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SOME CHARACTERIZATION THEOREMS FOR THE DISCRETE HOLOMETRIC SPACE

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ABSTRACT. This paper extends our earlier results on the properties of the discrete holometric space. Here, a characterization of domains is given and proved that domains are invariant under D-isometries. Also, we introduce the notion of D-kernel and some of its properties are studied.

1. The discrete holometric space

: consider the discrete subset of the complex plane $H = \{(q^m x_0, q^n y_0); m, n \in \mathbb{Z}\}$ ere $q \in (0,1)$ is fixed and (x_0, y_0) is a fixed point in the first quadrant, x_0, $y_0 \neq 0$. e points $z = (q^m x_0, q^n y_0)$ are called lattice points.

The discrete plane H was first considered by Harman [4] in 1972 to evolve the crete analogue of analytic function theory. Earlier, in [6,7] we have introduced and estigated the properties of D-linear sets, r-sets, etc. which are discrete analogues segments and circles. Some special type of discrete transformations also have been :died in [8].

We shall now consider the following fundamental concepts.

:finition 1. Let $z \in H$ and consider

$$N(z) = \{(q^{m+1} x_0, q^n y_0), (q^m x_0, q^{n+1} y_0), (q^{m-1} x_0, q^n y_0), (q^m x_0, q^{n-1} y_0)\}.$$

discrete curve joining any two points z_1 and z_t in H is a finite sequence of points H, $C = \langle z_1, z_2, \ldots, z_t \rangle$ where $z_{i+1} \in N(z_i)$, for $i = 1, 2, \ldots, t-1$. A discrete curve between two given points consisting of a minimum number of lattice points is cal a path joining them.

Definition 2. Let $z = (q^m x_0, q^n y_0) \in H$. Then

$$S(z) = \{(q^m x_0, q^n y_0), (q^{m+1} x_0, q^n y_0), (q^{m+1} x_0, q^{n+1} y_0), (q^m x_0, q^{n+1} y_0)\}$$

is called the basic set associated with z. A finite union of basic sets is called a regi R. If R can be expressed as a union of basic sets, $\cup_{i=1}^{t} B_i$ with $B_i \cap B_{i+1} \neq$ $i = 1, 2, \ldots, t-1$, then it is called a domain, denoted by D.

Definition 3. Consider two points $z_1 = (q^{m_1} x_0, q^{n_1} y_0)$ and $z_2 = (q^{m_2} x_0, q^{n_2} y_0) \in$ The distance d between z_1 and z_2 is defined as $d(z_1, z_2) = N - 1$, where N is number of lattice points of a path joining them. Equivalently,

$$d(z_1, z_2) = |m_1 - m_2| + |n_1 - n_2|.$$

This metric takes only integral values and $H = (H, d)$ is called the discrete holome space.

Definition 4. Let A be a finite subset of H. A is said to be D-linear if we can la the points of A as $A = \{z_1, z_2, \ldots, z_n\}$ such that

$$d(z_1, z_n) = \sum_{i=1}^{n-1} d(z_i, z_{i+1}).$$

Note 5. When we write the D-linear set $A = \{z_1, z_2, \ldots, z_n\}$ we mean t $z_1, z_2, \ldots, z_n$ are in that order in which

$$d(z_1, z_n) = \sum_{i=1}^{n-1} d(z_i, z_{i+1}).$$

2. A characterization of domains

In this section, we shall give an equivalent definition of domains, following [1] prove that they are invariant under D-isometries.

Definition 6. Consider two basic sets B_1 and B_2. Then,

$$\min\{d(z_1, z_2) : z_1 \in B_1, \ z_2 \in B_2\}$$

defined as the distance between B_1 and B_2, denoted by $d(B_1, B_2)$.

efinition 7. Two basic sets B_1 and B_2 are adjacent if there are two pairs of points $, z_1' \in B_1$, $z_2, z_2' \in B_2$ such that

$$d(z_1, z_2) = d(z_1', z_2') = d(B_1, B_2) = 1.$$

efinition 8. $z_1 = (q^{m_1} x_0, q^{n_1} y_0)$ and $z_2 = (q^{m_2} x_0, q^{n_2} y_0)$ are in the same horintal (vertical) set if $n_1 = n_2$ $(m_1 = m_2)$.

ote 9. If $B_1 = S(z_1)$ and $B_2 = S(z_2)$ are two adjacent basic sets, then z_1, z_2 belong the same horizontal or vertical set. Consequently, for a given basic set, there are ur basic sets adjacent to it.

heorem 10. *If $D = \bigcup_{i=1}^{t} B_i$ is such that B_i and B_{i+1} are adjacent for $i = 2, \ldots, t-1$, then D is a domain.*

roof. Let $D = \bigcup_{i=1}^{t} B_i$ such that B_i and B_{i+1} are adjacent for $i = 1, 2, \ldots, t-1$. hen B_{i+1} falls in one of the four cases mentioned in note 9. In any case, we can id a basic set (say) B, such that $B_i \cap B \neq \emptyset$ and $B_{i+1} \cap B \neq \emptyset$. Include B also in ır collection of basic sets and, proceeding like this, D can be expressed as a union basic sets $\{B_i'\}_{i=1}^{T}$ with $B_i' \cap B_{i+1}' \neq \emptyset$, where $T > t$. Hence D is a domain.

ote 11. In [1], Bajaj has defined connectedness in an integer-valued metric space follows. Let (A, d') be an integer-valued metric space. Then (A, d') is connected there do not exist non-empty disjoint sets A_1 and A_2, $A_1 \subset A$, $A_2 \subset A$, such that $_1 \cup A_2 = A$ and

$$\min\{d'(x, y) : x \in A_1,\ y \in A_2\} > 1.$$

a theorem, he further establishes that (A, d') is connected if and only if given any ir x, y of distinct points in A, there are points $x = x_1, x_2, \ldots, x_p = y$ in A, such at

$$d'(x_i, x_{i+1}) = 1 \qquad \text{for } i = 1, 2, \ldots, p-1.$$

heorem 12. *Consider a union of basic sets,*

$$R = \bigcup_{i=1}^{t} B_i.$$

hen, R is connected if and only if $\{B_i\}_{i=1}^{t}$ can be relabelled as $\{B_i'\}$ such that

$$d(B_i', B_{i+1}') \leq 1.$$

Proof. Let $R = \bigcup_{i=1}^{t} B_i$ be connected. That is, given any two points z and ξ of
there are points $z = z_1,\ z_2,\ \dots,\ z_n = \xi$ such that

$$d(z_i, z_{i+1}) = 1 \qquad \text{for } i = 1, 2, \dots, n-1.$$

Consider B_1 and choose all other basic sets B_i in $\{B_2, B_3, \dots, B_t\}$ such that

$$d(B_1, B_i) \leq 1.$$

By tracing back if necessary at each step to B_1, these basic sets together with B_1 be relabelled as $B_1',\ B_2', \dots,\ B_t'$ such that

$$d(B_i', B_{i+1}') \leq 1 \qquad \text{for } i = 1, 2, \dots, t-1.$$

If no such basic sets exist, then every other basic set B_s is such that $d(B_1, B_s) >$ Choose one such B_s. So by definition every pair of points $z_1 \in B_1$ and $z_2 \in B$ with $d(z_1, z_2) \geq 2$. For any such pair, we cannot find a sequence of points satisf the hypothesis and hence the supposition that there are no basic sets with the ab property leads us to a contradiction. Now, in the remaining basic sets of R, if tl is at least one basic set B_r' which is at distance ≤ 1, with at least one among $B_2', \dots,\ B_t'$ (say) B_p', then we can similarly relabel the collection of all such basic together with those already relabelled, by tracing back if necessary at each ste B_p' such that the distance is less than or equal to 1. Thus, proceeding likewise basic sets constituting R can be relabelled as $\{B_i'\}_{i=1}^{t}$ such that $d(B_i', B_{i+1}') \leq 1$.

The converse can be proved easily. Hence the theorem.

Note 13. In the labelling $\{B_i'\}$ mentioned, if further $B_i',\ B_{i+1}'$ are adjacent then R domain. Conversely if R is a domain, then it is connected and there is labelling B_i' s that $B_i',\ B_{i+1}'$ are adjacent for $i = 1, 2, \dots$. Thus we have a metric characterizatio domains.

Using the above results, we prove the following theorem establishing the pre vence of the property of being a domain under D-isometries [8].

Theorem 14. *If D is a domain and $T : H \to H$ is a D-isometry, then $T(D)$ is a domain.*

Proof. Let $D = \bigcup_{i=1}^{t} B_i$ be a domain. Then the basic sets B_i and B_{i+1} are adja and for any two points z, ξ of D , there are points $z = z_1,\ z_2,\ \dots,\ \xi = z_n$ in D s that $d(z_i, z_{i+1}) = 1,\ i = 1, 2, \dots, n-1$. Since T is a D-isometry, by definition, points $z_1, z_2, \dots, z_n$ with $d(z_i, z_{i+1}) = 1$ will be mapped onto points $w_1, w_2, \dots$ with $d(w_i, w_{i+1}) = 1$ and further the adjacency of basic sets will also be preser Hence $T(D)$ is also a domain.

3. D-kernel

ɔoley [2], German [3], Soltan [5] and many others have studied convexity and related ncepts in metric spaces. In [7], we have defined D-convexity for subsets of H and ıdied some of its properties. In this section, we introduce the notion of D-kernel r subsets of H and some characterization theorems are proved.

efinition 15. Let A be a nonempty finite subset of H. Then, A is D-convex if for ery $z_1, z_2 \in A$,

$$[z_1, z_2] = \{z \in H : B(z_1, z, z_2)\} \subset A.$$

Here, $B(z_1, z, z_2)$ means that z is holometrically between z_1 and z_2, and satisfies

$$d(z_1, z_2) = d(z_1, z) + d(z, z_2).$$

ɔte that the empty set is trivially D-convex and the definition could be extended to finite subsets of H also.

efinition 16. Let A be a subset of H. Then the set of all the $z_i \in A$ such that for ery $z_j \in A$, all the D-linear sets with z_i and z_j as end points are contained in A, is lled the D-kernel of A, denoted by $\mathrm{D}-\ker(A)$.

ĸamples 17. (1) Let

$$A_1 = \{z_0 = (x_0, y_0), z_1 = (q\,x_0, y_0), z_2 = (q\,x_0, q\,y_0), z_3 = (x_0, q\,y_0),$$
$$z_4 = (q^{-1}\,x_0, y_0), z_5 = (q^{-1}\,x_0, q^{-1}\,y_0), z_6 = (x_0, q^{-1}\,y_0)\}.$$

ıen,

$$\mathrm{D}-\ker(A_1) = \{z_0\}.$$

) Let

$$A_2 = A_1 \cup \{z_7 = (q\,x_0, q^{-1}\,y_0)\}.$$

hen,

$$\mathrm{D}-\ker(A_2) = \{z_0, z_1, z_6, z_7\}.$$

) Let

$$A_3 = A_2 \cup \{z_8 = (q^{-1}\,x_0, q\,y_0)\}.$$

hen,

$$\mathrm{D}-\ker(A_3) = A_3.$$

In the above examples, it turns out that $\mathrm{D}-\ker(A_2)$ is D-convex, A_3 is D-ınvex and its D-kernel is itself. This is true in general and is proved in the following .eorems.

Theorem 18. *For any nonempty subset A of H, $\mathrm{D}-\ker(A)$ is always D-convex*

Proof. Consider two points $z_1, z_2 \in \mathrm{D}-\ker(A)$. That is, for every $z_j \in A$, all D-linear sets joining z_1 to z_j and z_2 to z_j are contained in A. Required to prove t all the points between z_1 and z_2 are in $\mathrm{D}-\ker(A)$. That is, if ξ is any such point, t all the D-linear sets joining ξ and z_j, for every $z_j \in A$ are contained in A? Supp not. That is, there exists at least one point $\eta \in A$ such that the D-linear set joir ξ and η is not contained in A. That is, there exists a D-linear set (say)

$$L_1 = \{\xi, \alpha_1, \alpha_2, \ldots, \eta\}$$

joining η and ξ of A containing some points not in A. Now ξ is a point betweer and z_2. So there exists a D-linear set joining z_1 and ξ viz.,

$$L_2 = \{z_1, \beta_1, \beta_2, \ldots, \xi\}.$$

Now

$$L_3 = \{z_1, \beta_1, \beta_2, \ldots, \xi, \alpha_1, \alpha_2, \ldots, \eta\}$$

gives a D-linear set joining z_1 and η (this works since η is a point between z_1 and which is not contained in A and $\eta \in A$, which implies that $z_1 \notin \mathrm{D}-\ker(A)$. leads us to a contradiction. Hence the theorem.

Theorem 19. *Let A be a nonempty subset of H. Then, $\mathrm{D}-\ker(A) = A$ if and if A is D-convex.*

Proof. Let $\mathrm{D}-\ker(A) = A$. Then, by the previous theorem, A is D-convex.

Conversely, let A be D-convex. So by definition, for every $z_i, z_j \in A$,

$$\{z \in H : B(z_i, z, z_j)\} \subset A.$$

Now, $\mathrm{D}-\ker(A)$ is equal to the set of all the $z_i \in A$ such that for every $z_j \in$ all the D-linear sets joining z_i and z_j are contained in A, and equal to A since D-convex. Hence $\mathrm{D}-\ker(A) = A$ if and only if A is D-convex.

Theorem 20. *Let A and B be two D-convex sets. Then, $A \cap B \subset \mathrm{D}-\ker(A \cup$*

Proof. Since A and B are D-convex sets, $A \cap B$ is also D-convex. Let $z \in A$ To prove that $z \in \mathrm{D}-\ker(A \cup B)$. That is, to prove that for any $z_i \in A \cup B$, e D-linear set from z to z_i is contained in $A \cup B$. Without loss of generality, let z_i Then $z, z_i \in B$ and B is D-convex. So the result follows.

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