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APPROXIMATE SOLUTION OF A CERTAIN CLASS OF NONLINEAR SINGULAR INTEGRAL EQUATIONS

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ABSTRACT. In this paper we solve a nonlinear singular integral equation with Hilbert kernel by the method of mechanical quadrature in generalized Hölder space. We also determine the rate of convergence for the approximate solution.

Introduction

In this paper we shall solve the following nonlinear singular integral equation

$$u(y) = \frac{\lambda}{2\pi} \int_0^{2\pi} F(y, s, u(s)) \cot \frac{s-y}{2} \, ds \tag{0.1}$$

in the generalized Hölder spaces $H_{\phi,m}$ [2] and $H_{\phi,m}^{(N)}$ [8]. Here the function $F(y, s, u(s))$ is 2π-periodic in y and s where $y, s \in [0, 2\pi]$ and $u \in [-M, M]$, $(M > 0)$.

The function $F(y, s, u)$ and its derivative $F'_y(y, s, u)$ satisfy the following two conditions respectively:

$$|F(y_1, s_1, u_1) - F(y_2, s_2, u_2)| \leq A_1 \left[\phi_1(|y_1 - y_2|) + \phi(|s_1 - s_2|) + |u_1 - u_2| \right], \tag{0.2}$$

$$|F'_y(y_1, s_1, u_1) - F'_y(y_2, s_2, u_2)| \leq A_2 \left[\phi(|y_1 - y_2|) + \phi(|s_1 - s_2|) + |u_1 - u_2| \right] \tag{0.3}$$

where $\phi, \phi_1 \in \Phi$, A_1, A_2 are constants, $y_i, s_i \in [0, 2\pi]$ and $u_i \in [-M, M]$; $(i = 1, 2)$.

Our objective is to determine the rate of convergence for the approximate solution of the equation (0.1).

1. The solution in the space $H_{\phi,m}$

Definition 1.1.

(a) We define the class Φ to be the class of all continuous almost increasing functions ϕ defined on $(0,\pi]$ such that

$$\phi(t) > 0, \qquad \lim_{t\to 0+} \phi(t) = 0.$$

(b) The class Φ^m is the class of all functions $\phi \in \Phi$ such that $0 < t_1 < t_2 < \pi$ implies

$$t_1^m\, \phi(t_2) \le c(m)\, t_2^m\, \phi(t_1),$$

where m is a natural number.

(c) We denote by $c_{2\pi}$ the space of 2π periodic continuous functions with norm

$$\|u\|_c = \max_{x\in[-\pi,\pi]} |u(x)|.$$

(d) For a natural number m we define

$$H_{\phi,m} = \{u \in c_{2\pi} : W_u^m(\delta) = O(\phi(\delta)),\ \phi \in \Phi^m\}$$

where $W_u^m(\delta)$ is the modulus of continuity of order m of u.

(e) For $u \in H_{\phi,m}$ we define

$$\|u\|_{\phi,m} = \|u\|_c + \sup_{0<\delta\le\pi} \frac{W_u^m(\delta)}{\phi(\delta)}$$

and

$$H_{\phi,m}(M) = \{u \in H_{\phi,m} : \|u\|_{\phi,m} \le M,\ M > 0\}.$$

(Please see [2], [6]).

Theorem 1.2. [6] *Let $\phi \in H\Phi^m$, then the operator*

$$(Au)(x) = \tilde{u}(x) = \frac{1}{2\pi}\int_0^{2\pi} u(y)\, \cot\frac{y-x}{2}\, dy$$

transforms $H_{\phi,m}(M)$ into $H_{\phi,m}(\tilde{M})$ where

$$\tilde{M} = M\left(c_1(m)\int_0^{\pi} \frac{\phi(\xi)}{\xi}\, d\xi + c_1(m) + c_2(m)\,\tilde{c}(m)\right).$$

Lemma 1.3. *Let the function $F(y,s,u(s))$ be a 2π-periodic in $y,s \in [0,2\pi]$ and $u \in [-M,M]$, $(M>0)$, has $(m-1)$ partial derivatives. If the function $F(y,s,u(s))$ satisfies the following condition for arbitrary $y_n, s_n \in [0,2\pi]$ and $u_n \in [-M,M]$, $(n=1,2)$:*

$$\left|\frac{\partial^l}{\partial y^i\,\partial s^j\,\partial u^k}F(y_1,s_1,u_1)-\frac{\partial^l}{\partial y^i\,\partial s^j\,\partial u^k}F(y_2,s_2,u_2)\right| \le c(l)\left(W(|y_1-y_2|)+W^*(|s_1-s_2|)+|u_1-u_2|\right), \tag{1.1}$$

for $i+j+k=l$, $l=0,\dots,m-1$, where $W(\delta)$, $W^*(\delta)$ are nondecreasing functions from $(0,\pi]$ into $\mathbf{R}_+$ and $W(0)=0$, $W^*(0)=0$.

Then

$$W_F^m(\delta)\le c(m)\begin{cases} W(\delta)+W^*(\delta)+W_u^1(\delta) & \text{at } m=1\\ W_u^m(\delta)+W_u^{m-2}(\delta)\,\delta\,W(\delta)+W_u^{m-2}(\delta)\,\delta\,W^*(\delta) & \text{at } m\ge 2.\end{cases}$$

Proof. For $m=1$, the lemma is true.

For $m=2$ we have

$$\Delta_h^2 F(y,s,u(s)) = F(y+2h,s+2h,u(s+2h)) - 2F(y+h,s+h,u(s+h)) + F(y,s,u(s)).$$

Using Lagrange's formula we get

$$\begin{aligned}\Delta_h^2 F(y,s,u(s)) = \Bigg|\int_0^1 &\big[F_y'(y+h+\theta h, s+2h, u(s+2h))\\ &- F_y'(y+\theta h, s+h, u(s+h))\big]\,h\,d\theta\\ +\int_0^1 &\big[F_s'(y+h, s+h+\theta h, u(s+2h))\\ &- F_s'(y, s+\theta h, u(s+h))\big]\,h\,d\theta\\ +\int_0^1 &\big[F_u'(y+h,s+h,u(s+h)+\theta(u(s+2h)-u(s+h)))\\ &- F_u'(y+h,s+h,u(s)+\theta(u(s+h))-u(s)))\big]\\ &(u(s+2h)-u(s+h))\,d\theta\\ +\int_0^1 &\big[F_u'(y+h,s+h,u(s)+\theta(u(s+h)-u(s)))\\ &- F_u'(y,s,u(s)+\theta(u(s+h)-u(s)))\big]\,(u(s+2h)-u(s+h))\,d\theta\\ +\int_0^1 &\big[F_u'(y,s,u(s)+\theta(u(s+h)-u(s)))\\ &(u(s+2h)-2u(s+h)+u(s))\,d\theta\Bigg|,\end{aligned}$$

for $0 \le \theta \le 1$.

On applying condition (1.1) we get

$$\left|F'_u(y,s,u(s)+\theta(u(s+h)-u(s)))\right| \le c(2)\,M + K_1$$

where

$$K_{m-1} = \max_{y,s\in[0,2\pi]} |F_u^{m-1}(y,s,0)|.$$

Then

$$W_F^2(\delta) \le c(2)\left[W_u^2(\delta) + \delta\,W(\delta) + \delta\,W^*(\delta)\right].$$

By induction we see that

$$\begin{aligned}\Delta_h^m F(y,s,u(s)) = \sum_{\nu=0}^{m-1} c_\nu^{m-1} \int_0^1 \Delta_h^\nu F'_u(y+h,s+h,u(s) \\ + \theta(u(s+h)-u(s)))\Delta_h^{m-\nu}u(s+\nu\,h)\,d\theta \\ + h\int_0^1 \Delta_h^{m-1}F'_y(y+\theta\,h,s,u(s))\,d\theta \\ + h\int_0^1 \Delta_h^{m-1}F'_s(y+h,s+\theta\,h,u(s))\,d\theta.\end{aligned}$$

The functions $F'_u(y+h,s+h,u(s)+\theta(u(s+h)-u(s)))$, $F'_y(y+\theta\,h,s,u(s))$ and $F'_s(y+h,s+\theta\,h,u(s))$ have partial derivatives to $(\nu-1)$ order, $(\nu = \overline{1,m-1})$ and these derivatives satisfy condition (1.1), then by the hypothesis we get

$$\left|\Delta_h^m F(y,s,u(s))\right| \le c(m)\left[W_u^m(h) + W_u^{m-2}(h)\,h\,W(h) + W_u^{m-2}(h)\,h\,W^*(h)\right].$$

That is

$$W_F^m(\delta) \le c(m)\left[W_u^m(\delta) + W_u^{m-2}(\delta)\,\delta\,W(\delta) + W_u^{m-2}(\delta)\,\delta\,W^*(\delta)\right].$$

Remark. We say that $(W, W^*, \phi) \in BH\Phi^m$ if

$$\delta^{m-1}\,W(\delta)\int_\delta^\pi \frac{\phi(\xi)}{\xi^{m-1}}\,d\xi = O(\phi(\delta))$$

and

$$\delta^{m-1}\,W^*(\delta)\int_\delta^\pi \frac{\phi(\xi)}{\xi^{m-1}}\,d\xi = O(\phi(\delta))$$

where $\phi \in H\Phi^m$.

Lemma 1.4. *If the condition* (1.1) *is satisfied and* $(W, W^*, \phi) \in BH\Phi^m$ *and if* $u(y) \in H_{\phi,m}$ *then* $F(y, s, u(s)) \in H_{\phi,m}$.

Proof. This follows from the above remark, Lemma (1.3) and Marscho's theorem [8].

Theorem 1.5. *Let the function* $F(y, s, u(s))$ *satisfy the condition (1.1) and for* $(W, W^*, \phi) \in BH\Phi^m$, *then for*

$$|\lambda| < |\lambda_0| \qquad (\lambda_0 \text{ arbitrary small}),$$

the equation

$$U(y) = \frac{\lambda}{2\pi} \int_0^{2\pi} F(y, s, u(s)) \cot \frac{s-y}{2} \, ds \tag{1.2}$$

has a unique solution in $H_{\phi,m}(M)$. *The solution is uniformly convergent and can be evaluated by the method of successive approximations.*

Proof. Let $u \in H_{\phi,m}(M)$, then by Lemma (1.3), (1.4) and Theorem (1.2) the operator

$$(Au)(y) = \frac{\lambda}{2\pi} \int_0^{2\pi} F(y, s, u(s)) \cot \frac{s-y}{2} \, ds$$

transforms $H_{\phi,m}(M)$ into $H_{\phi,m}(|\lambda| R')$.

Therefore $|\lambda| R' \leq M$, hence the operator transforms $H_{\phi,m}(M)$ into itself.

On using M. Riesz' theorem [3] we have

$$\|\tilde{u}\|_{L_p} \leq c(p) \|u\|_{L_p}, \qquad 1 < p < \infty$$

where

$$\tilde{u}(y) = \frac{1}{2\pi} \int_0^{2\pi} u(s) \cot \frac{s-y}{2} \, ds.$$

Now,

$$\begin{aligned}
&\|Au_1 - Au_2\|_{L_p} \\
&= \left[\int_0^{2\pi} \left| \frac{\lambda}{2\pi} \int_o^{2\pi} [F(y, s, u_1(s) - F(y, s, u_2(s))] \cot \frac{s-y}{2} \, ds \right|^p dy \right]^{1/p} \\
&\leq |\lambda| \, c(p) \, \|F(y, s, u_1(s)) - F(y, s, u_2(s))\|_{L_p} \\
&= |\lambda| \, c(p) \left[\int_0^{2\pi} |F(y, s, u_1(s)) - F(y, s, u_2(s))|^p \, ds \right]^{1/p} \\
&\leq |\lambda| \, c(p) \, c(0) \, \|u_1 - u_2\|_{L_p}.
\end{aligned}$$

If $|\lambda|\, c(p)\, c(0) < 1$, then the operator A is a contraction mapping. Thus we have

$$|\lambda| < \lambda_0 = \min\left\{\frac{M}{R'}, \frac{1}{c(p)\, c(0)}\right\}.$$

From the completeness of $H_{\phi,m}(M)$ in L_p, $1 < p < \infty$, the equation (1.2) has a unique solution in $H_{\phi,m}(M)$ and it can be found by the method of successive approximations.

2. The solution in the space $H_{\phi,m}^{(N)}$

For the integral

$$(Ju)(s) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(y) \cot\frac{y-s}{2}\, dy, \qquad u(y) \in H_{\phi,m} \tag{2.1}$$

the quadrature formula takes the following form, [4],

$$(Ju)(s) = \frac{1}{N}\sum_{k=0}^{2N-1} U_k \sin^2\frac{s-s_k}{2} \cot\frac{s_k - s}{2}, \tag{2.2}$$

where

$$U_k = U(s_k), \qquad s_k = \frac{k\,\pi}{N}.$$

Formula (2.2) at node points s_k takes the form

$$(Ju)(s_j) = \frac{1}{2N}\sum_{\substack{k=0\\k\neq j}}^{2N-1} U_k\left(1-(-1)^{k-j}\right) \cot\frac{s_k - s_j}{2}. \tag{2.3}$$

Applying the quadrature formula (2.3) to (1.2) at node points we get

$$U(y_l) = \frac{\lambda}{2n}\sum_{\substack{k=0\\k\neq l}}^{2n-1} F(y_l, y_k, u(y_k))\left(1-(-1)^{k-l}\right) \cot\frac{y_k - y_l}{2} + R_N(F, y_l),$$

where $R_N(F, y_l)$ is the remainder term, $l = \overline{0, 2N-1}$.

If we put $u(y_l) = z_l$ we obtain the following system of nonlinear algebraic equations

$$z_l = \frac{\lambda}{2N}\sum_{\substack{k=0\\k\neq l}}^{2N-1} F(y_l, y_k, z_k)\left(1-(-1)^{k-l}\right) \cot\frac{y_k - y_l}{2} \tag{2.4}$$

Lemma 2.1. *If the function $F(y,s,u)$ and its derivative $F'_y(y,s,u)$ satisfy the conditions* (0.2) *and* (0.3), *then the function*

$$\psi(y,s,u) = F(y,s,u) - F(s,s,u)$$

satisfies the following condition

$$|\psi(y,s,u) - \psi(y,s,v)| \le A_2\,|y-s|\,|u-v| \tag{2.5}$$

where $u, v \in [-M, M]$ and A_2 is a constant.

Proof. See [5].

Theorem 2.2. *Let the function $F(y,s,u)$ satisfy the conditions (0.2) and (0.3), then the system of nonlinear algebraic equations (2.4) for arbitrary $N \ge 3$ has a unique solution in $H^{(N)}_{\phi,m}(M)$ and this solution can be found by the method of successive approximations.*

Proof. Let

$$H^{(N)}_{\phi,m}(M) = \left\{ z \in H^{(N)}_{\phi,m} : \|z\|^{(N)}_{\phi,m} \le M \right\}.$$

$$z = (z_0, z_1, \dots z_{2N-1}) \in H^{(N)}_{\phi,m}(M),$$

$$Gz = \left(F(y_0, s_0, z_0), \dots, F(y_{2N-1}, s_{2N-1}, z_{2N-1})\right),$$

and since the space $H^{(N)}_{\phi,m}(M)$ of vectors of bounded norms is a closed subspace of $L^{(N)}_p$ and the function $F(y,s,z)$ satisfies the conditions of Theorem (1.5) and Lemma (1.3), (1.4), then $Gz \in H^{(N)}_{\phi,m}(R)$.

Taking

$$B^{(N)}z = (B^{(N)}_0 z, \dots, B^{(N)}_{2N-1} z),$$

where

$$B^{(N)}_l z = \frac{\lambda}{2N} \sum_{\substack{k=0 \\ k\ne l}}^{2N-1} F(y_l, y_k, z_k)\left(1 - (-1)^{k-l}\right) \cot\frac{y_k - y_l}{2}.$$

In other words

$$B^{(N)}z = \lambda A^{(N)} Gz,$$

where

$$A^{(N)}z = (A^{(N)}_0 z, \dots, A^{(N)}_{2N-1} z)$$

and

$$\|A^{(N)}\|_{H^{(N)}_{\phi,m}} \le c(m) \qquad \text{([8], theorem 3).}$$

We get

$$\left\|B^{(N)}z\right\|_{H^{(N)}_{\phi,m}} \le |\lambda|\, R\, c(m).$$

Now, let

$$z^{(i)} = (z_0^{(i)}, z_1^{(i)}, \ldots, z_{2N-1}^{(i)}) \in H^{(N)}_{\phi,m}(M), \qquad (i = 1, 2).$$

We get

$$\begin{aligned}
\left\|B^{(N)}z^{(1)} - B^{(N)}z^{(2)}\right\|_{L_p^{(N)}} &= \left\{ \frac{\pi}{N} \sum_{l=0}^{2N-1} \left| \frac{\lambda}{2N} \sum_{\substack{k=0\\k\neq l}}^{2N-1} \left[F(y_l, y_k, z_k^{(1)}) \right.\right.\right. \\
&\quad \left.\left.\left. - F(y_l, y_k, z_k^{(2)})\right] \left(1 - (-1)^{k-l}\right) \cot \frac{y_k - y_l}{2} \right|^p \right\}^{1/p} \\
&\le |\lambda| \left\{ \frac{\pi}{N} \sum_{l=0}^{2N-1} \frac{1}{2N} \left| \sum_{\substack{k=0\\k\neq l}}^{2N-1} \left[F(y_k, y_k, z_k^{(1)}) \right.\right.\right. \\
&\quad \left.\left.\left. - F(y_k, y_k, z_k^{(2)})\right] \left(1 - (-1)^{k-l}\right) \cot \frac{y_k - y_l}{2} \right|^p \right\}^{1/p} \\
&\quad + |\lambda| \left\{ \frac{\pi}{N} \sum_{l=0}^{2N-1} \left| \frac{1}{2N} \sum_{\substack{k=0\\k\neq l}}^{2N-1} \left[F(y_l, y_k, z_k^{(1)}) \right.\right.\right. \\
&\quad - F(y_k, y_k, z_k^{(1)}) - F(y_l, y_k, z_k^{(2)}) \\
&\quad \left.\left.\left. + F(y_k, y_k, z_k^{(2)})\right] \left(1 - (-1)^{k-l}\right) \cot \frac{y_k - y_l}{2} \right|^p \right\}^{1/p} \\
&= Q_1 + Q_2
\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}
Q_1 = |\lambda| &\left\{ \frac{\pi}{N} \sum_{l=0}^{2N-1} \frac{1}{2N} \left| \sum_{\substack{k=0\\k\neq l}}^{2N-1} \left[F(y_k, y_k, z_k^{(1)}) \right.\right.\right. \\
&\quad \left.\left.\left. - F(y_k, y_k, z_k^{(2)})\right] \left(1 - (-1)^{k-l}\right) \cot \frac{y_k - y_l}{2} \right|^p \right\}^{1/p},
\end{aligned}$$

and

$$Q_2 = |\lambda| \left\{ \frac{\pi}{N} \sum_{l=0}^{2N-1} \frac{1}{2N} \left| \sum_{\substack{k=0 \\ k \neq l}}^{2N-1} \left[F(y_l, y_k, z_k^{(1)}) \right. \right. \right.$$

$$- F(y_k, y_k, z_k^{(1)}) - F(y_l, y_k, z_k^{(2)})$$

$$\left. \left. \left. + F(y_k, y_k, z_k^{(2)}) \right] \left(1 - (-1)^{k-l}\right) \cot \frac{y_k - y_l}{2} \right|^p \right\}^{1/p}.$$

Since the operator $B^{(N)}$ is bounded on $L_p^{(N)}$, [7], and by using the condition (0.2) we get

$$Q_1 \le |\lambda| \left\{ \frac{\pi}{N} \sum_{k=0}^{2N-1} \left| F(y_k, y_k, z_k^{(1)}) - F(y_k, y_k, z_k^{(2)}) \right|^p \right\}^{1/p}$$

$$\le |\lambda| A_1 \left\| z^{(1)} - z^{(2)} \right\|_{L_p^{(N)}} \tag{2.7}$$

Also,

$$Q_2 \le |\lambda| \left\{ \frac{\pi}{N} \sum_{l=0}^{2N-1} \left[\frac{1}{N} \sum_{k=0}^{2N-1} \left| F(y_l, y_k, z_k^{(1)}) \right. \right. \right.$$

$$- F(y_k, y_k, z_k^{(1)}) - F(y_l, y_k, z_k^{(2)})$$

$$\left. \left. \left. + F(y_k, y_k, z_k^{(2)}) \right| \left| \cot \frac{y_k - y_l}{2} \right| \right]^p \right\}^{1/p}.$$

On using Lemma (2.1) we have

$$Q_2 \le A_2 |\lambda| \left\{ \frac{\pi}{N} \sum_{l=0}^{2N-1} \left[\frac{1}{N} \sum_{\substack{k=0 \\ k \neq l}}^{2N-1} \eta_{l,k} \right]^p \right\}^{1/p}$$

$$\le A_2 |\lambda| \left\{ \frac{\pi}{N} \sum_{l=0}^{[N/2]} (a_l + b_l)^p + \frac{\pi}{N} \sum_{l=[N/2]+1}^{2N-1} (c_l + d_l)^p \right\}^{1/p}, \tag{2.8}$$

where

$$\eta_{l,k} = \left| z_k^{(1)} - z_k^{(2)} \right| \left| \frac{y_l - y_k}{\sin((y_l - y_k)/2)} \right|,$$

$$a_l = \frac{1}{N} \sum_{k=0}^{[3N/2]} \eta_{l,k},$$

$$b_1 = \frac{1}{N} \sum_{k=[3N/2]+1}^{2N-1} \eta_{l,k},$$

$$c_1 = \frac{1}{N} \sum_{k=0}^{[N/4]} \eta_{l,k}$$

and

$$d_1 = \frac{1}{N} \sum_{k=[N/4]+1}^{2N-1} \eta_{l,k}.$$

Now, to determine a_1, we apply Hölder's inequality and we deduce that

$$a_1 \le \frac{2}{\pi^{1/p}} \left(\frac{\pi}{N} \sum_{k=0}^{[3N/2]} \left| z_k^{(1)} - z_k^{(2)} \right|^p \right)^{1/p} \left[\frac{1}{N} \left(\sum_{k=0}^{l-1} + \sum_{k=l+1}^{[3N/2]} \right) \left(\frac{(y_l - y_k)/2}{\sin((y_l - y_k)/2))} \right)^q \right]^{1/q}.$$

Putting $l - k = \lambda$ in the first $\sum$ and $k - l = \lambda$ in the second $\sum$ we get

$$\begin{aligned} a_1 &\le \frac{2}{\pi} \left\| z^{(1)} - z^{(2)} \right\|_{L_p^{(N)}} \left[\frac{\pi}{N} \left(\sum_{\lambda=1}^{l} + \sum_{\lambda=1}^{[3N/2]-l} \right) \left(\frac{t_\lambda/2}{\sin(t_\lambda/2)} \right)^q \right]^{1/q} \\ &\le \frac{2}{\pi} \left\| z^{(1)} - z^{(2)} \right\|_{L_p^{(N)}} (\Omega_1 + \Omega_2)^{1/q}, \end{aligned} \tag{2.9}$$

where

$$\Omega_1 = \frac{\pi}{N} \sum_{\lambda=1}^{l} \left(\frac{t_\lambda/2}{\sin(t_\lambda/2)} \right)^q$$

and

$$\Omega_2 = \frac{\pi}{N} \sum_{\lambda=1}^{[3N/2]-l} \left(\frac{t_\lambda/2}{\sin(t_\lambda/2)} \right)^q.$$

For $N \ge 3$ we have

$$t_{l+1} \le \frac{\pi}{N} + \frac{\pi}{2} = \frac{5\pi}{6} \qquad \text{at } 0 \le l \le \left[\frac{N}{2} \right],$$

then

$$\Omega_1 = \frac{\pi}{N}\sum_{\lambda=1}^{l}\left(\frac{t_\lambda/2}{\sin(t_\lambda/2)}\right)^q$$
$$\leq \sum_{\lambda=1}^{l}\int_{t_\lambda}^{t_\lambda+1}\left(\frac{x/2}{\sin(x/2)}\right)^q dx$$
$$\leq \int_0^{5\pi/6}\left(\frac{x/2}{\sin(x/2)}\right)^q dx, \tag{2.10}$$

$$\Omega_2 = \frac{\pi}{N}\sum_{\lambda=1}^{[3N/2]-l}\left(\frac{t_\lambda/2}{\sin(t_\lambda/2)}\right)^q$$
$$\leq \sum_{\lambda=1}^{[3N/2]-l}\int_{t_\lambda}^{t_\lambda+1}\left(\frac{x/2}{\sin(x/2)}\right)^q dx$$
$$\leq \int_0^{11\pi/6}\left(\frac{x/2}{\sin(x/2)}\right)^q dx. \tag{2.11}$$

From (2.10) and (2.11) in (2.9) we have

$$a_1 \leq \frac{2^{1+1/q}}{\pi}\left\|z^{(1)} - z^{(2)}\right\|_{L_p^{(N)}}\left[\int_0^{11\pi/6}\left(\frac{x/2}{\sin(x/2)}\right)^q dx\right]^{1/q}, \tag{2.12}$$

for $0 \leq l \leq [N/2]$.

Similarly we can show that

$$b_2 \leq \frac{2}{\pi}\left\|z^{(1)} - z^{(2)}\right\|_{L_p^{(N)}}\left[\int_0^{\pi}\left(\frac{x/2}{\sin(x/2)}\right)^q dx\right]^{1/q}, \tag{2.13}$$

for $0 \leq l \leq [N/2]$,

$$c_1 \leq \frac{2}{\pi}\left\|z^{(1)} - z^{(2)}\right\|_{L_p^{(N)}}\left[\int_0^{7\pi/4}\left(\frac{x/2}{\sin(x/2)}\right)^q dx\right]^{1/q}, \tag{2.14}$$

for $[N/2] + 1 \leq l \leq 2N - 1$, and

$$d_1 \leq \frac{2^{1+1/q}}{\pi}\left\|z^{(1)} - z^{(2)}\right\|_{L_p^{(N)}}\left[\int_0^{7\pi/4}\left(\frac{x/2}{\sin(x/2)}\right)^q dx\right]^{1/q}, \tag{2.15}$$

for $[N/2]+1 \le l \le 2N-1$.

Using (2.12)-(2.15) into (2.8) we have

$$Q_2 \le \frac{9\,|\lambda|\,A_2}{\pi^{1/q}} \left[\int_0^{11\pi/6} \left(\frac{x/2}{\sin(x/2)} \right)^q dx \right]^{1/q} \left\| z^{(1)} - z^{(2)} \right\|_{L_p^{(N)}}. \tag{2.16}$$

Substituting (2.7) and (2.16) in (2.6) we get

$$\left\| B^{(N)} z^{(1)} - B^{(N)} z^{(2)} \right\|_{L_p^{(N)}} \le |\lambda| \left[A_1 + \frac{9\,A_2}{\pi^{1/q}} \left(\int_0^{11\pi/6} \left(\frac{x/2}{\sin(x/2)} \right)^q dx \right)^{1/q} \right] \left\| z^{(1)} - z^{(2)} \right\|_{L_p^{(N)}}. \tag{2.17}$$

From boundedness of $B^{(N)}$ in $L_p^{(N)}$ and using the contraction mapping principle at

$$|\lambda| < \min \left\{ \frac{M}{Rc(m)}, \left(A_1 + \frac{9\,A_2}{\pi^{1/q}} \left(\int_0^{11\pi/6} \left(\frac{x/2}{\sin(x/2)} \right)^q dx \right)^{1/q} \right)^{-1} \right\},$$

the system (2.4) for arbitrary $N \ge 3$ has a unique solution in $H_{\phi,m}^{(N)}(M)$ and the theorem is proved.

3. The rate of convergence of the approximate solution

Let

$$|\lambda| < \min \left\{ \frac{M}{Rc(m)}, \left(A_1 + \frac{9\,A_2}{\pi^{1/q}} \left(\int_0^{11\pi/6} \left(\frac{x/2}{\sin(x/2)} \right)^q dx \right)^{1/q} \right)^{-1} \right\}, \tag{3.1}$$

then the equation (1.2) has a unique solution $u^*(y) \in H_{\phi,m}(M)$ and the system (2.4) at arbitrary $N \ge 3$ has a unique solution

$$z^* = (z_0^*, z_1^*, \ldots, z_{2N-1}^*) \in H_{\phi,m}^{(N)}(M).$$

The function

$$u_N^*(y) = \frac{\lambda}{2N} \sum_{k=0}^{2N-1} F(y, y_k, z_k^*) \sin^2 \frac{y - y_k}{2} \cot \frac{y_k - y}{2} \tag{3.2}$$

at $y = y_l$, (k and l do not take the same value simultaneously) is called the approximate solution of (1.2).

It is easy to deduce that $u_N^*(y) = z_l^*$ at $y = y_l$, ($l = \overline{0, 2N-1}$). The norm of the difference of the vectors z^* and

$$u^* = (u(y_0), u(y_1), \dots, u(y_{2N-1}))$$

in $L_p^{(N)}$ may be found as follows.

Applying the quadrature formula (2.2) to (1.2) at node points y_l, we get

$$u^*(y_l) = \frac{\lambda}{2\pi} \sum_{\substack{k=0 \\ k \neq l}}^{2N-1} F(y_l, y_k, u^*(y_k)) \left(1 - (-1)^{k-l}\right) \cot \frac{y_k - y_l}{2} + R_N(F, y_l) \tag{3.3}$$

where

$$F = F(y, s, u(s)).$$

Put $z^{(1)} = u^*$ and $z^{(2)} = z^*$ in (2.17) and using (3.1), we get

$$\|u^* - z^*\|_{L_p^{(N)}} \leq |\lambda| \, \|R_N(F, y)\|_c \left\{ 1 - |\lambda| \left[A_1 + \frac{9 A_2}{\pi^{1/q}} \left(\int_0^{11\pi/6} \left(\frac{x/2}{\sin(x/2)} \right)^q dx \right)^{1/q} \right] \right\}^{-1}. \tag{3.4}$$

Now, we evaluate $\|u^*(y) - u_N^*(y)\|_c$; from (3.2) and by applying (2.2) on the equation (1.2) we get

$$u^*(y) - u_N^*(y) = \frac{\lambda}{N} \sum_{k=0}^{2N-1} \left[F(y, y_k, u^*(y_k)) - F(y, y_k, z_k^*) \right] \sin^2 \frac{y - y_k}{2} \cot \frac{y_k - y}{2} + |\lambda| \, R_N(F, y),$$

from Lemma (5.2) in [1] and condition (0.2) we deduce the following

$$\begin{aligned} \|u^*(y) - u_N^*(y)\|_c &\leq A_1 \, |\lambda| \max_l |u^*(y_l) - z_l^*| \left(\frac{1}{N} \sum_{k=0}^{2N-1} \left| \sin^2 \frac{y - y_k}{2} \cot \frac{y_k - y}{2} \right| \right) + |\lambda| \, \|R_N(F, y)\|_c \\ &\leq 2 \, |\lambda| \, A_1 \, (1 + \pi) \, (1 + \ln 2N) \max_l |u^*(y_l) - z_l^*| \\ &\quad + |\lambda| \, \|R_N(F, y)\|_c. \end{aligned} \tag{3.5}$$

From Theorems (1.5) and (2.2) and since $u^*(y) \in H_{\phi,m}(M)$ then

$$u^* = (u(y_0), u(y_1), \ldots, u(y_{2N-1})) \in H_{\phi,m}^{(N)}(M)$$

and

$$u^* - z^* \in H_{\phi,m}^{(N)}(2\,M).$$

From the inequalities (3.4), and

$$\max_l |u^*(y_l) - z_l^*| \le c(m, M) \left[\left(\frac{N}{h}\right)^{1/p} \|u^* - z^*\|_{L_p^{(N)}} + \phi\left(\frac{\pi h}{N}\right)\right], \tag{3.6}$$

$$\|R_N(F,y)\|_c \le c(m)\,\phi\left(\frac{\pi}{N}\right) \ln N \qquad ([6],[7]), \tag{3.7}$$

we get

$$\max_l |u^*(y_l) - z_l^*| \le c(m, M) \min_{2 \le h \le (N/2(m+1))} \left[\left(\frac{N}{h}\right)^{1/p} \phi\left(\frac{\pi}{N}\right) \ln N + \phi\left(\frac{\pi h}{N}\right)\right].$$

Taking $h = N^\alpha$, $0 < \alpha < 1$, then

$$\max_l |u^*(y_l) - z_l^*| \le \text{const} \left[\frac{\ln N}{N^{(\alpha-1)/p}} \phi\left(\frac{\pi}{N}\right) + \phi\left(\frac{\pi}{N^{1-\alpha}}\right)\right]. \tag{3.8}$$

Consequently from (3.5), (3.7) and (3.8) we can conclude that

$$\begin{aligned} \|u^* - u_N^*\|_c &\le 2\,|\lambda|\,A_1\,(1+\pi)\,(1+\ln 2N) \left[\frac{\ln N}{N^{(\alpha-1)/p}} \phi\left(\frac{\pi}{N}\right) + \phi\left(\frac{\pi}{N^{1-\alpha}}\right)\right] \\ &\quad + |\lambda|\,c(m)\,\phi\left(\frac{\pi}{N}\right) \ln N \\ &\le \text{const} \left(\frac{\ln^2 N}{N^{(\alpha-1)/p}}\right) \end{aligned}$$

for $1 < p < \infty$.

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