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Classification, through coordinates, of M-bases in separable Banach spaces

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Abstract

We give an outlook to some properties concerning the expansive sequence of a vector with respect to an M-basis, in a Banach space. From this point of view, we analyze some kinds of M-bases.

1. Previous concepts. Background

Let $S = (a_n)_{n\in N}$ denote a linearly independent sequence in a separable Banach space B. $[\cdot]$ stands for "closed linear span". Associated to S we define the following two closed subspaces:

1) *Kernel* of S:

$$K(S) = \bigcap_{n\in\mathbb{N}} [a_n, a_{n+1}, \ldots];$$

2) *Strict kernel* of S:

$$K_s(S) = \bigcap \{K(S') : S' \text{ infinite subsequence of } S\}.$$

S is said to be *complete* if $[S] = B$.

S is said to be *minimal* if $a_n \notin [a_{n+1}, a_{n+2}, \ldots]$ $(n \in \mathbb{N})$: equivalently if there exists in $[S]^*$ (dual of $[S]$) a sequence $(a_n^*)_{n\in\mathbb{N}}$, called conjugate, such that $a_m^*(a_n) = \delta_{nm}$ (Kronecker indices).

A minimal complete sequence with zero kernel is a *Markushevich basis* (M-basis) of B.

For $T \subset \mathbb{N}$, let $W_T = [a_t; t \in T]$ and

$$W_T^* = \bigcap_{k \notin T} [a_1, \ldots, a_{k-1}, a_{k+1}, \ldots].$$

An M-basis S is a *strong M-basis* if $W_T^* = W_T$ $(T \subset \mathbb{N})$ [8, §8].

An M-basis $S = (a_n)_{n\in\mathbb{N}}$ is a *basis with brackets* of B if there exists an increasing sequence of natural numbers $(p_n)_{n\in\mathbb{N}}$ such that

$$x = \lim_n \left(\sum_{i=1}^{p_n} a_i^*(x)\, a_i \right),$$

for every $x \in B$.

If we can take $(p_n)_{n\in\mathbb{N}}$ as the whole $\mathbb{N}$, the sequence is a *basis* (or Schauder basis) of B.

If S is a basis such that the series

$$\sum_{i=1}^{\infty} a_i^*(x) a_i$$

converges unconditionally to x, for every x in B, then S is an *unconditional basis* (or absolute basis) of B. That is to say that, for every rearrangement $\sigma : \mathbb{N} \to \mathbb{N}$, the sequence $(a_{\sigma(n)})_{n\in\mathbb{N}}$ is a Schauder basis.

We say that a sequence $S = (a_n)_{n\in\mathbb{N}}$ in B is weakly convergent to x, if, for every $f \in B^*$ (dual of B), is $f(x) = \lim_n f(a_n)$.

Theorem 1.1

(The weak basis theorem). Let $S = (a_n)_{n\in\mathbb{N}}$ be an M-basis of B. Then S is a basis if and only if, for every $x \in B$, the sequence

$$\left(\sum_{i=1}^{n} a_i^*(x)\, a_i \right)_{n\in\mathbb{N}}$$

is weakly convergent to x.

Proof. See [3]. For generalizations, see [4]. □

Using a similar theorem on the concept of Schauder decompositions, the following can be proved [3,4]

Proposition 1.2

Let $S = (a_n)_{n\in\mathbb{N}}$ be an M-basis of B. Then S is a basis with brackets if and only if there exists an increasing sequence of natural numbers, say $(p_n)_{n\in\mathbb{N}}$, such that for any $x \in B$, the sequence

$$\left(\sum_{i=1}^{p_n} a_i^*(x)\, a_i\right)_{n\in\mathbb{N}}$$

is weakly convergent to x.

The next implications are all strict [8,9]:

unconditional basis $\Rightarrow$ basis $\Rightarrow$

basis with brackets $\Rightarrow$ strong M-basis $\Rightarrow$ M-basis.

With respect to weak convergence, the following lemmas shall be needed in the sequel:

Lemma 1.3

If S is weakly convergent to x, then $x \in K_s(S)$.

Lemma 1.4

In B reflexive, given an M-basis $(a_n)_{n\in N}$ and a general sequence $(b_n)_{n\in\mathbb{N}}$, the following statements are equivalent :

(i) $\sup_n \|b_n\| < \infty$, and $\lim_n a_m(b_n) = \beta_m$ ($n \in \mathbb{N}$).

(ii) The sequence $(b_n)_{n\in N}$ is weakly convergent to a vector b such that $a_m(b) = \beta_m$ ($m \in \mathbb{N}$).

Proof. See [7, p. 195]. □

2. Coordinates and expansive sequences

DEFINITION 2.1. Given an M-basis $S = (a_n)_{n\in\mathbf{N}}$ in B, with conjugate $(a_n^*)_{n\in\mathbf{N}}$, and a vector x in B, the sequence $\left(a_n^*(x)\right)_{n\in\mathbf{N}}$ is called the *coordinate sequence* of x relative to S. Let $T_x = \{k \in \mathbf{N} : a_k^*(x) \neq 0\}$. If $T_x = (p_n)_{n\in\mathbf{N}}$ (infinite), then the sequence

$$S_x = \left(\sum_{i=1}^{n} a_{p_i}^*(x)\, a_{p_i}\right)_{n\in\mathbf{N}}$$

is called the *expansive sequence* of x relative to S.

DEFINITION 2.2. Given an M-basis $S = (a_n)_{n\in\mathbf{N}}$ of B, the set

$$D(S) = \bigcup_{T\subset\mathbf{N}} (W_T^* - W_T)$$

is called the *deficiency* of S. (This set gives an idea of how far is the M-basis of being a strong M-basis).

Related to this concept, we have the

Proposition 2.3

$x \in D(S)$ if and only if T_x is infinite and $K(S_x) = \{0\}$. Equivalently, when T_x is infinite, $x \notin D(S)$ if and only if $K(S_x) = [x]$.

Proof. See [1] or [6]. □

Remark. Notice that, T_x being infinite, only two cases for $K(S_x)$ may arise: $K(S_x) = \{0\}$ or $K(S_x) = [x]$.

3. A first classification

The expansive sequences of vectors $x \in B$ characterize the strong M-bases and the bases among the M-bases. This result was first stated for B reflexive in [6]. The same without the restriction of reflexivity was stated in [2]. Finally, using a similar idea, bases with brackets are also characterized.

The main result is given by

Theorem 3.1

Let $S = (a_n)_{n\in\mathbf{N}}$ be an M-basis of B. Then
(a) S is a strong M-basis if and only if $K(S_x) = [x]$, for every x with T_x infinite.
(b) S is a basis if and only if $K_s(S_x) = [x]$ for every x with T_x infinite.

With respect to basis with brackets, consider the following construction: Given an increasing sequence $(q_n)_{n\in\mathbf{N}}$ of natural numbers, and x with $T_x = (p_n)_{n\in\mathbf{N}}$, set, for every $n \in \mathbf{N}$, $h(n)$ as the greatest integer such that $p_{h(n)} < q_n$. (If there is no $h(n)$ in that situation, pass to the next n and define $h(n) = h(n+1)$). We have now the

Proposition 3.2

Let $S = (a_n)_{n\in\mathbf{N}}$ be an M-basis of B and $x \in B$ with $T_x = (p_n)_{n\in\mathbf{N}}$ infinite. Then S is a basis with brackets if and only if there exists an increasing sequence of natural numbers $(q_n)_{n\in\mathbf{N}}$ such that the sequence

$$S'_x = \left(\sum_{i=1}^{h(n)} a^*_{p_i}(x)\, a_{p_i}\right)_{n\in\mathbf{N}}$$

verifies $K_s(S'_x) = [x]$ (for any x with T_x infinite).

The ideas above may be used to construct M-basic sequences which are not strongly M-basic. (A completely different construction of such sequences appears in [8, §8]).

For this purpose, we need a strong M-basis not being a basis, in some Banach space B. That is not difficult: For instance, being $S = (a_n)_{n\in\mathbf{N}}$ a basic sequence in a reflexive Banach space, it can be proved [1] that the sequence $(a_n - a_{n+1})_{n\in\mathbf{N}}$ is a strong M-basis, but not a basis, of its closed linear span.

Example 3.3 [5]. Let $S = (a_n)_{n\in\mathbf{N}}$ be a strong M-basis but not a basis of B. By 3.1, take $x \in B$ with T_x infinite and $K_s(S_x) \neq [x]$. For the sake of simplicity, take T_x as $\mathbf{N}$. By 2.3 there exists a subsequence S'_x of S_x such that $K(S'_x) = \{0\}$. Set $S'_x = (a'_n)_{n\in\mathbf{N}}$, where

$$a'_n = \sum_{i=1}^{q_n} a^*_i(x) a_i,$$

and $q_1 < q_2 < \cdots < q_n < \cdots$.

Then, there exists $r \in \mathbf{N}$ such that $x \notin [a'_r, a'_{r+1}, \ldots]$. Set now $\hat{S} = (b_n)_{n\in\mathbf{N}}$, where

$$b_n = \sum_{i=q_r+1}^{n} a_i^*(x)a_i \qquad \text{for } q_r < n \le q_{r+1}.$$

Clearly, $\hat{S}$ is an M-basis of $[a_n; n \in \mathbf{N}]$ and (relative to $\hat{S}$) , $x \in W_T^* - W_T$, with $T = \{q_r, q_{r+1}, \ldots\}$ (infinite). Therefore, $\hat{S}$ is not a strong M-basis.

4. On boundedness of expansive sequences, in reflexive Banach spaces

Let $S = (a_n)_{n\in\mathbf{N}}$ be an M-basis in B reflexive. Let $x \in B$ with T_x infinite, and $S_x = (b_n)_{n\in\mathbf{N}}$ (we shall write "b_n" for short). Paying attention to the boundedness of S_x , we distinguish three cases :

1) $\sup_n \|b_n\| < \infty$,

2) $\sup_n \|b_n\| = \infty$, but $(\|b_n\|)_{n\in\mathbf{N}}$ does not tend to infinity,

3) $\lim_n \|b_n\| = \infty$.

In case 1), by 1.4, it follows that S_x is weakly convergent to x, so by 1.3, $x \in K_s(S_x)$.

In case 2) there exists a subsequence $(b_{r_n})_{n\in\mathbf{N}}$ of $(b_n)_{n\in\mathbf{N}}$, weakly convergent to x.

In case 3) it is straightforward to notice that $(b_n/\|b_n\|)_{n\in\mathbf{N}}$ is weakly convergent to zero, therefore $K_s(S_x) = 0$ since B is reflexive [9].

First case carries some consequences, including a characterization of bases, in reflexive B, equivalent to that in 3.1(b). We have :

Corollary 4.1

Let $S = (a_n)_{n\in N}$ be an M-basis of B reflexive, and $x \in B$ with $T_x = (p_n)_{n\in\mathbf{N}}$ infinite. Then if $x \in D(S)$, for any rearrangement $\sigma : \mathbf{N} \to \mathbf{N}$ it holds that

$$\lim_n \left\| \sum_{i=0}^{n} a^*_{\sigma(p_i)}(x) a_{\sigma(p_i)} \right\| = \infty$$

(we set $p_0 = 1$).

Corollary 4.2

Let $S = (a_n)_{n\in\mathbf{N}}$ be an M-basis of B reflexive. Then S is a basis if and only if, for every x with T_x infinite, the sequence $S_x = (b_n)_{n\in\mathbf{N}}$ is bounded, that is $sup_n\|b_n\| < \infty$.

Proof. It follows directly from 3.1 and the definition of basis. □

Remark. Similarly to 4.2, a characterization of bases with brackets might be given, using the boundedness on S'_x (see 3.2).

EXAMPLE 4.3. Let $B = \ell_2$ (separable Hilbert space) and $S = (e_n)_{n\in\mathbb{N}}$ an orthonormal basis. Set $\hat{S} = (e_n - e_{n+1})_{n\in\mathbb{N}}$ ($\hat{S}$ is a strong M-basis, but not a basis, of ℓ_2).

Let x be the vector

$$x = \sum_{n=1}^{\infty} \frac{e_n}{n}.$$

Its coordinate sequence relative to $\hat{S}$ is

$$\left(\sum_{i=1}^{n} \frac{1}{i}\right)_{n\in\mathbb{N}}.$$

So $\hat{S}_x = (a_n)_{n\in\mathbb{N}}$, where

$$a_n = \sum_{i=1}^{n} \left(1 + \frac{1}{2} + \cdots + \frac{1}{i}\right)(e_i - e_{i+1}).$$

and

$$\|a_n\| > 1 + \frac{1}{2} + \cdots + \frac{1}{n} \xrightarrow{n} \infty.$$

Let now $\sigma : \mathbb{N} \to \mathbb{N}$ be a rearrangement and call $\hat{S}^\sigma = (e_{\sigma(n)} - e_{\sigma(n)+1})_{n\in\mathbb{N}}$. Then $\hat{S}^\sigma_x = (b_n)_{n\in\mathbb{N}}$, where

$$b_n = \sum_{k=1}^{n} \left(\sum_{i=1}^{\sigma(k)} \frac{1}{i}\right)(e_{\sigma(k)} - e_{\sigma(k)+1}).$$

It follows again that

$$\|b_n\| > 1 + \frac{1}{2} + \cdots + \frac{1}{\sigma(n)} \xrightarrow{n} \infty.$$

This example suggests the

DEFINITION 4.4. A strong M-basis $S = (a_n)_{n\in\mathbb{N}}$ is said to be *perfect* if for every x with $T_x = (p_n)_{n\in\mathbb{N}}$ infinite, there exists a rearrangement $\sigma : \mathbb{N} \to \mathbb{N}$ such that, setting $p_0 = 1$, the sequence

$$\left(\sum_{i=0}^{n} a^*_{\sigma(p_i)}(x)a_{\sigma(p_i)}\right)_{n\in N}$$

does not tend to infinity.

EXAMPLE 4.5. Let $B = \ell_2$ and $(e_n)_{n\in\mathbb{N}}$ an orthonormal basis. Set

$$S = \left(\sum_{i=1}^{n} e_i\right)_{n\in\mathbb{N}} .$$

S is a strong M-basis, but not a basis of ℓ_2.

If

$$x = \sum_{i=1}^{\infty} x_i\, e_i,$$

the coordinate sequence of x relative to S is $(x_n - x_{n+1})_{n\in\mathbb{N}}$.

Since

$$\|(x_1-x_2)e_1+\ldots+(x_n-x_{n+1})(e_1+\ldots+e_n)\|^2 = \sum_{i=1}^{n} x_i^2+nx_{n+1}^2-2x_{n+1}(x_1+\ldots+x_n),$$

and

(i)

$$\sum_{i=1}^{n} x_i^2 < \|x\|^2 \qquad (n \in \mathbb{N})$$

(Bessel's inequality),

(ii) $nx_{n+1}^2 < 1$, for an infinite number of n's,

(iii)

$$|x_{n+1}x_1 + \ldots + x_{n+1}x_n| \le \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} \left(nx_{n+1}^2\right)^{1/2},$$

it is easy to conclude that S is a perfect strong M-basis.

The second case has a particular similarity with the third case. Moreover, this yields a characterization of unconditional bases in B reflexive. We have :

Theorem 4.6

Let $S = (a_n)_{n\in\mathbf{N}}$ be an M-basis in B reflexive, and $x \in B$ with T_x infinite and $S_x = (b_n)_{n\in\mathbf{N}}$ such that $\sup_n \|b_n\| = \infty$.

Then, there exists a rearrangement $\sigma : \mathbf{N} \to \mathbf{N}$ such that

$$\lim_n \left\| \sum_{i=1}^{n} a^*_{\sigma(i)}(x) a_{\sigma(i)} \right\| = \infty.$$

The proof leans on the concept of *inclination* $\nu(E, F)$ between two closed subspaces E and F of B:

$$\nu(E, F) = \inf_{\substack{x\in E, \|x\|=1 \\ y\in F}} \|x + y\|.$$

In a reflexive B, being E^p a subspace of finite dimension p, it is not difficult to notice that, for any M-basis $(a_n)_{n\in\mathbf{N}}$, it follows

$$\lim_n \nu\big(E^p, [a_n, a_{n+1}, \ldots]\big) = 1.$$

Proof of Theorem 4.6. Only in order to make easy the notation, put $T_x = N$. Let $0 < K_1 < K_2 < \ldots < K_n < K_{n+1} < \ldots$ with $\lim_n K_n = \infty$. By hypothesis, there exists $p_1 \in \mathbf{N}$ such that

$$\left\| \sum_{i=1}^{p_1} a^*_i(x) a_i \right\| > K_1.$$

Now

$$\nu\left(\left[\sum_{i=1}^{p_1} a_i(x) a_i \right], [a_n, a_{n+1}, \ldots] \right) \xrightarrow{n} 1,$$

so that there exists $p_2 > p_1$ with

$$\nu\left(\left[\sum_{i=1}^{p_1} a^*_i(x) a_i \right]. [a_{p_2}. a_{p_2+1}, \ldots] \right) > \frac{1}{2}.$$

Hence for any $i \geq p_2$, we have

$$\left\|\sum_{h=1}^{p_1} a_h^*(x)a_h + \sum_{j=p_2}^{i} a_j^*(x)a_j\right\| \geq \left\|\sum_{h=1}^{p_1} a_h^*(x)a_h\right\| \times \nu\left(\left[\sum_{h=1}^{p_1} a_h^*(x)a_h\right], \left[\sum_{j=p_2}^{i} a_j^*(x)a_j\right]\right)$$
$$\geq \left\|\sum_{h=1}^{p_1} a_h^*(x)a_h\right\| \times \nu\left(\left[\sum_{h=1}^{p_1} a_h^*(x)a_h\right], [a_{p_2}, a_{p_2+1}, \ldots]\right) > \frac{K_1}{2}.$$

Now, consider

$$M = \max\left\{\|a_{p_1+1}^*(x)a_{p_1+1}\|, \ldots, \left\|\sum_{i=p_1+1}^{p_2-1} a_i^*(x)a_i\right\|\right\}$$

Since

$$\sup_{i \geq p_2}\left\|\sum_{j=1}^{p_1} a_j^*(x)a_j + \sum_{h=p_2}^{i} a_h^*(x)a_h\right\| = \infty,$$

fix $p_3 > p_2$ such that

$$\left\|\sum_{j=1}^{p_1} a_j^*(x)a_j + \sum_{h=p_2}^{p_3} a_h^*(x)\right\| > K_2 + M.$$

We have obtained the following conditions:

$$\left\|\sum_{i=1}^{p_1} a_i^*(x)a_i\right\| > K_1 > \frac{K_1}{2},$$

$$\left\|\sum_{i=1}^{p_1} a_i^*(x)a_i + a_{p_2}^*(x)a_{p_2}\right\| > \frac{K_1}{2},$$

$$\dots\dots\dots$$

$$\left\|\sum_{i=1}^{p_1} a_i^*(x)a_i + \sum_{h=p_2}^{p_3} a_h^*(x)a_h\right\| > K_2 + M \geq K_2 > \frac{K_1}{2},$$

$$\left\|\sum_{i=1}^{p_1} a_i^*(x)a_i + \sum_{h=p_2}^{p_3} a_h^*(x)a_h + a_{p_1}^*(x)a_{p_1}\right\| > (K_2+M) - M = K_2 > \frac{K_1}{2},$$

$$\dots\dots\dots$$

$$\left\|\sum_{i=1}^{p_1} a_i^*(x)a_i + \sum_{h=p_2}^{p_3} a_h^*(x)a_h + \sum_{j=p_1+1}^{p_2-1} a_j^*(x)a_j\right\| > (K_2+M) - M = K_2 > \frac{K_1}{2}.$$

We iterate the above process starting from

$$b_2 = \sum_{i=1}^{p_3} a_i^*(x)a_i,$$

whose norm is bigger than K_2, and continue to

$$b_3 = \sum_{h=1}^{p_5} a_h^*(x)a_h = b_2 + \sum_{j=p_4}^{p_5} a_j^*(x)a_j + \sum_{r=p_3+1}^{p_4-1} a_r^*(x)a_r.$$

In this step, the succesive partial sums have a norm bigger than $K_2/2$, and the last one, b_3, has a norm exceeding K_3.

The process goes on indefinitely.

Finally, the rearrangement

$$\sigma = \begin{pmatrix} 1,\dots,p_1,p_1+1,\dots,p_2,p_2+1,\dots,p_3,p_3+1,\dots,p_4,p_4+1,\dots,p_5,\dots \\ 1,\dots,p_1,p_2,\dots,p_3,p_1+1,\dots,p_2-1,p_4,\dots,p_5,p_3+1,\dots,p_4-1,\dots \end{pmatrix}$$

satisfies the conditions required by 4.6. □

Finally we characterize unconditional bases :

Corollary 4.7

Let $S = (a_n)_{n\in\mathbf{N}}$ be an M-basis in B reflexive. The following statements are equivalent:

(i) S is an unconditional basis,

(ii) For every $x \in B$ and any rearrangement $\sigma : \mathbf{N} \to \mathbf{N}$, it follows that

$$\sup_n \left\| \sum_{i=1}^n a^*_{\sigma(i)}(x) a_{\sigma(i)} \right\| < \infty,$$

(iii) For every $x \in B$ and any rearrangent $\sigma : \mathbf{N} \to \mathbf{N}$ it follows that the sequence

$$\left(\left\| \sum_{i=1}^n a_{\sigma(i)}(x) a_{\sigma(i)} \right\| \right)_{n\in\mathbf{N}}$$

does not tend to infinity.

Proof. It follows directly from 4.5 . □

5. Conclusion

Applying succesively stronger properties on expansive sequences, we classify (from M-basis to unconditional basis) the M-bases in B reflexive. If B is not reflexive a classification is given, from the M-bases to the Schauder basis, passing through the basis with brackets.

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