Collect. Math. **39** (1988), 31–40

The differential equation $y' = fy$ in the algebras $H(D)$

Alain Escassut

Mathématiques Pures, Université Blaise Pascal, F-63170 Aubière, France

Marie-Claude Sarmant

Mathématiques, Université Pierre et Marie Curie, 4 Place Jussieu, F-75230 Paris 05, France

Received 11/OCT/88

Abstract

Let D be an a clopen bounded infraconnected set in an algebraically closed complete ultrametric valued field, and $H(D)$ the Banach algebra of the analytic elements in D [10,11,3]. Let f be an element of $H(D)$; we show that if the differential equation $y' = fy$ has a solution g invertible in $H(D)$, then the space of the solutions in $H(D)$ has dimension 1. We prove that a solution g has no zero isolated in D and that if g is not invertible, it is strictly annulled by a T-filter [6]. At last we prove that if $H(D)$ has no divisor of zero the space has dimension 0 or 1.

Introduction and theorems

Let $\mathbf{K}$ be an algebraically closed field of characteristic 0 provided with an ultrametric absolute value $|\cdot|$ for which it is complete. For any set D in $\mathbf{K}$ we will denote by $R(D)$ the $\mathbf{K}$-algebra of the rational functions $h(x) \in \mathbf{K}(x)$ with no pole in D. When D is closed and bounded, the algebra $R(D)$ is provided with the norm of the uniform convergence on D denoted by $\|\cdot\|_D$ [3] that makes it a normed $\mathbf{K}$- algebra. Its completion for that norm is then a $\mathbf{K}$-Banach algebra denoted by $H(D)$, the elements of which are called the *analytic elements on* D [1,3,4,11].

A set D is said to be *infraconnected* if for all $a \in D$, the adherence of the set $\{|x-a| : x \in D\}$ in $\mathbb{R}$ is an interval. In a previous article [8] we saw that a clopen bounded set D is infraconnected if and only if the only analytic elements on D whose derivative is identically null are the constants.

Here we take a clopen bounded infraconnected set D, an f in $H(D)$, we consider the differential equation $(\mathcal{E})$ $y' = fy$ with $y \in H(D)$, and we denote by $\mathcal{S}$ the space of the solutions $g \in H(D)$ of $(\mathcal{E})$.

By classical results, we know that $\mathcal{S}$ may be reduced to $\{0\}$. (For example, if D is the disk $|x| \leq 1$, it is easily seen that the equation $y' = y$ has no solution in $H(D)$). Here we will give sufficient conditions on the algebra $H(D)$ to have $\mathcal{S}$ of dimension 1 or 0. In another article we will see that the dimension of $\mathcal{S}$ sometimes may be greater than 1 when $H(D)$ has divisors of zero.

In the three theorems that follow, D is a clopen bounded infraconnected set, f belongs to $H(D)$, $(\mathcal{E})$ denotes the differential equation $y' = fy$ and $\mathcal{S}$ is the linear space of the solutions of $(\mathcal{E})$ in $H(D)$.

The notions of *T-filter* and *strictly annulled element* involved in Theorem 2 will be recalled below.

Theorem 1

If $(\mathcal{E})$ has at least one solution g invertible in $H(D)$ then $\mathcal{S}$ has dimension 1.

Theorem 2

We assume that $(\mathcal{E})$ has at least one solution g non identically null. Then g has no isolated zeros in D. Besides

a) either g is invertible in $H(D)$, or

b) g is strictly annulled by a T-filter on D.

Theorem 3

If $H(D)$ has no divisor of zero, then $\mathcal{S}$ has dimension 0 or 1.

The proof of Theorem 1 is easily obtained.

Proof of Theorem 1. Let g be a solution of $(\mathcal{E})$ invertible in $H(D)$, and let h be another solution. We verify that h/g is a constant in $H(D)$. Indeed, by hypothesis, h/g does belong to $H(D)$. Then

$$\left(\frac{h}{g}\right)' = \frac{h'g - hg'}{g^2} = \frac{fhg - hfg}{g^2} = 0,$$

and then by [8, Theorem 5] we know that h/g is a constant in D. □

Now we have to recall the definitions linked to the monotonous filters.

Technical definitions and proof of Theorem 2

The technique used in the proofs of the Theorems requires a lot of classical definitions previously given [4,5,6,7,9].

We will denote by "log" a real logarithm function of base $\omega > 1$ and by v the valuation defined on $\mathbf{K}$ by $v(x) = \log|x|$.

Now we have to define the monotonous filters. Henceforth, *D will denote a closed bounded infraconnected set* we will specify when it is supposed to be open; f will denote an element of $H(D)$ and $(\mathcal{E})$ is the equation $y' = fy$ with $y \in H(D)$.

For all $a \in \mathbf{K}$, $r \in \mathbb{R}_+$, $d(a,r)$ denotes the disk $\{x \in \mathbf{K} : |x-a| \leq r\}$, $d^-(a,r)$ is the disk $\{x \in \mathbf{K} : |x-a| < r\}$, and $C(a,r)$ is the circle $\{x \in \mathbf{K} : |x-a| = r\}$.

For $a \in \mathbf{K}$, $r', r'' \in \mathbb{R}_+$ with $0 < r' < r''$, we will denote by $\Gamma(a, r', r'')$ the set $\{x \in \mathbf{K} : r' < |x-a| < r''\}$.

Let $a \in D$, let r be the diameter of D, let $\tilde{D}$ be the disk $d(a,r)$. Then $\tilde{D} \setminus D$ admits a partition into a unique family $(T_i)_{i \in I}$ where each T_i is a disk $d^-(a_i, r_i)$ and r_i is maximal. The T_i are called the *holes* of D.

We call an *increasing filter* (resp. a *decreasing filter*) *of center $a \in \tilde{D}$ and diameter r* the filter on D that admits as a base the family of sets $\Gamma(a,s,r) \cap D$ with $0 < s < r$ (resp. $\Gamma(a,r,s) \cap D$ with $r < s$).

We call a *decreasing filter with no center on D* a filter that admits as a base a sequence D_n in the form $D_n = d(a_n, r_n) \cap D$ with

$$d(a_{n+1}, r_{n+1}) \subset d(a_n, r_n), \qquad \lim_{n \to \infty} r_n > 0, \qquad \bigcap_{n=1}^{\infty} d(a_n, r_n) = \emptyset,$$

and the limit of (r_n) is called the *diameter* of the filter.

We call a *monotonous filter* a filter that is either increasing or decreasing.

We know that if $\mathcal{F}$ is a monotonous filter on D and if $f \in H(D)$, then the function defined on D by $|f(x)|$ has a limit along the filter $\mathcal{F}$ and the mapping $f \mapsto \lim_{\mathcal{F}} |f(x)|$ is a multiplicative semi-norm on $H(D)$ continuous with respect to the norm $\|\cdot\|_D$ [5,9].

If $\mathcal{F}$ is a monotonous filter of center a and diameter r, we also have

$$\lim_{\mathcal{F}} |f(x)| = \lim_{\substack{|x-a| \to r \\ |x-a| \neq r \\ x \in D}} |f(x)|.$$

For convenience we introduce the valuation function $v_a(f,\mu)$ defined by

$$v_a(f,-\log r) = \lim_{\substack{|x-a|\to r\\ |x-a|\neq r\\ x\in D}} v(f(x)) \qquad \text{if} \qquad \lim_{\substack{|x-a|\to r\\ |x-a|\neq r\\ x\in D}} |f(x)| \neq 0$$

and

$$v_a(f,-\log r) = +\infty \qquad \text{if} \qquad \lim_{\substack{|x-a|\to r\\ |x-a|\neq r\\ x\in D}} f(x) = 0.$$

Let R be the diameter of D. Then for all $a \in \tilde{D}$, the function $\mu \mapsto v_a(f,\mu)$ is continuous and piecewise linear on its interval of definition I. If a does not belong to a hole of D, I is $[-\log R, +\infty[$. If a belongs to a hole $T = d^-(a,\rho)$, then $I = [-\log R, -\log \rho]$.

When $a = 0$ we will only write $v(f,\mu)$ for $v_0(f,\mu)$.

For $\mu < v(a-b)$ we have $v_a(f,\mu) = v_b(f,\mu)$ for all $f \in H(D)$ [4,5].

By the definition of $v_a(f,\mu)$ it is easily seen that $-\log \|f\|_D \leq v_a(f,\mu)$ for all $a \in D$, and $\mu \geq -\log R$. In particular, *if f and g are such that $-\log\|f-g\|_D < v_a(f,\mu)$, then $v_a(f,\mu) = v_a(g,\mu)$.*

Let f belong to $H(D)$. f is said to be *strictly annulled by an increasing filter (resp. a decreasing filter) of center a and diameter r*, if there exists $\lambda < -\log r$ (resp. $\lambda > -\log r$) such that $v_a(f,\mu) < +\infty$ whenever $\mu \in]-\log r, \lambda]$ (resp. whenever $\mu \in [\lambda, -\log r[$) and if $\lim_{\mathcal{F}} f(x) = 0$.

f is said to be *strictly annulled by a decreasing filter $\mathcal{F}$ with no center, of diameter r, of base (D_n) with $D_n = d(a_n, r_n) \cap D$*, if there exists $\lambda > -\log r$ such that $v_{a_n}(f,\mu) < +\infty$ whenever $\mu \in [\lambda, -\log r_n]$, whenever $n \in \mathbb{N}$, and if $\lim_{\mathcal{F}} f(x) = 0$.

Now recall that a monotonous filter is called a T-filter if the holes of the elements of its bases form a sequence that satisfies a condition given in [6] (we won't explicitly need it in the present work). Then we know that *given a monotonous filter $\mathcal{F}$, there exist elements $f \in H(D)$ strictly annulled by $\mathcal{F}$ if and only if $\mathcal{F}$ is a T-filter* [6].

An element $f \in H(D)$ is said to be *quasi-invertible* if it factorizes in the form $P(x)g(x)$ with P a polynomial the zeros of which are in the interior of D, and g an invertible element in $H(D)$.

Then if D is a clopen bounded infraconnected set, an element $f \in H(D)$ is not quasi-invertible if and only if it is annulled by a T-filter on D [6].

Proof of Theorem 2. Let us assume that g has an isolated zero a in D. Since D is open we know that g factorizes in the form $(x-a)^q h(x)$ with $h \in H(D)$ and $h(a) \neq 0$ [3,4], hence

$$g' = (x-a)^{q-1}\big(qh + (x-a)h'\big),$$

hence

$$qh = (x-a)(f-h'),$$

which contradicts the hypothesis $h(a) \neq 0$, (since $q \neq 0$). Thus g has no isolated zero in D.

Now suppose that g is not invertible; since it has no isolated zero, it is not quasi-invertible, and since D is open, that implies that g is strictly annulled by a T-filter on D [5,6]. □

Beaches, integrity and proof of Theorem 3

Let $\mathcal{F}$ be an increasing (resp. a decreasing) filter of center a and diameter $r > 0$. The set of the $x \in D$ such that $|x-a| \geq r$ (resp. $|x-a| \leq r$) is called *the beach* of $\mathcal{F}$, denoted by $\mathcal{P}(\mathcal{F})$. The beach $\mathcal{P}(\mathcal{F})$ of a decreasing filter $\mathcal{F}$ with no center is the empty set $\emptyset$. We denote by $\mathcal{C}(\mathcal{F})$ the set $D \setminus \mathcal{P}(\mathcal{F})$, by $\mathcal{J}(\mathcal{F})$ the ideal of the $f \in H(D)$ such that $\lim_{\mathcal{F}} f(x) = 0$ and by $\mathcal{J}_0(\mathcal{F})$ the ideal of the $f \in \mathcal{J}(\mathcal{F})$ such that $f(x) = 0$ whenever $x \in \mathcal{P}(\mathcal{F})$. *Then $\mathcal{J}(\mathcal{F})$ and $\mathcal{J}_0(\mathcal{F})$ are closed prime ideals* [5,6,7].

Two monotonous filters $\mathcal{F}$ and $\mathcal{G}$ on D are said to be *complementary* if $\mathcal{P}(\mathcal{F}) \cup \mathcal{P}(\mathcal{G}) = D$.

The Banach algebra $H(D)$ has no divisors of zero if and only if D is infraconnected with no couple of complementary T-filters [7].

In all the following lemmas *D will denote a closed bounded infraconnected set* and we will specify when it is open.

Lemma A

Let $a \in D$ and let $r \in \mathbb{R}_+$. Assume $f(x) = 0$ whenever $x \in d(a,r) \cap D$. Assume that there exists $b \in D$ such that $f(b) \neq 0$. Then there exists a T-filter $\mathcal{F}$ on D such that $b \in \mathcal{C}(\mathcal{F})$ and $d(a,r) \subset \mathcal{P}(\mathcal{F})$ [7].

Lemma B

Let $\mathcal{F}$ be a T-filter on D with no complementary T-filter. Then $\mathcal{J}(\mathcal{F}) = \mathcal{J}_0(\mathcal{F})$.

Proof of Lemma B. The equality $\mathcal{J}(\mathcal{F}) = \mathcal{J}_0(\mathcal{F})$ is trivial when $\mathcal{P}(\mathcal{F}) = \emptyset$; hence we will assume that $\mathcal{F}$ has center a. Let r be its diameter and let $\theta = -\log r$. Let $f \in \mathcal{J}(\mathcal{F})$ and let us show $f \in \mathcal{J}_0(\mathcal{F})$. For this, let us assume $f \notin \mathcal{J}_0(\mathcal{F})$ and let $b \in \mathcal{P}(\mathcal{F})$ be such that $f(b) \neq 0$.

Let $\lambda = v(a-b)$.

1) Assume that $\mathcal{F}$ is increasing.

1)α) Assume first $v_a(f,\lambda) < +\infty$.

By hypothesis since $f \in \mathcal{J}(\mathcal{F})$, we know $v_a(f,\theta) = +\infty$. Hence there exists $\gamma \in [\theta,\lambda]$ such that $v_a(f,\gamma) = +\infty$ and $v_a(f,\mu) < +\infty$ whenever $\mu \in [\gamma,\lambda]$. Then f is strictly annulled by the decreasing filter $\mathcal{G}$ of center a and diameter $s = \omega^{-\gamma}$. This filter $\mathcal{G}$ is then a T-filter complementary to $\mathcal{F}$ which contradicts the hypothesis.

1)β) Assume now $v_a(f,\lambda) = +\infty$. We know $v_b(f,\lambda) = v_a(f,\lambda)$ since $\lambda = v(a-b)$ and therefore $v_b(f,\lambda) = +\infty$, while $v_b(f,\mu) < +\infty$ when μ approaches $+\infty$ because $f(b) \neq 0$.

Then it exists $\gamma \geq \lambda$ such that $v_b(f,\mu) < +\infty$ whenever $\mu > \gamma$ and $v_b(f,\gamma) = +\infty$. Hence f is strictly annulled by the increasing filter of center b and diameter $s = \omega^{-\gamma}$. This filter is then a T-filter $\mathcal{G}$. Since $\max(r,s) \leq |a-b|$, $\mathcal{G}$ is complementary to $\mathcal{F}$, which contradicts the hypothesis.

2) Now, let us assume that $\mathcal{F}$ is decreasing. Then a and b belong to $\mathcal{P}(\mathcal{F}) = d(a,r) \cap D$; therefore $|a-b| \leq r$, hence $v_b(f,\theta) = +\infty$. Then it exists $\gamma > \theta$ such that $v_b(f,\gamma) = +\infty$ and $v_b(f,\mu) < +\infty$ for all $\mu > \gamma$, hence the increasing filter of center b and diameter $s = \omega^{-\gamma} < r$ is a T-filter complementary to $\mathcal{F}$, which ends the proof of Lemma B. □

Corollary C

If $H(D)$ has no divisor of zero then for every T-filter $\mathcal{F}$ on D, $\mathcal{J}(\mathcal{F}) = \mathcal{J}_0(\mathcal{F})$.

Lemma D

We assume that D has a family of T-filters $(\mathcal{F}_i)_{i\in I}$ such that

$$\bigcap_{i\in I} \mathcal{P}(\mathcal{F}_i) \neq \emptyset.$$

Let $j \in I$ and let $f \in \mathcal{J}(\mathcal{F}_j)$. Then

$$f(x) = 0 \qquad \text{whenever} \qquad x \in \bigcap_{i\in I} \mathcal{P}(\mathcal{F}_i).$$

Proof of Lemma D. Let

$$\Delta = \left(\bigcap_{i\in I} \mathcal{P}(\mathcal{F}_i)\right) \cup \mathcal{C}(\mathcal{F}_j).$$

It is easily seen that $\mathcal{F}_j$ is a T-filter on Δ with no complementary T-filter and, by Lemma B, $f \in \mathcal{J}_0(\mathcal{F}_j)$; hence $f(x) = 0$ whenever $x \in \mathcal{P}(\mathcal{F}_j) \cap \Delta$, hence

$$f(x) = 0 \qquad \text{whenever} \qquad x \in \bigcap_{i\in I} \mathcal{P}(\mathcal{F}_i). \ \square$$

DEFINITION. Let $g \in H(D)$. We call *support* of g the set Σ of the $x \in D$ such that $g(x) \neq 0$, and Σ will be *reinforced* if for every $a, b \in \Sigma$, the function $\mu \mapsto v_a(f,\mu)$ is bounded on the interval $[v(a-b), +\infty[$.

Proposition E

Assume that $H(D)$ has no divisor of zero. Then every $f \in H(D) \setminus \{0\}$ has a reinforced support.

Proof. Let $f \in H(D)$, let Σ be the support of f, and $a, b \in \Sigma$. Let us show that $v_a(f,\mu)$ is bounded when $\mu \in [v(a-b), +\infty[$. Indeed assume that it is not. Since $a \in \Sigma$, $f(a) \neq 0$, hence there exists $\gamma \in \mathbf{R}$ such that $v_a(f,\mu) = v(f(a))$ whenever $\mu \geq \gamma$. Since $v_a(f,\cdot)$ is a continuous function, if it is not bounded on $[v(a-b), +\infty[$, there exists $\lambda \geq v(a-b)$ such that $v_a(f,\mu) < +\infty$ whenever $\mu > \lambda$ and $v_a(f,\lambda) = +\infty$, so that D has an increasing T-filter $\mathcal{F}$ of center a and diameter $r = \omega^{-\lambda}$.

Assume first $v_a\big(f, v(a-b)\big) < +\infty$. Then there exists $\alpha \in]v(a-b), \lambda]$ such that $v_a(f,\mu) < +\infty$ whenever $\mu \in]v(a-b), \alpha[$, and $v_a(f,\alpha) = +\infty$, which means that D has a decreasing T-filter $\mathcal{G}$ of center a and diameter $\omega^{-\alpha} > r$. Then $\mathcal{G}$ is complementary to $\mathcal{F}$, which contradicts the hypothesis "$H(D)$ has no divisor of zero". By then we have proven $v_a\big(f, v(a-b)\big) = +\infty$, and $v_b\big(f, v(a-b)\big) = +\infty$. Reasoning as above, one can show the existence of an increasing T-filter $\mathcal{G}$ of center b and diameter $s < \omega^{-v(a-b)}$ hence $\mathcal{G}$ is complementary to $\mathcal{F}$, which contradicts again the hypothesis "$H(D)$ has no divisor of zero". Thus $v_a(f,\mu)$ is finally bounded on $[v(a-b), +\infty[$ and that ends the proof of Proposition E. $\square$

Lemma F

Let A and B be infraconnected closed bounded sets such that $\tilde{A} = \tilde{B}$. Then $A \cup B$ is infraconnected.

Proof. Let $d(\alpha, R) = \tilde{A} = \tilde{B}$. Let $a \in A$. Since $\widetilde{A \cup B} = d(\alpha, R) = d(a, R)$ the set $I(a) = \{|x - a| : x \in A \cup B\}$ is included in $[0, R]$. Since A is infraconnected, of diameter R, the set $\{|x - a| : x \in A\}$ is dense in $[0, R]$, hence $I(a)$ is dense in $[0, R]$. In the same way, when $a \in B$, $I(a)$ is still dense in $[0, R]$, and that finishes proving Lemma F. □

Proposition G

Assume that D is open. Let $f \in H(D)$ and assume that the support Σ of f is reinforced. Then for every couple $(a, b) \in \Sigma \times \Sigma$, there exists a clopen bounded infraconnected set $\Omega_a^b \subset \Sigma$ with $a, b \in \Omega_a^b$ and a number $\delta > 0$ such that $|f(x)| \geq \delta$ whenever $x \in \Omega_a^b$.

Proof. Let $r = |a - b|$. By hypothesis there exists $M \in \mathbf{R}_+$ such that $v_a(f, \mu) \leq M$ and $v_b(f, \mu) \leq M$ for all $\mu \geq v(a - b)$. Then the equality

$$v(f(x)) = v_a(f, v(x - a)) \qquad (\text{resp. } v(f(x)) = v_b(f, v(x - b)))$$

is true in all $D \cap d(a, r)$ (resp. $D \cap d(b, r)$), except maybe in a finite number of circles of center a (resp. b) and radii $\rho \leq r$. [5].

Let $C(a, \rho_i)_{1 \leq i \leq m}$ (resp. $C(b, \sigma_j)_{1 \leq j \leq n}$) be the circles of center a (resp. b) that contain points $x \in D$ such that $v(f(x)) \neq v_a(f, v(x - a))$ (resp. $v(f(x)) \neq v_b(f, v(x - b))$) and let

$$\Delta_a^b = (d(a, r) \cap D) \setminus \left(\bigcup_{i=1}^{m} C(a, \rho_i) \right)$$

$$\left(\text{resp.} \qquad \Delta_b^a = (d(b, r) \cap D) \setminus \left(\bigcup_{j=1}^{n} C(b, \sigma_j) \right) \right).$$

Then Δ_a^b (resp. Δ_b^a) is clearly infraconnected and clopen.

Moreover by hypothesis we have

$$v(f(x)) = v_a(f, v(x - a)) \leq M$$

on all Δ_a^b and

$$v(f(x)) = v_b(f, v(x - b)) \leq M$$

on all Δ_b^a. Let us put $\Omega_a^b = \Delta_a^b \cup \Delta_b^a$. Then $v(f(x)) \leq M$ whenever $x \in \Omega_a^b$ hence we can take $\delta = \omega^{-M}$ to obtain the relation $|f(x)| \geq \delta$ in Ω_a^b.

Now Ω_a^b is clearly clopen. At last by Lemma F, Ω_a^b is infraconnected because Δ_a^b and Δ_b^a are infraconnected sets such that $\widetilde{\Delta_a^b} = \widetilde{\Delta_b^a} = d(a, r)$. Proposition G is then proven. □

Proposition H

Let D be clopen, let $f \in H(D)$ and let $(\mathcal{E})$ be the differential equation $y' = fy$. We assume that $(\mathcal{E})$ has a solution g whose support is reinforced. Let h be another solution of $(\mathcal{E})$ Then there exists $\lambda \in \mathbf{K}$ such that $h(x) = \lambda g(x)$ whenever $x \in \Sigma$.

Proof. Since D is open, Σ is clearly open in $\mathbf{K}$, hence for every $a \in \Sigma$ there exists a disk $\Delta(a)$ included in Σ. Let $(\mathcal{E}_a)$ be the equation $y' = f(x)y$ for $x \in \Delta(a)$; then $(\mathcal{E}_a)$ has non null solutions (like the restriction of g to $\Delta(a)$), hence the space of the solutions has dimension one by classical results (and by Theorem 1). It only remains to show that $\lambda(a)$ is constant when a runs in Σ.

Let us fix a and b in Σ. By Proposition G, there exists a clopen bounded infraconnected set $\Omega_a^b \subset \Sigma$, with $a, b \in \Omega_a^b$, and $\delta > 0$ such that $|g(x)| \geq \delta$ whenever $x \in \Omega_a^b$.

The restriction $\tilde{g}$ of g to Ω_a^b is then invertible in $H(\Omega_a^b)$. Hence the restriction $\widetilde{h/g}$ of h/g to Ω_a^b is a locally constant element of $H(\Omega_a^b)$. Since Ω_a^b is clopen and infraconected, by [8, Theorem 5] we know that h/g is a constant in $H(\Omega_a^b)$, hence $(h/g)(b) = (h/g)(a)$ and then Proposition H is proved. □

Proof of Theorem 3. Assume that $(\mathcal{E})$ has a non identically null solution g. By Proposition E, the support Σ of g is reinforced. Let h be another non identically null solution. Since $H(D)$ has no divisor of zero, the support Σ' of h does have common points with Σ. By Proposition H there exists $\lambda \in \mathbf{K}$ such that $h(x) = \lambda g(x)$ whenever $x \in \Sigma$. Since $\Sigma \cap \Sigma' \neq \emptyset$, λ can't be zero. Hence $h(x) \neq 0$ whenever $x \in \Sigma$, therefore $\Sigma' \supset \Sigma$. By the same reasoning we just have $\Sigma' \subset \Sigma$, hence $\Sigma' = \Sigma$. The relation $h(x) = \lambda g(x)$ is then true on Σ, and it is trivially true on $D \setminus \Sigma$ where $h(x) = g(x) = 0$. Theorem 3 is then proved. □

References

1. Y. Amice, *Les Nombres p–adiques*, P. U. F., Paris, 1975.
2. B. Dwork, *Lectures on p–adic Differential Equations*, Springer, New York-Heidelberg-Berlin, 1982.
3. A. Escassut, Algèbres de Krasner, *Comptes Rendus Acad. Sci. Paris* **272** (1971), 598–601.
4. A. Escassut, Algèbres de éléments analytiques en analyse non archimédienne, *Indag. Math.* **36** (1974), 339-351
5. A. Escassut, Éléments analytiques et filtres percés sur un ensemble infraconnexe, *Annali Mat. Pura ed Appl. Bologna* **110** (1976), 335–352.
6. A. Escassut, T-filtres, ensembles analytiques et transformation de Fourier p–adique, *Ann. Inst. Fourier* **25** (1975), 45–80.

7. A. Escassut, Algèbres de Krasner intègres et noethériennes, *Proceedings Koninklijke Nederlandse Akademie van Wetenschappen* A **78** (1976), 109–130.
8. A. Escassut, Derivative of analytic elements on infraconnecetd clopen sets, *Proceedings Koninklijke Nederlandse Akademie van Wetenschappen*, to appear.
9. G. Garandel, Les semi-normes multiplicatives sur les algèbres d'éléments analytiques au sens de Krasner, *Indag. Math.* **37** (1975), 327–341.
10. M. Krasner, Prolongement analytique dans les corps valués complets: préservation de l'analyticité par la convergence uniforme et par la dérivation; théorème de Mittag-Leffler généralisé pour les éléments analytiques, *Comptes Rendus Acad. Sci. Paris* **244** (1957), 2570–2573.
11. M. Krasner, Prolongement analytique uniforme et multiforme dans les corps valués complets, in *Les Tendences Géométriques en Algèbre et Théorie des Nombres*, Colloques Internationaux du C. N. R. S. 143, C. N. R. S., Paris, 1964, pp. 97-141.
12. E. Motzkin and Ph. Robba, Prolongement analytique en analyse $p-$adique, *Séminaire de Théorie des Nombres*, Exp. 3, Faculté des Sciences de Bordeaux, Bordeaux, 1968–69.
13. Ph. Robba, Fonctions anlytiques sur les corps valués ultramétriques complets, *Astérisque* **10** (1973), 109–220.
14. M.-C. Sarmant and A. Escassut, T-suites idempotentes, *Bull. Sci. Math.* **106** (1982), 289–303.