_1)\} =$$

$$= \begin{cases} \dfrac{J}{(2m+1)\pi^2}\left(\dfrac{6\pi^2}{V}\right)^{(2m+1)/3} N^{(2m+1)/3} + o(N^{(2m+1)/3}) & \text{if } m > -1/2, \\ \dfrac{J}{3\pi^2} \ln\left(\dfrac{6\pi^2}{V}N\right) + o\left(\ln\left(\dfrac{6\pi^2}{V}N\right)\right) & \text{if } m = -1/2. \end{cases} \tag{3.13}$$

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