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TORSIONAL RIGIDITY OF CIRCULAR SECTOR BARS

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ABSTRACT. Various results for the torsional rigidity of circular sector bars were obtained for various angles of the sector. The results obtained differ slightly in most cases from the values given by Saint-Venant [1] and Timoshenko [2], [3].

Sheng [4] obtained the torsional rigidity for semi-circular bars only. In this paper we shall show that the resulting series can be summed up not only for the semi-circular section but also for some other values of the sector angle.

The general stress function ψ for the torsion of circular cross section is given by

$$\psi = \frac{1}{2}\left[-r^2\left(1-\frac{\cos 2\theta}{\cos 2\beta}\right)+\frac{64\,a^2\beta^2}{\pi^3} \times \sum_{n=1,3,5,\ldots}^{\infty}(-1)^{(n+1)/2}\left(\frac{r}{a}\right)^{\frac{n\pi}{2\beta}}\frac{\cos\left(\frac{n\pi\theta}{2\beta}\right)}{n\left(n+\frac{4\beta}{\pi}\right)\left(n-\frac{4\beta}{\pi}\right)}\right] \tag{1}$$

where r, θ are the polar coordinates, a the radius of the sector and 2β the angle of the sector. The applied Torque T is given by

$$T = 2\mu\alpha\int_S \psi\, dS = 2\mu\alpha\int\int \psi\, r\,\theta\, dr,$$

where the integration is taken over the cross-sectional area, μ denotes the modulus of rigidity and α the angle of twist per unit length of the bar. Substituting the value of

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ψ from (1) and affecting the integration we obtain

$$T = -\mu\alpha \left[\int_{-\beta}^{\beta} \int_0^a r^3 \left(1 - \frac{\cos 2\theta}{\cos 2\beta}\right) dr\, d\theta + \frac{64\, a^3 \beta^2}{\pi^3} \right.$$
$$\left. \times \sum_{n=1,3,5,\ldots}^{\infty} \frac{(-1)^{(n+1)/2}}{n\left(n^2 - \frac{16\,\beta^2}{\pi^2}\right)} \int_{-\beta}^{\beta} \int_0^a \left(\frac{r}{a}\right)^{\frac{n\pi}{2\beta}+1} \cos \frac{n\pi\theta}{2\beta}\, dr\, d\theta \right].$$

From which we get

$$\frac{T}{\mu\alpha a^4} = \frac{1}{4} \tan 2\beta - \frac{1}{2}\beta - \frac{512\,\beta^4}{\pi^5} \left[\sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{n^2\left(n - \frac{4\beta}{\pi}\right)\left(n + \frac{4\beta}{\pi}\right)^2} \right]$$

or

$$\frac{T}{\mu\alpha a^4} = \frac{1}{4} \tan 2\beta - \frac{1}{2}\beta - \frac{8\beta}{\pi^2} \sum_{n=1,3,5,\ldots} \left[-\frac{1}{n^2} + \frac{\pi}{16\beta}\left\{ \frac{1}{n - \frac{4\beta}{\pi}} - \frac{1}{n + \frac{4\beta}{\pi}} \right\} \right.$$
$$\left. + \frac{\pi}{4\beta n} - \frac{\pi}{4\beta\left(n + \frac{4\beta}{\pi}\right)} - \frac{1}{2\left(n + \frac{4\beta}{\pi}\right)^2} \right] \ldots \tag{2}$$

It is known that

$$\sum_{m=0}^{\infty} \left\{ \frac{1}{m+x} - \frac{1}{m+1-x} \right\} = \pi \cot \pi x.$$

Hence we shall have

$$\sum_{n=1,3,5,.} \left\{ \frac{\pi}{16\beta\left(n - \frac{4\beta}{\pi}\right)} - \frac{\pi}{16\beta\left(n + \frac{4\beta}{\pi}\right)} \right\}$$
$$= \frac{-\pi}{32\beta} \sum_{m=0}^{\infty} \left\{ \frac{1}{m + \frac{1}{2} + \frac{2\beta}{\pi}} - \frac{1}{m + \frac{1}{2} - \frac{2\beta}{\pi}} \right\}.$$
$$= -\frac{\pi^2}{32\beta} \cot\left(\frac{\pi}{2} + 2\beta\right)$$
$$= \frac{\pi^2}{32\beta} \tan 2\beta. \tag{3}$$

Also it is well known that

$$\sum_{n=1,3,5,\ldots} \frac{1}{n^2} = \frac{\pi^2}{8} . \tag{4}$$

Substituting (3), (4), in (2) we get

$$\frac{T}{\mu\alpha a^4} = \frac{\beta}{2} - \frac{2\beta}{\pi^2} \sum_{n=1,3,5,\ldots} \left\{ \frac{\pi}{\beta} \left(\frac{1}{n} - \frac{1}{n + \frac{4\beta}{\pi}} \right) - \frac{2}{\left(n + \frac{4\beta}{\pi} \right)^2} \right\} .$$

Let $k = 4\beta/\pi$, then

$$\frac{T}{\mu\alpha a^4} = \frac{k\pi}{8} - \frac{k}{2\pi} \sum_{n=1,3,5,\ldots} \left\{ \frac{4}{n(n+k)} - \frac{2}{(n+k)^2} \right\} . \tag{5}$$

When

$$\beta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi,$$

i.e. $k = 1, 2, 3, 4$, the sum of the resulting series can be found exactly.

For the significance of the resulting cases we shall consider each case separately.

a. *Quadrantal cross-section:* In this case $\beta = \pi/4$, i.e. $k = 1$, and substituting in (5) we easily find

$$\frac{T}{\mu\alpha a^4} = \frac{\pi}{6} - \frac{2}{\pi} \ln 2 = 0.0823276 .$$

b. *Semi-circular cross-section:* In this case $\beta = \pi/2$, i.e. $k = 2$, and substituting in (5) we easily find

$$\frac{T}{\mu\alpha a^4} = \frac{\pi}{2} - \frac{4}{\pi} = 0.2975568 .$$

c. *Circular cross-section with a quadrantal notch:* In this case $\beta = 3\pi/4$, i.e. $k = 3$, and substituting in (5) we easily find

$$\frac{T}{\mu\alpha a^4} = \frac{\pi}{2} - \frac{7}{4\pi} - \frac{2}{\pi} \ln 2 = 0.5724828 .$$

d. *Circular cross-section with complete radial slit:* In this case $\beta = \pi$, i.e. $k = 4$, and substituting in (5) we easily find

$$\frac{T}{\mu\alpha a^4} = \pi - \frac{64}{9\pi} = 0.8780557 .$$

The corresponding results of St. Venant are 0.0825, 0.296, 0.528, 0.878 respectively. The slight difference between these and the calculated results is obviously seen.

References

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