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MUTATIONS OF C*-ALGEBRAS AND QUASIASSOCIATIVE JB*-ALGEBRAS

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Introduction

ıe centroid of an algebra A is the largest ring over which A can be regarded as an ʒebra. In case A is a C*-algebra, the centroid of A also has a natural structure of C*-ʒebra and, for f in the centroid of A with $0 \leq f \leq 1$, the f–mutation of A (denoted $^{f)}$) with the same norm as A is a (complete) normed algebra in the classical sense at the norm is submultiplicative (see [4], section 2). To be more precise, the algebras $^{f)}$ as above are examples of noncommutative JB*-algebras (see [2] for definition) ıich are split quasiassociative over their centroids. In this note we prove that there e no other examples, thus answering by the desired negative a problem posed in . Our proof is strongly based in the main result in [4] and the Dauns-Hofmann eorem.

1. Mutations of C*-algebras

:t R be a (unital associative and commutative) ring and let A be an algebra over R 'e assume

$$r \in R,\ r\,A = 0 \Longrightarrow r = 0).$$

ɔr any fixed r in R, the r–mutation of A (denoted $A^{(r)}$) is defined as the new algebra th the same R–module as A and product

$$(a, b) \longrightarrow r\,a\,b + (1 - r)\,b\,a.$$

ıe centroid of A (denoted by $C(A)$) is defined as the set of those additive mappings from A into A satisfying

$$f(a\,b) = a\,f(b) = f(a)\,b \qquad \text{for } a, b \in A.$$

When A has zero annihilator, i.e.

$$a \in A, \; a\,A = A\,a = 0 \Longrightarrow a = 0,$$

it is well known that $C(A)$ is a ring so that A can be regarded as an algebra c $C(A)$, and R can be imbedded in $C(A)$ in an obvious way. If A is a complete norr algebra with zero annihilator, then $C(A)$ is a closed (commutative) subalgebra of Banach algebra $BL(A)$ of all continuous linear operators on A (use the closed gr theorem to verify the inclusion $C(A) \subset BL(A)$), and, if actually A is a C*-algel then $C(A)$ is a C*-algebra under the involution defined by

$$f^*(a) = (f(a^*))^* \qquad \text{for } f \in C(A), \; a \in A.$$

Usually, for a C*-algebra A, the centroid $C(A)$ appears in the literature unde concrete realization as closed self-adjoint subalgebra of the bidual of A, namely as center of the algebra of multipliers of A (see, for example, [3]).

In this section we deal with the following question. Given a C*-algebra A, de mine the elements f in the centroid of A for which the norm of A is submultiplica in the f-mutation of A. As asserted in the introduction, elements f with $0 \leq f$ are examples of such a situation.

Lemma. *Assume that the C*-algebra A is not commutative and let λ be in $\mathbb{C}$ s that*

$$||\lambda\, a\, b + (1-\lambda)\, b\, a|| \leq ||a||\,||b|| \qquad \text{for } a, b \in A.$$

Then $0 \leq \lambda \leq 1$.

Proof. By Kaplansky's theorem ([1], Appendix III, Theorem B), there exists a i with $||a|| = 1$ and $a^2 = 0$. Let α, β be in $\mathbb{C}$ such that

$$|\alpha| = |\beta| = 1, \qquad \alpha\,\lambda = |\lambda| \qquad \text{and} \qquad \beta\,(1-\lambda) = |1-\lambda|,$$

and write $b := \alpha\, a^*\, a + \beta\, a\, a^*$. Then $||b|| = 1$ and

$$\begin{aligned} 1 &= ||a||\,||b|| \\ &\geq ||\lambda\, a\, b + (1-\lambda)\, b\, a|| \\ &= (|\lambda| + |1-\lambda|)\,||a\, a^*\, a|| \\ &= |\lambda| + |1-\lambda|. \end{aligned}$$

Therefore $0 \leq \lambda \leq 1$, as required.

ıeorem 1. *Let g be in the centroid of a C^*-algebra A and assume that*

$$\|g\,a\,b + (1-g)\,b\,a\| \leq \|a\|\,\|b\| \qquad \text{for } a,b \in A.$$

ıen there exists f in the centroid of A with $0 \leq f \leq 1$ and

$$f\,a\,b + (1-f)\,b\,a = g\,a\,b + (1-g)\,b\,a \qquad \text{for } a,b \in A.$$

oof. Let Prim(A) denote the set of primitive ideals of A endowed with the Jacobson ›ology. By the Dauns-Hofmann theorem ([3], Corollary 4.4.8) for h in $C(A)$ and t Prim(A), there is a unique complex number $\hat{h}(t)$ such that

$$h\,a - \hat{h}(t)\,a \in t, \qquad \text{for } a \in A, \tag{1}$$

d the mapping $h \longrightarrow \hat{h}$ is a *–isomorphism from $C(A)$ onto the C*-algebra (Prim(A)) of all complex valued bounded continuous functions on Prim(A). Now, ːor t in Prim(A) we denote by π_t the canonical mapping $A \longrightarrow A/t$, by (1) we have

$$\begin{aligned}\|\hat{g}(t)\,\pi_t(a)\,\pi_t(b) + (1-\hat{g}(t))\pi_t(b)\,\pi_t(a)\| &= \|\pi_t(g\,a\,b + (1-g)\,b\,a)\| \\ &\leq \|g\,a\,b + (1-g)\,b\,a\| \\ &\leq \|a\|\,\|b\|\end{aligned}$$

· all $a,b \in A$. Changing a by $a' \in \pi_t(a)$ and b by $b' \in \pi_t(b)$ and taking infimum, we tain

$$\|\hat{g}(t)\,\pi_t(a)\,\pi_t(b) + (1-\hat{g}(t))\pi_t(b)\,\pi_t(a)\| \leq \|\pi_t(a)\|\,\|\pi_t(b)\|.$$

t is not 1-codimensional in A, then A/t is not commutative so, by the Lemma, $\leq \hat{g}(t) \leq 1$ in this case. Let f be in $C(A)$ such that

$$\hat{f}(t) = \max\{0, \min\{1, \operatorname{Re}\hat{g}(t)\}\} \qquad \text{for } t \in \text{Prim}(A).$$

ıen clearly $0 \leq f \leq 1$ and $\hat{f} - \hat{g}$ vanishes in the set of primitive ideals of A which ɜ not 1-codimensional so in particular, using (1), we have $(f-g)\,[A,A] \subset t$ for y primitive ideal t which is not 1-codimensional. But if the primitive ideal t is ːodimensional, then the inclusion $[A,A] \subset t$ is obvious so

$$(f-g)\,[A,A] \;(= [(f-g)\,A,A]) \;\subset t,$$

ιo. Therefore

$$(f-g)\,[a,b] = 0 \qquad \text{for } a,b \in A.$$

›w

$$\begin{aligned}f\,a\,b + (1-f)\,b\,a &= b\,a + f\,[a,b] \\ &= b\,a + g\,[a,b] \\ &= g\,a\,b + (1-g)\,b\,a.\end{aligned}$$

2. Quasiassociative JB*-algebras

An algebra B over a ring R is said to be a split quasiassociative algebra (over R there are an associative algebra A over R and an r in R such that $B = A^{(r)}$. S quasiassociative algebras are particular types of noncommutative Jordan algeb which are defined as those nonassociative algebras satisfying

$$a(ba^2) = (ab)a^2$$

$$a(ba) = (ab)a$$

for all a, b in the algebra. Recall ([2]) that a noncommutative JB*-algebra i complete normed noncommutative Jordan complex algebra (say B) with algebra volution * such that

$$\|U_b(b^*)\| = \|b\|^3$$

for all b in B, where U_b is defined by

$$U_b(c) = (bc + cb)b - cb^2$$

for all c in B. (Associative) C*-algebras and (commutative) JB*-algebras are p ticular types of noncommutative JB*-algebras. Also, for any noncommutative J algebra B, B^+ (the 1/2–mutation of B) is a JB*-algebra under the same norm involution as B.

For next reference we state the main result in [4] which is a particular non tri converse to the last assertion.

Proposition ([4], Theorem 2). *Let A be an associative complex algebra and ass that A^+ is a JB*-algebra for suitable norm and involution. Then A, with the s norm and involution, is a C*-algebra.*

Note that the given involution on A^+ is not assumed to be an algebra involut on A.

If A is a C*-algebra and f is in $C(A)$ with $0 \leq f \leq 1$, then $A^{(f)}$ (with the s norm as A) is a noncommutative JB*-algebra which is split quasiassociative over centroid (see [4]). Now we can prove the converse result.

Theorem 2. *Let B be a noncommutative JB*-algebra which is split quasiassocia over its centroid. Then there are a C*-algebra A and an f in $C(A)$ with $0 \leq f$ such that $B = A^{(f)}$.*

Proof. By the definition of split quasiassociative algebras there are an associa algebra A over $C(B)$ and g in $C(B)$ such that $B = A^{(g)}$. Since A^+ $(= B^+)$

-algebra, it follows from the above Proposition that A, with the same norm and olution as B, is a C-algebra. Clearly g belongs to $C(A)$. Now, if the product of s denoted by juxtaposition, then, since B is a normed algebra, we have

$$||g\,a\,b + (1-g)\,b\,a|| \leq ||a||\,||b||$$

all a, b in the C*-algebra A. By Theorem 1 there exists f in $C(A)$ such that $\leq f \leq 1$ and $(B =)\ A^{(g)} = A^{(f)}$.

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