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ERGODIC THEOREMS FOR CERTAIN POWER BOUNDED OPERATORS

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ABSTRACT. We consider invertible power bounded operators T on an Orlicz space such that T or T^{-1} is positive or T separates supports. For a wide class of Orlicz spaces we prove individual ergodic theorems and dominated ergodic theorems, and study the ergodic Hilbert transforms.

1. Introduction

t $(X, \mathcal{M}, \mu)$ be a σ–finite measure space and $L_\phi \equiv L_\phi(X, \mathcal{M}, \mu)$ an Orlicz space sociated to an N-function ϕ. In this paper we consider *invertible linear operators* : $L_\phi \to L_\phi$ such that

.1) $$\int_X \phi\left(|T^k f|\right) d\mu \leq C \int_X \phi(|f|)\, d\mu \qquad (f \in L_\phi) \quad (k \in \mathbb{Z}),$$

th $C > 0$ independent of f and k, and such that either T or T^{-1} is *positive* or else *separates supports* (that is, T maps functions with disjoints supports to functions th disjoint supports). We prove that for a wide class of N-functions ϕ, *the almost erywhere convergence* and *the norm convergence* ot the *Césàro-averages*

.2) $$\frac{1}{n} \sum_{i=0}^{n-1} T^i f$$

.3) $$\frac{1}{2n+1} \sum_{i=-n}^{n} T^i f$$

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and of the sequences

$$\sum_{0<|i|\leq n} \frac{T^i f}{i} \tag{1.4}$$

hold for every $f \in L_\phi$.

The limit function $H_T f \in L_\phi$ of the sequence (1.4) is called the *ergodic Hilb transform of f with respect to T*. We shall prove that the operator $f \to H_T f$ *bounded* in L_ϕ; more precisely, there exists a constant $C > 0$ such that

$$\int_X \phi(|Hf|)\, d\mu \leq C \int_X \phi(|f|)\, d\mu$$

for every $f \in L_\phi$.

In order to obtain the convergence of the averages defined by (1.2) and (1 (*individual ergodic theorem*) we shall prove that the *ergodic maximal operator* Λ defined by

$$M_T f = \sup_{m,n\geq 0} \left| \frac{1}{m+n+1} \sum_{i=-m}^{n} T^i f \right| \tag{1.5}$$

is bounded in L_ϕ (*dominated ergodic theorem*), which is of interest by itself. Likewi for the existence of the ergodic Hilbert transform, we shall prove that the *ergo maximal Hilbert transform* H_T^* is also bounded, where

$$H_T^* f = \sup_{n\geq 1} \left| \sum_{0<|i|\leq n} \frac{T^i f}{i} \right|. \tag{1.6}$$

In the L_p–case, $1 < p < \infty$, and assuming that T and T^{-1} are positive, respective dominated ergodic theorem is proved by A. de la Torre in [11]. Likewi for L_p with $1 < p < \infty$, the boundedness of H_T^* is obtained by R. Sato in [9] and [1

Now, we shall present the basic definitions and results concerning N-functi and Orlicz spaces which shall be used in this paper.

An N-function is a continuous and convex function $\phi : [0,\infty) \to \mathbb{R}$ such t $\phi(s) > 0$, for $s > 0$, $s^{-1}\phi(s) \to 0$ for $s \to 0$ and $s^{-1}\phi(s) \to \infty$ for $s \to \infty$.

The function ϕ is an N-function if and only if it has the representation

$$\phi(s) = \int_0^s \varphi$$

re $\varphi : [0,\infty) \to \mathbf{R}$ is continuous from the right, non decreasing, such that $\varphi(s) > 0$ $s > 0$, $\varphi(0) = 0$ and $\varphi(s) \to \infty$ for $s \to \infty$. More precisely φ is the right derivative and will be called the *density function of* ϕ.

Associated to φ we have the function $\rho : [0,\infty) \to \mathbf{R}$ defined by

$$\rho(t) = \sup\{s : \varphi(s) \leq t\}$$

ch has the same aforementioned properties of φ. The N-function ψ defined by

$$\psi(t) = \int_0^t \rho$$

alled the *complementary N-function of* ϕ. Thus, if $\phi(s) = p^{-1} s^p$, $p > 1$, then $) = q^{-1} t^q$ where $p\,q = p + q$.

An N-function ϕ is said to satisfy the Δ_2*–condition in* $[0,\infty)$ (or merely the -condition) if

$$\sup_{s>0} \frac{\phi(2s)}{\phi(s)} < \infty.$$

If φ is the density function of ϕ, then ϕ satisfies Δ_2 if and only if there exists onstant $\alpha > 1$ such that $s\,\varphi(s) < \alpha\,\phi(s)$, $s > 0$. The complementary N-function ψ satisfies the Δ_2–condition if and only if there exists a constant $\beta > 1$ such t $\beta\,\phi(s) < s\,\varphi(s)$, $s > 0$. As examples of N-functions which, together with their nplementary N-functions, satisfy the Δ_2–condition we have

$$\phi_1(s) = s^p, \quad p > 1;$$

$$\phi_2(s) = s^p\,(1 + \log(1+s)), \quad p > 1;$$

$$\phi_3(s) = s^p \ \log^k(1+s), \quad p > 1 \text{ and } k > 0;$$

$$\phi_4(s) = \int_0^s \rho$$

ere $\rho : [0,\infty) \to [0,\infty)$ is defined by $\rho(0) = 0$, $\rho(t) = 2^{-n}$ if $t \in [2^{-n}, 2^{-n+1})$ and $) = 2^{n-1}$ if $t \in [2^{n-1}, 2^n)$, n a positive integer.

If $(X, \mathcal{M}, \mu)$ is a σ–finite measure space the *Orlicz spaces* $L_\phi \equiv L_\phi(X, \mathcal{M}, \mu)$ and $\equiv L^*_\phi(X, \mathcal{M}, \mu)$ are defined by

$$L_\phi = \left\{ f \in \mathfrak{M} : \int_X \phi(|f|)\, d\mu < \infty \right\}$$

$$L^*_\phi = \{ f \in \mathfrak{M} : f\,g \in L_1 \text{ for all } g \in L_\psi \},$$

where ψ is the complementary N-function of ϕ. We have $L_\phi \subset L_\phi^*$ and if ϕ satis
Δ_2 then $L_\phi = L_\phi^*$. We have that L_ϕ^* is a Banach space with the norms

$$\|f\|_\phi = \sup\left\{\int_X |f\, g|\, d\mu :\ g \in S_\psi\right\},$$

where

$$S_\psi = \left\{g \in L_\psi :\ \int_X \psi(|g|)\, d\mu \leq 1\right\},$$

and

$$\|f\|_{(\phi)} = \inf\left\{\lambda > 0 :\ \int_X \phi(\lambda^{-1}\, |f|\, d\mu \leq 1\right\}$$

which are called the *Orlicz norm* and the *Luxemburg norm* respectively. Both no
are equivalent, actually

$$\|f\|_{(\phi)} \leq \|f\|_\phi \leq 2\, \|f\|_{(\phi)}.$$

If $\phi(s) = s^p$ with $p > 1$ then $L_\phi^* = L_\phi = L_p$, $\|f\|_{(\phi)} = \|f\|_p$ and $\|g\|_\psi = |$
where $p\, q = p + q$.

The convergence $f_n \to f$ in $[L_\phi^*, \|\cdot\|_\phi]$ implies the mean convergence

$$\lim_{n\to\infty} \int_X (|f_n - f|)\, d\mu = 0$$

but, in general, mean convergence only implies norm convergence when ϕ satisfies

If ϕ and ψ satisfy Δ_2 then $[L_\phi, \|\cdot\|_{(\phi)}]$ is *reflexive*.

The proof of most of above-mentioned results can be found in [5] or in IV-1
[8].

We shall also use in this paper the following *interpolation theorem*:

Theorem 1.7. *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{F}, \nu)$ be two σ-finite measure spaces, ‹
N-function satisfying, together with its complementary N-function, the Δ_2-cond
and let $T : L_r + L_s \to \mathfrak{M}(Y)$ be a quasi-additive operator which is simultaneous
weak type (r, r) and (s, s) where $1 \leq r < q_\phi$, $p_\phi < s \leq \infty$ and q_ϕ, p_ϕ are given b*

$$q_\phi^{-1} = \lim_{s\to 0+} \frac{-\log h_\phi(s)}{\log s} = \inf_{0<s<1} \frac{-\log h_\phi(s)}{\log s}$$

$$p_\phi^{-1} = \lim_{s\to\infty} \frac{-\log h_\phi(s)}{\log s} = \sup_{s>1} \frac{-\log h_\phi(s)}{\log s},$$

where

$$h_\phi(s) = \sup_{t>0} \frac{\phi^{-1}(t)}{\phi^{-1}(s\, t)}.$$

ıen, T maps $L_\phi(\mu)$ into $L_\phi(\nu)$ and there exists a constant C such that

$$\int_Y \phi(|Tf|)\,d\nu \le C \int_X \phi(|f|)\,d\mu$$

· every $f \in L_\phi(\mu)$.

A direct proof of Theorem 1.7 can be found in [2].

In the following, we shall always assume that

.8) $(X, \mathcal{M}, \mu)$ is a σ–finite measure space and ϕ, together with its complementary function, satisfy the Δ_2–condition.

2. Results in the case either T or T^{-1} positive

heorem 2.1. (Dominated ergodic theorems) *Assume* (1.8) *and let* $T : L_\phi \to L_\phi$ *be* ı *invertible linear operator such that either* T *or* T^{-1} *is positive and*

.2) $$\int_X \phi(|T^k f|)\,d\mu \le C \int_X \phi(|f|)\,d\mu \qquad (f \in L_\phi) \quad (k \in \mathbb{Z}),$$

ith constant $C > 0$ *independent of* f *and* k. *Then, there exists a constant* A *such* *at*

.3) $$\int_X \phi(M_T f)\,d\mu \le A \int_X \phi(|f|)\,d\mu \qquad (f \in L_\phi)$$

.4) $$\int_X \phi(H_T^* f)\,d\mu \le A \int_X \phi(|f|)\,d\mu \qquad (f \in L_\phi),$$

here M_T *and* H_T^* *are defined by* (1.5) *and* (1.6) *respectively.*

roof. In this proof we follow the idea given by de la Torre in [11].

It is enough to prove the theorem in the case T^{-1} positive.

For every integer $L \ge 1$ we consider the truncated operator $M_{T,L}$ defined by

.5) $$M_{T,L} f = \max_{0 \le m,n \le L} \left| \frac{1}{m+n+1} \sum_{i=-m}^{n} T^i f \right| .$$

$k \ge 0$ and $0 \le m, n \le L$ we have $M_{T,L} f \le T^{-k} M_{T,L}(T^k f)$ and therefore it follows ɔm (2.2) that

:.6) $$\int_X \phi(M_{T,L} f)\,d\mu \le C \int_X \phi(M_{T,L}(T^k f))\,d\mu \qquad (k \ge 0)$$

and consequently for every integer $N \geq 1$ we get

$$\int_X \phi(M_{T,L}f)\, d\mu \leq C\, N^{-1} \sum_{k=1}^{N} \int_X \phi(M_{T,L}(T^k f))\, d\mu.$$

For a given function F defined in $\mathbb{Z}$ and $k \in \mathbb{Z}$ let

$$MF(k) = \sup_{m,n \geq 0} \left| \frac{1}{m+n+1} \sum_{i=-m}^{n} F(k+i) \right| ,$$

$$M_L F(k) = \max_{0 \leq m,n \leq L} \left| \frac{1}{m+n+1} \sum_{i=-m}^{n} F(k+i) \right| .$$

For a given $f \in L_\phi$ and $x \in X$ let F_x be defined by $F_x(k) = T^k f(x)$. Then, we ha

$$\int_X \phi(M_{T,L}f)\, d\mu \leq C\, N^{-1} \int_X \sum_{k=1}^{N} \phi(M_L F_x(k))\, d\mu(x). \tag{2.7}$$

The operator M is of weak type $(1,1)$ with respect to counting measure on since $\tau : \mathbb{Z} \to \mathbb{Z}$, $\tau(k) = k+1$ preserves the measure, and on the other hand i obvious that M is bounded on ℓ_∞. Then, it follows from the interpolation theor 1.7 that there exists a constant C' such that

$$\sum_{k \in \mathbb{Z}} \phi(MF(k)) \leq C' \sum_{k \in \mathbb{Z}} \phi(|F(k)|) \qquad \text{for } F \in \mathfrak{M}(\mathbb{Z}).$$

If $1 \leq k \leq N$ then

$$M_L F(k) = M_L(F\, \chi_{[-L+1, L+N]})(k) \leq M(F\, \chi_{[-L+1, L+N]})(k)$$

and therefore

$$\sum_{k=1}^{N} \phi(M_L F(k)) \leq C' \sum_{k=-L+1}^{L+N} \phi(|F(k)|), \qquad \text{for } F \in \mathfrak{M}(\mathbb{Z}). \tag{2.8}$$

It follows from (2.2), (2.7) and (2.8) that for every $L \geq 1$ and $f \in L_\phi$ we hav

$$\int_X \phi(M_{T,L}f)\, d\mu \leq C^2\, C'\, N^{-1}\, (2\,L + N) \int_X \phi(f)\, d\mu$$

and thus we obtain (2.3) with $A = C^2\, C'$.

The proof of (2.4) is an adaptation of the proof of (2.3). For this, for every $L \geq 1$ consider the truncated operator $H^*_{T,L}$ defined by

$$H^*_{T,L}f = \max_{1 \leq n \leq L} \left| \sum_{0<|i|\leq n} \frac{T^i f}{i} \right|$$

d, as in the case $M_{T,L}$, for every integer $N \geq 1$ we obtain

9) $$\int_X \phi(H^*_{T,L}f)\, d\mu \leq C\, N^{-1} \sum_{k=1}^{N} \int_X \phi(H^*_{T,L}(T^k f))\, d\mu.$$

Now, we consider the operators on $\mathfrak{M}(\mathbb{Z})$ defined by

$$H^* F(k) = \sup_{n \geq 1} \left| \sum_{0<|i|\leq n} \frac{F(k+i)}{i} \right|,$$

$$H^*_L F(k) = \max_{1 \leq n \leq L} \left| \sum_{0<|i|\leq n} \frac{F(k+i)}{i} \right|.$$

is well-known that H^* is bounded on every ℓ_p with $1 < p < \infty$ (see [3]). Then, it lows again from interpolation theorem (1.7) that there is a constant C'' such that

$$\sum_{k \in \mathbb{Z}} \phi(H^* F(k)) \leq C'' \sum_{k \in \mathbb{Z}} \phi(|F(k)|), \qquad \text{for } F \in \mathfrak{M}(\mathbb{Z})$$

d therefore (2.9) shows that

$$\begin{aligned} \int_X \phi(H^*_{T,L}f)\, d\mu &\leq C\, N^{-1} \int_X \sum_{k=1}^{N} \phi(H^*_L F_x(k))\, d\mu(x) \\ &\leq C\, C''\, N^{-1} \int_X \sum_{k=1}^{L+N} \phi(|F_x(k)|)\, d\mu(x) \\ &\leq C^2\, C''\, N^{-1}\, (L+N) \int_X \phi(|f|)\, d\mu, \end{aligned}$$

ere $F_x(k) = T^k f(x)$, so that we obtain (2.4) with $A = C^2\, C''$. Thus, the proof is nplete.

ıeorem 2.10. *In the conditions of theorem 2.1, for every $f \in L_\phi$ there exist $\bar{f}$, f^* d $H_T f$ in L_ϕ such that the following conditions are satisfied:*

a)

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{i=0}^{n-1}T^i f-\bar{f}\right\|_{(\phi)}=0,$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}T^i f(x)=\bar{f}(x)\qquad\text{a.e.}$$

b)

$$\lim_{n\to\infty}\left\|\frac{1}{2n+1}\sum_{i=-n}^{n}T^i f-f^*\right\|_{(\phi)}=0,$$

$$\lim_{n\to\infty}\frac{1}{2n+1}\sum_{i=0}^{n}T^i f(x)=f^*(x)\qquad\text{a.e.}$$

c)

$$\lim_{n\to\infty}\left\|\sum_{0<|i|\le n}\frac{T^i f}{i}-H_T f\right\|_{(\phi)}=0,$$

$$\lim_{n\to\infty}\sum_{0<|i|\le n}\frac{T^i f(x)}{i}=H_T f(x)\qquad\text{a.e.}$$

Moreover, there exists a constant A such that

$$\int_X\phi(|H_T f|)\,d\mu\le A\int_X\phi(|f|)\,d\mu\qquad\text{for } f\in L_\phi. \tag{2.11}$$

Proof. It follows from (2.2) that

$$\|T^k f\|_{(\phi)}\le\max(1,C)\,\|f\|_{(\phi)},\qquad\text{for } f\in L_\phi \text{ and } k\ge 0.$$

Since T is a power bounded linear operator, that is, the powers T^k, $k\ge 0$, uniformly bounded in $V=[L_\phi,\|\cdot\|_{(\phi)}]$, and V is a reflexive space, then, the Cé averages

$$R_n f=\frac{1}{n}\sum_{k=0}^{n-1}T^k f$$

converge in norm for all $f\in L_\phi$ (see Theorem 2.1.2 in [6]).

The norm convergence implies that L_ϕ is the closure of the direct sum of th of fixed points and the space $(I-T)L_\phi$ (see 2.1 in [6]). On the other hand, since function complementary of ϕ satisfies Δ_2, there exists a constant $\beta > 1$ such that $\phi(s) < s\,\varphi(s)$ for $s > 0$, where φ is the density function of ϕ, which implies that the ıction $s \to s^{-\beta}\,\phi(s)$ increases strictly for $s > 0$ and consequently $\phi(s\,t) \leq s^{\beta}\,\phi(t)$ $0 \leq s \leq 1$ and $t \geq 0$, so that

$$\int_X \sum_{n=1}^{\infty} \phi(|n^{-1}\,T^n g|)\,d\mu \leq \sum_{n=1}^{\infty} n^{-\beta} \int_X \phi(|T^n g|)\,d\mu \leq C\left(\sum_{n=1}^{\infty} n^{-\beta}\right) \int_X \phi(|g|)\,d\mu < \infty.$$

nce $n^{-1}T^n g(x) \to 0$ a.e. for $n \to \infty$ and thus $R_n f \to 0$ a.e. if $f = g - Tg$ since ıt $R_n f = n^{-1}(g - T^n g)$.

Therefore, we have that $\{R_n f\}$ converges almost everywhere for all f in a dense ɔset of L_ϕ. Then, the almost everywhere convergence for every $f \in L_\phi$ follows from 3) and the *Banach principle* since

$$\lim_{\lambda\to\infty} \sup_{\|f\|_{(\phi)}\leq 1} \mu\{x \in X : \; R_T^* f(x) > \lambda\} \leq \lim_{\lambda\to\infty} \frac{A}{\phi(\lambda)} = 0,$$

ıere R_T^* is defined by

$$R_T^* f = \sup_{n=1} \left| \frac{1}{n} \sum_{i=0}^{n-1} T^i f \right|.$$

Since T^{-1} satisfies the same hypothesis as T, then b) follows from a).

Now, for $f \in L_\phi$ let $\{S_n f\}$ be the sequence given by (1.4). It follows from (2.4) ıt for almost everywhere convergence it is enough to prove that $\{S_n f\}$ converges :. for all f in a dense subset of $[L_\phi, \|\cdot\|_{(\phi)}]$.

It is easy to verify that if $f = g - Tg$, with $g \in L_\phi$, then

$$S_n f = g + Tg - n^{-1}\,(T^{n+1}g + T^{-n}g) - \sum_{i=1}^{n-1} \frac{1}{i\,(i+1)}\,(T^{i+1}g + T^{-i}g).$$

We have

$$\int_X \sum_{n=1}^{\infty} \phi(|n^{-1}(T^{n+1}g + T^{-n}g)|)\,d\mu \leq \frac{\alpha}{2} \sum_{n=1}^{\infty} n^{-\beta} \int_X (\phi(|T^{n+1}g|) + \phi(|T^{-n}g|))\,d\mu \leq \alpha\,C\left(\sum_{n=1}^{\infty} n^{-\beta}\right) \int_X \phi(|g|)\,d\mu < \infty,$$

where α is a constant in the Δ_2–condition for ϕ, and hence

$$\lim_{n\to\infty} n^{-1}(T^{n+1}g + T^{-n}g)(x) = 0 \qquad \text{a.e.}$$

On the other hand, let $h_n \in L_\phi$ and $h \in \mathfrak{M}$ be defined by

$$h_n = \sum_{i=1}^{n-1} \frac{|T^{i+1}g + T^{-i}g|}{i(i+1)},$$

$$h(x) = \sum_{i=1}^{\infty} \frac{|T^{i+1}g(x) + T^{-i}g(x)|}{i(i+1)}.$$

For every positive integer n we have

$$\|h_n\|_\phi \le C' \|g\|_\phi \sum_{i\in\mathbb{N}} i^{-2} = b < \infty,$$

where $C' = 4\max(1, C)$, and therefore

$$\int_X |h\, v|\, d\mu = \lim_{n\to\infty} \int_X |h_n\, v|\, d\mu \le b$$

for every v such that

$$\int_X \psi(|v|)\, d\mu \le 1,$$

where ψ is the N-function complementary of ϕ. This proves that $h \in L_\phi$ and cc quently $\{h_n\}$ converges almost everywhere. Therefore we get the almost everyw convergence of $\{S_n f\}$ for every $f \in L_\phi$.

Now, let

$$H_T f(x) = \lim_{n\to\infty} S_n f(x);$$

it follows from (2.4) that $H_T f \in L_\phi$ and $\phi(|S_n f - H_T f|)$ is dominated by $\phi(2\, H_T^*$ L_1; therefore, the Lebesgue dominated convergence theorem shows that

$$\lim_{n\to\infty} \int_X \phi(|S_n f - H_T f|)\, d\mu = 0$$

and consequently

$$\lim_{n\to\infty} \|S_n f - H_T f\| = 0,$$

since ϕ satisfies Δ_2.

Lastly, (2.11) follows trivially from (2.4). Thus, the proof is complete.

In [4] Kan gives a non trivial example of an operator that satisfies the correspond- g hypothesis of Theorem 2.1 in the case $\phi(s) = s^p$, $1 < p < \infty$; exactly $T : \ell_p \to \ell_p$, $= U + t\,S$ where $U\{x_k\} = \{x_{k+1}\}$, t is a real with $0 < t < 1$ and $S\{x_k\} = \{y_k\}$ iere $y_k = 0$ if $k \neq -1$ and $y_k = x_1$ if $k = -1$ (ℓ_p is the L_p–space on the set of tegers with counting measure). The operator T is positive but T^{-1} is not. This ample is valid in the case ℓ_ϕ; exactly, if $x = \{x_k\}$ we have

$$\sum_{k\in\mathbb{Z}} \phi(|(T^n x)_k|) \le 2^{-1}\,\alpha\,(1+t) \sum_{k\in\mathbb{Z}} \phi(|x_k|),$$

$$\sum_{k\in\mathbb{Z}} \phi(|(T^{-n} x)_k|) \le \alpha^m \sum_{k\in\mathbb{Z}} \phi(|x_k|),$$

iere α is a constant in the Δ_2–condition of ϕ and m is such that $(1-t)^{-1} \le 2^m$.

3. Results in the case of a Lamperti operator

is well-known that *every positive linear operator on a normed space of functions* $\mathfrak{M}$ *that has a positive inverse separates supports.* On the other hand there exist erators T satisfying the hypothesis of the theorems of Section 2 such that T does t separate supports (in the infinite dimensional case); precisely, the operator T of e example above is in this situation. Hence, it is of interest to obtain the same sults for the case in which T separates supports.

A bounded linear operator separating supports is called a *Lamperti operator*.

Certain *isometries* on L_ϕ are Lamperti operators. For example, *if* $T : L_\phi \to L_\phi$ *a positive linear operator such that*

$$\int_X \phi(|Tf|)\,d\mu = \int_X \phi(|f|)\,d\mu \qquad \text{for } f \in L_\phi \tag{.1}$$

ien T is a Lamperti operator.

Indeed, condition (3.1) implies that $\|Tf\|_{(\phi)} = \|f\|_{(\phi)}$ for every $f \in L_\phi$ and thus is an isometry. On the other hand, given f and g in L_ϕ^+ with disjoint supports we ive

$$\int_X \phi(Tf + Tg)\,d\mu = \int_X (\phi(Tf) + \phi(Tg))\,d\mu$$

id therefore $\phi(Tf + Tg) = \phi(Tf) + \phi(Tg)$ since $\phi(s+t) \ge \phi(s) + \phi(t)$ for $s,t \ge 0$. his shows that the supports of Tf and Tg are disjoint since $\phi(s+t) = \phi(s) + \phi(t)$ and only if $s\,t = 0$. Thus T is a Lamperti operator.

The same result holds for linear operators satisfying (3.1) not necessarily posi but with ϕ such that $s \to \phi(\sqrt{s})$ is *strictly convex or concave* (this follows fr Theorem 2.1 in [7]).

In order to obtain the same results as in Section 2 for operators separating s ports, we need an structural theorem for Lamperti operators which permits to use methods used in the proof of Theorem 2.1. In Theorem 4.1 of [7], Lamperti proves structural theorem for certain isometries. In [4] Kan notices that Lamperti's re may be adapted to Lamperti operators on L_p, $1 \leq p \leq \infty$. We observe that the sa happens for Lamperti operators on L_ϕ carrying out a similar method; exactly we h

Proposition 3.2. *For every Lamperti operator T on $[L_\phi, \|\cdot\|]$ there exist a endomorphism τ of the measure algebra $(X, \mathcal{M}, \mu)$ and a $\mathcal{M}$-measurable functior with support τX, such that $Tf = h\,Pf$ for every $f \in L_\phi$, where P is the posi linear operator, on the space of $\mathcal{M}$-measurable functions, induced by τ (in the c $\mu(X) < \infty$, τ is defined by $\tau E = \operatorname{supp} T\chi_E$ and $h = T1$). Conversely, if $Tf = h$ $f \in L_\phi$, then T separates supports.*

The operator P is characterized by condition $P\chi_E = \chi_{\tau E}$, $E \in \mathcal{M}$. Ot properties of P, which will be used later, are $|Pf| = P|f|$ and $P(f\,g) = Pf\,Pg$, f g in $\mathfrak{M}$.

Theorem 3.3. *Assume* (1.8) *and let $T : L_\phi \to L_\phi$ be an invertible linear opera separating supports such that*

$$\int_X \phi(|T^k f|)\,d\mu \leq C \int_X \phi(|f|)\,d\mu,$$

with constant $C > 0$ independent of $f \in L_\phi$ and $k \in \mathbb{Z}$. Then, the results of Theore 2.1 *and* 2.10 *hold.*

Proof. It follows from Proposition 3.2 that $Tf = h\,Pf$ for every $f \in L_\phi$, wh h is a $\mathcal{M}$-measurable function and P is the positive linear operator induced b σ-endomorphism τ of $(X, \mathcal{M}, \mu)$ and thus, for every $k \geq 0$ and $f \in L_\phi$, we have t $T^k f = h_k\,P^k f$, where $h_k = h\,Ph\,P^2h\,\ldots\,P^{k-1}h$. For a given $L \geq 1$ let $M_{T,L}$ be operator defined by (2.5). We can decompose X into a finite family of disjoint s $\{E_{m,n}\}$ such that

$$M_{T,L} f = \sum_{0 \leq m,n \leq L} \chi_{E_{m,n}} |R_{m,n} f|,$$

$$R_{m,n} f = \frac{1}{m+n+1} \sum_{i=-m}^{n} T^i f.$$

Given $f \in L_\phi$ and $k \geq 0$ we have

$$\begin{aligned} P^k(M_{T,L}f) &= \sum_{0\leq m,n\leq L} \chi_{\tau^k(E_{m,n})}|P^k(R_{m,n}f)| \\ &\leq \max_{0\leq m,n\leq L} |P^k(R_{m,n}f)|, \end{aligned}$$

ıce the sets $\tau^k(E_{m,n})$ are disjoint. Therefore

$$\begin{aligned} |T^k(M_{T,L}f)| &= |h_k|\, P^k(M_{T,L}) \\ &\leq \max_{0\leq m,n\leq L} |h_k\, P^k(R_{m,n}f)| \\ &= \max_{0\leq m,n\leq L} |R_{m,n}(T^k f)| \\ &= M_{T,L}(T^k f) \end{aligned}$$

d thus we get

$$\begin{aligned} \int_X \phi(M_{T,L}f)\, d\mu &\leq C \int_X \phi(|T^k(M_{T,L}f)|)\, d\mu \\ &\leq C \int_X \phi(M_{T,L}(T^k f))\, d\mu. \end{aligned}$$

In this way, we obtain the same inequality (2.6) obtained in the proof of Theorem l and the argument used there can be used here to obtain (2.3). The same happens the proof of (2.4), once we get inequality (2.9), and in order to obtain (2.9) it is ough to prove that

$$|T^k(H^*_{T,L}f)| \leq H^*_{T,L}(T^k f) \qquad \text{for every } f \in L_\phi \text{ and } k \geq 0,$$

ıich is obtained carrying out the same ideas that in case $M_{T,L}$.

Once we have the boundedness of M_T and H^*_T, then, the almost everywhere nvergence and the norm convergence of the averages defined by (1.2), and of the quences defined by (1.4), are proved following the proof of Theorem 2.10. Taking o account that T^{-1} also separates supports, we have the convergence of the averages fined by (1.3). Last, the boundedness of the ergodic Hilbert transform follows vially from the boundedness of H^*_T.

A justification of the assumption that T^{-1} separates supports is the following. t h and P be the given in Proposition 3.2. We have that $h(x) \neq 0$ a.e. since composing X into an at most countable family of sets E_m with $\mu(E_m) < \infty$ and nsidering $A_m = \{x \in E_m : h(x) = 0\}$ we get that for every m there exists a function ı $\in L_\phi$ such that $\chi_{A_m} = h\, Pf_m$ and therefore $\mu(A_m) = 0$. Then, it follows from

Proposition 3.2 that for every f and g in L_ϕ we have that $f\,g = T^{-1}((Tf\,Tg)/h)$ a thus if Tf and Tg have disjoint supports the same holds for f and g, which pro that T^{-1} separates supports.

Final remark. The results obtained in this paper can be applied to certain *isometr* on L_ϕ. For example, if T is an invertible linear operator such that either T or T is positive and T satisfies (3.1), then the conclusions of Theorems 2.1 and 2.10 ho Likewise, it can be proved that if T satisfies (3.1) and T is positive (not necessar invertible), then, we have the boundedness (2.3) for the maximal ergodic opera R_T^* and thus we obtain the almost everywhere convergence of the averages defined (1.2). However, this does not yield anything new since the only Orlicz spaces, wh have non-trivial isometries, are the L_p-spaces (see [1]).

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