ıllect. Math. 38 (1987), 99–115

GLOBAL STABILITY OF SETS FOR SYSTEMS WITH IMPULSES*

G. K. Kulev and D. D. Bainov

ABSTRACT. In the present paper the question of global stability of sets of sufficiently general type with respect to systems of differential equations with impulses is considered. It is proved that the existence of piecewise continuous functions of the type of Lyapunov's functions with certain properties is a sufficient condition for various types of global asymptotic stability.

1. Introduction

the recent years systems of differential equations with impulses have been an object numerous investigations ([1]–[11]), related to the applications of these systems in ysics, biology, control theory, etc. The necessity of consideration of such systems ses from the study of real processes and phenomena which during their evolution ; subject to short-time perturbations in the form of impulses. The duration of the tion of these perturbations is negligibly small in comparison with the total duration the process.

In the present paper the problem of global stability of sets with respect to sysns of differential equations with impulses is considered. In the investigations acmplished piecewise continuous auxiliary functions are used which are an analogue the classical Lyapunov's functions. It is proved that the existence of such functions th certain properties is a sufficient condition for various types of global asymptotic ıbility.

* Investigation accomplished with the financial support of the Committee for Scice of the Council of Ministers of the People's Republic of Bulgaria, Grant No. 61

We shall note that results related to the stability and global stability of sets systems of ordinary differential equations without impulses have been obtaine[12]–[16].

2. Preliminary notes and definitions

Let $\mathbb{R}^n$ be an n–dimensional Euclidian space with a norm $\|\cdot\|$, scalar product and distance $d(\cdot,\cdot)$ and let $I = [0, \infty]$.

Consider the following system of differential equations with impulses

$$\frac{dx}{dy} = f(t,x), \qquad \text{for } (t,x)\overline{\in}\sigma_i;$$

$$\Delta x_{|(t,x)\in\sigma_i} = I_i(x), \qquad \text{for } i = 1, 2, \ldots$$

where $x \in \mathbb{R}^n$, $f : I \times \mathbb{R}^n \to \mathbb{R}^n$, $I_i : \mathbb{R}^n \to \mathbb{R}^n$,$\tau_i : \mathbb{R}^n \to \mathbb{R}$,

$$\sigma_i = \{(t,x) \in I \times \mathbb{R}^n : \ t = \tau_i(x)\},$$

$$\Delta x_{|(t,x)\in\sigma_i} = x(t+0) - x(t-0).$$

Such systems are characterized by the fact that under the action of a short-t force (impact, impulse) the mapping point $(t, x(t))$ from the extended phase sp $I \times \mathbb{R}^n$ by meeting some of the hypersurfaces σ_i is transferred momentarily from position $(t, x(t))$ into the position $(t, x(t) + I_i(x(t)))$. Each solution $x(t)$ of system is a piecewise continuous function with points of discontinuity of first type at wl it is left continuous, i.e. at the moment t_i when the integral curve $(t, x(t))$ meets hypersurface σ_i the following relations hold:

$$x(t_i - 0) = x(t_i),$$

$$\Delta x_{|t=t_i} = x(t_i + 0) - x(t_i - 0) = I_i(x(t_i)).$$

Let $M \subset I \times \mathbb{R}^n$. Introduce the notations:

$$M(t) = \{x \in \mathbb{R}^n : \ (t,x) \in M\}, \qquad \text{for } t \in I;$$

$$M(t,\epsilon) = \{x \in \mathbb{R}^n : \ d(x, M(t)) < \epsilon\}, \qquad \text{for } \epsilon > 0,$$

where

$$d(x, M(t)) = \inf_{y\in M(t)} \|x - y\|$$

the distance between x and the set $M(t)$;

$$G_i = \{(t,x) \in I \times \mathbb{R}^n : \tau_{i-1}(x) < t < \tau_i(x)\}, \qquad \text{for } i = 1,2,\dots;$$

$$(\tau_0(x) \equiv 0 \qquad \text{for } x \in \mathbb{R}^n)$$

$$G = \bigcup_{i=1}^{\infty} G_i;$$

$$S_\alpha = \{x \in \mathbb{R}^n : \|x\| \le \alpha\}, \qquad \text{for } \alpha > 0;$$

$$D_{\alpha,t} = \{x \in \mathbb{R}^n : d(x, M(t)) \le \alpha\}, \qquad \text{for } \alpha > 0,\ t \in I.$$

Let $t_0 \in I$, $x_0 \in \mathbb{R}^n$. Denote by $x(t;t_0,x_0)$ the solution of system (1) satisfying e initial condition $x(t_0+0;t_0,x_0) = x_0$ and by $J^+(t_0,x_0)$ denote the maximal ;erval of the form (t_0,ω) in which the solution $x(t;t_0,x_0)$ is defined.

We shall say that conditions (A) are satisfied if the following conditions hold:

A$_1$. The function $f(t,x)$ is continuous on $I \times \mathbb{R}^n$ and Lipschitz continuous with respect to its second argument.

A$_2$. $\|f(t,x)\| \le L < \infty$ for $(t,x) \in I \times \mathbb{R}^n$ $(L \ge 0)$.

A$_3$. For any $i = 1,2,\dots$ the following inequality holds

$$\|I_i(x_1) - I_i(x_2)\| \le d\,\|x_1 - x_2\|, \qquad \text{for } x_1, x_2 \in \mathbb{R}^n \qquad (d \ge 0).$$

A$_4$. The functions $\tau_i(x)$ $(i = 1,2,\dots)$ are continuous and the following relations hold

$$0 < \tau_1(x) < \tau_2(x) < \cdots, \qquad \text{for } x \in \mathbb{R}^n,$$

$$\lim_{i\to\infty} \tau_i(x) = \infty \qquad \text{uniformly on } \mathbb{R}^n,$$

$$\inf_{\mathbb{R}^n} \tau_i(x) - \sup_{\mathbb{R}^n} \tau_{i-1}(x) \ge \theta > 0.$$

We assume that for system (1) the phenomenon "beating" is not observed, i.e. at the following condition (B) holds:

B. The integral curve of each solution of system (1) meets each of the hypersurfaces $\{\sigma_i\}$ at most once.

Effective sufficient conditions which guarantee the absence of the phenomenon ›eating" are given in [10] and [11].

We shall note that condition (B) is satisfied, for instance, in the particular case hen $\tau_i(x) = t_i$, $i = 1,2,\dots$, $x \in \mathbb{R}^n$, i.e. the impulse effect is realized at fixed oments.

We shall say that conditions (C) are satisfied if the following conditions hold

C_1. For each $t \in I$ the set $M(t)$ is not empty.

C_2. There exists a compact set $Q \subset \mathbf{R}^n$ such that $M(t) \subset Q$ for any $t \in I$.

C_3. For any compact subset F of $I \times \mathbf{R}^n$ there exists a constant $K > 0$ depend on F so that if $(t, x), (t', x) \in F$ then the following inequality holds

$$|d(x, M(t)) - d(x, M(t'))| \leq K\,|t - t'|.$$

We shall say that condition (D) is satisfied if the following condition holds:

D. Each solution $x(t; t_0, x_0)$ of system (1) which satisfies the estimate

$$d(x(t, ; t_0, x_0), M(t)) \leq h < \infty \qquad \text{for } t \in J^+(t_0, x_0)$$

is defined in the interval (t_0, ∞).

We shall give definitions of stability of the set M with respect to system (1).

Definition 1. The set M is called:

a) *Stable with respect to system* (1) if

$$(\forall t_0 \in I)\,(\forall \alpha > 0)\,(\forall \epsilon > 0)\,(\exists \delta = \delta(t_0, \alpha, \epsilon) > 0)\,(\forall x_0 \in S_\alpha \cap M(t_0, \delta))$$

$$(\forall t \in J^+(t_0, x_0)) : \; x(t; t_0, x_0) \in M(t, \epsilon).$$

b) *t (α)–uniformly stable with respect to system* (1) if the number δ from p‹ a) does not depend on t_0 (on α).

c) *Uniformly stable with respect to system* (1) if the number δ from point depends only on ϵ.

We shall give definitions of boundedness of the solutions of system (1) w respect to the set M.

Definition 2. The solutions of system (1) are called:

a) *Equi–M–bounded (equi-bounded with respect to the set M)* if:

$$(\forall t_0 \in I)\,(\forall \alpha > 0)\,(\forall \eta > 0)\,(\exists \beta = \beta(t_0, \alpha, \eta) > 0)$$

$$(\forall x_0 \in S_\alpha \cap D_{\eta, t_0})\,(\forall t \in J^+(t_0, x_0)) : \; x(t; t_0, x_0) \in M(t, \beta).$$

b) *t (α)–uniformly M–bounded* if the number β from point a) does not dep‹ on t_0 (on α).

c) *Uniformly M–bounded* if the number β from point a) depends only on η.

Finally we shall give definitions of global asymptotical stability of the set M with spect to system (1).

efinition 3. The set M is called:

a) *Globally equi-attractive with respect to system* (1) if

$$(\forall t_0 \in I)\,(\forall \alpha > 0)\,(\forall \eta > 0)\,(\forall \epsilon > 0)\,(\exists \sigma = \sigma(t_0, \alpha, \eta, \epsilon) > 0)$$

$$(\forall x_0 \in S_\alpha \cap D_{\eta,t_0})\,(\forall t \geq t_0 + \sigma, t \in J^+(t_0, x_0)) :\ x(t; t_0.x_0) \in M(t, \epsilon).$$

b) *t (α)–uniformly globally attractive with respect to system* (1) if the number σ from point a) does not depend on t_0 (on α).

c) *Uniformly globally attractive with respect to system* (1) if the number σ from point a) depends only on η and ϵ.

efinition 4. The set M is called:

a) *Globally equi-asymptotically stable with respect to system* (1) if M is a stable set and a globally equi-attractive set of system (1) and the solutions of (1) are equi–M–bounded.

b) *t (α)–uniformly globally asymptotically stable with respect to system* (1) if M is a t (α)–uniformly stable and t (α)–uniformly globally attractive set of system (1) and if the solutions of system (1) are t (α)–uniformly M–bounded.

c) *Uniformly globally asymptotically stable with respect to system* (1) if M is a uniformly stable and uniformly globally attractive set of system (1) and if the solutions of system (1) are uniformly M–bounded.

d) *Exponentially globally asymptotically stable with respect to system* (1) if

$$(\exists c > 0)\,(\forall \alpha, \eta > 0)\,(\exists K(\alpha, \eta) > 0)\,(\forall t_0 \in I)\,(\forall x_0 \in S_\alpha \cap D_{\eta,t_0})$$

$$(\forall t \in J^+(t_0, x_0)) :\ d(x(t; t_0, x_0), M(t)) \leq K(\alpha, \eta)\, d(x_0, M(t_0))\, e^{-c(t-t_0)}$$

emark 1. If condition C_2 holds, then for any $\alpha > 0$ there exists a number $\eta > 0$ ıch that $S_\alpha \subset D_{\eta,t}$ for all $t \in I$ and, conversely, for any $\eta > 0$ one can find a number > 0 such that $D_{\eta,t} \subset S_\alpha$ for all $t \in I$. This shows that the M–boundedness of the ›lutions of system (1) defined in definition 2 is equivalent to the $I \times O$–boundedness f these solutions (O is the origin in $\mathbb{R}^n$), i.e. to boundedness in the usual sense [11].

Remark 2. If condition C_2 holds, then the numbers δ from definition 1, a), from definition 2, a) and σ from definition 3, a) can be chosen independent of t number α. Hence, for instance, if the set M is globally equi-asymptotically stal with respect to system (1), then M is an α–uniformly globally asymptotically stal set of system (1).

In the further considerations we shall use the class $\mathcal{V}_0$ of piecewise continuc auxiliary functions $V : I \times \mathbb{R}^n \to \mathbb{R}$ which are an analogue of Lyapunov's functic [8].

Definition 5. We shall say that *the function* $V : I \times \mathbb{R}^n \to \mathbb{R}$ *belongs to the class* if the following conditions hold:

1. The function V is continuous on G and is locally Lipschitz continuous w respect to its second argument on each of the sets G_i.
2. $V(t,x) = 0$ for $(t,x) \in M$ and $V(t,x) > 0$ for $(t,x) \notin M$.
3. For each $i = 1, 2, \ldots$ and for any point $(t_0, x_0) \in \sigma_i$ the limits

$$V(t_0 - 0, x_0) = \lim_{\substack{(t,x)\to(t_0,x_0) \\ (t,x)\in G_i}} V(t,x),$$

$$V(t_0 + 0, x_0) = \lim_{\substack{(t,x)\to(t_0,x_0) \\ (t,x)\in G_{i+1}}} V(t,x),$$

exist and are finite, and the equality

$$V(t_0 - 0, x_0) = V(t_0, x_0)$$

holds.

4. For any point $(t, x) \in \sigma_i$ $(i = 1, 2, \ldots)$ the following inequality holds

$$V(t + 0, x + I_i(x)) \leq V(t, x).$$

Let $V \in \mathcal{V}_0$. For $(t,x) \in G$ we set

$$\dot{V}_{(1)}(t,x) = \limsup_{h \to 0^+} h^{-1} \left[V(t+h, x + h\, f(t,x)) - V(t,x) \right].$$

From condition 1 of definition 5 it follows that if $x = x(t)$ is a solution of syst (1), then the equality

$$V(t,x) = D^+ V(t,x)$$

lds, where

$$D^+V(t,x) = \limsup_{h\to 0+} h^{-1}\left[V(t+h, x(t+h)) - V(t, x(t))\right]$$

the upper right Dini derivative of the function $V(t, x(t))$.

Moreover, if the function $V \in \mathcal{V}_0$ satisfies the condition

$$\dot{V}_{(1)}(t,x) \le 0 \qquad \text{for } (t,x) \in G$$

d if $x(t;t_0,x_0)$ is a solution of system (1), then the function $V(t, x(t;t_0,x_0))$ is ɔnotonely decreasing on $J^+(t_0,x_0)$.

Denote by $\mathcal{K}$ the class of all continuous and strictly increasing functions $a : I \to I$ ch that $a(0) = 0$.

3. Main results

the next considerations we shall use the following lemma.

ɜmma 1. *Let condition* A_4 *hold and let the function* $V : I \times \mathbb{R}^n \to \mathbb{R}$ *belong to e class* $\mathcal{V}_0$. *Then for any choice of the numbers* $t_0 \in I$, $\alpha > 0$, $\eta > 0$, *there exists a ɪmber* $K(t_0,\alpha,\eta) > 0$ *such that for* $(t_0,x_0) \in S_\alpha \cap D_\eta$ *the following inequality holds*

$$V(t_0+0, x_0) \le K(t_0,\alpha,\eta). \tag{3}$$

ˑoof. Suppose that the assertion is not true. Then there exist numbers $t_0 \in I$, $\alpha > 0$, > 0, and a sequence $\{x_k\}_1^\infty \subset \mathbb{R}^n$ so that $(t_0,x_k) \in S_\alpha \cap D_{\eta,t_0}$ for $k = 1,2,\ldots$ and e following inequality holds

$$V(t_0+0, x_k) > k, \qquad \text{for } k = 1,2,\ldots \tag{4}$$

Since the sequence $\{x_k\}$ is bounded, then out of it we can choose a convergent bsequence. We shall use the same notation for the members of this subsequence. t $\lim_{k\to\infty} x_k = x_0$.

Let $(t_0,x_0) \in G_i$ for some $i \in \mathbb{N}$, i.e.

$$\tau_{i-1}(x_0) < t_0 < \tau_i(x_0).$$

ıen since

$$\tau_{i-1}(x_0) = \lim_{k\to\infty} \tau_{i-1}(x_k)$$

and

$$\tau_i(x_0) = \lim_{k\to\infty} \tau_i(x_k),$$

for all sufficiently large values of k we have

$$\tau_{i-1}(x_k) < t_0 < \tau_i(x_0)$$

which shows that $(t_0, x_k) \in G_i$. From the continuity of $V(t,x)$ on G_i it follows th

$$\lim_{k\to\infty} V(t_0+0, x_k) = \lim_{k\to\infty} V(t_0, x_k) = V(t_0, x_0)$$

which contradicts inequality (4).

Let $(t_0, x_0) \in \sigma_i$ for some $i \in \mathbb{N}$, i.e. $\tau_i(x_0) = t_0$. Then the following relati hold

$$\begin{aligned}\lim_{k\to\infty} \tau_{i-1}(x_k) &= \tau_{i-1}(x_0)\\ &< \tau_i(x_0)\\ &= t_0\\ &= \tau_i(x_0)\\ &< \tau_{i+1}(x_0)\\ &= \lim_{k\to\infty} \tau_{i+1}(x_k).\end{aligned}$$

These relations show that the following three cases are possible:

1. For infinitely many members $\{x_{k_j}\}_{j=1}^{\infty}$ of the sequence $\{x_k\}$ we have $(t_0, x_{k}$ G_i. Then from the left continuity of the function $V(t,x)$ we obtain

$$\begin{aligned}\lim_{j\to\infty} V(t_0+0, x_{k_j}) &= \lim_{j\to\infty} V(t_0, x_{k_j})\\ &= V(t-0, x_0)\\ &= V(t_0, x_0),\end{aligned}$$

which contradicts inequality (4).

2. For infinitely many members $\{x_{k_j}\}_{j=1}^{\infty}$ of the sequence $\{x_k\}$ we have $(t_0, x_k$ G_{i+1}. Then

$$\begin{aligned}\lim_{j\to\infty} V(t_0+0, x_{k_j}) &= \lim_{j\to\infty} V(t_0, x_{k_j})\\ &= V(t_0+0, x_0)\\ &< \infty,\end{aligned}$$

which contradicts inequality (4).

3. Only a finite number of members $\{x_{k_j}\}_{j=1}^{\infty}$ of the sequence $\{x_k\}$ are that $(t_0, x_{k_j}) \in G_i$ or $(t_0, x_{k_j}) \in G_{i+1}$. Without loss of generality we can assume

$, x_k) \in \sigma_i$ for each $k \in \mathbb{N}$. From inequality (4) and the left continuity of the function $t, x)$ at the points (t_0, x_k) it follows that there exist sequences $\{(t_n^{(k)}, y_n^{(k)})\}_{n=1}^{\infty}$ $= 1, 2, \ldots)$ such that

$$\left(t_n^{(k)}, y_n^{(k)}\right) \in G_i,$$

$$\lim_{n\to\infty} \left(t_n^{(k)}, y_n^{(k)}\right) = (t_0, x_k)$$

d

$$V\left(t_n^{(k)} + 0, y_m^{(k)}\right) \geq k \qquad \text{for } k = 1, 2, \ldots.$$

en

$$\lim_{n\to\infty} \left(t_n^{(n)}, y_n^{(n)}\right) = (t_0, x_0).$$

nce

$$\begin{aligned}\lim_{n\to\infty} V(t_n^{(n)} + 0, y_n^{(n)}) &= \lim_{n\to\infty} V(t_n^{(n)}, y_n^{(n)}) \\ &= V(t_0 - 0, x_0) \\ &= V(t_0, x_0),\end{aligned}$$

ich contradicts the inequality

$$V(t_n^{(n)} + 0, y_n^{(n)}) \geq n.$$

This completes the proof of Lemma 1.

We shall find sufficient conditions for global asymptotical stability of the set M ;h respect to system (1).

ıeorem 1. *Let conditions* (A), (B), C$_1$), (C$_3$) *and* (D) *hold and functions* $V \in \mathcal{V}_0$ *ɗ* $a \in \mathcal{K}$ *exist such that*

(i) $a(d(x, M(t))) \leq V(t, x)$ *for* $(t, x) \in I \times \mathbb{R}^n$ *and* $a(r) \to \infty$ *as* $r \to \infty$.

(ii) $\dot{V}_{(1)}(t, x) \leq -c\, V(t, x)$ *for* $(t, x) \in G$ *where* $c > 0$ *is a constant.*

Then the set M *is globally equi-asymptotically stable with respect to system* (1).

ɔof. Let $\epsilon > 0$, $\alpha > 0$, $t_0 \in I$. From the condition $V(t_0, x) = 0$ for $x \in M(t_0)$ it lows that there exists a number $\delta = \delta(t_0, \alpha, \epsilon) > 0$ such that if $x \in S_\alpha \cap M(t_0, \delta)$, ɜn $V(t_0 + 0, x) < a(\epsilon)$.

Let $x_0 \in S_\alpha \cap M(t_0, \delta)$. Using successively (i), (ii) and (2) we obtain

$$\begin{aligned}a(d(x(t; t_0, x_0), M(t))) &\leq V(t, x(t; t_0, x_0)) \\ &\leq V(t_0 + 0, x_0) \\ &< a(\epsilon)\end{aligned}$$

for $t \in J^+(t_0, x_0)$. From condition (D) it follows that $J^+(t_0, x_0) = (t_0, \infty)$, he $x(t; t_0, x_0) \in M(t, \epsilon)$ for all $t > t_0$, i.e. the set M is stable with respect to system (

Let $\epsilon > 0$, $\alpha > 0$, $\eta > 0$ and $t_0 \in I$. From (ii) and (2) there follows the inequa

$$V(t, x(t; t_0, x_0)) \leq V(t_0 + 0, x_0) \exp[-c(t - t_0)].$$

Let

$$N(t_0, \eta, \alpha) = \sup\{V(t_0 + 0, x_0) : x_0 \in S_\alpha \cap D_{\eta, t_0}\}$$

and

$$\sigma = \sigma(t_0, \alpha, \eta, \epsilon) > \frac{1}{c} \ln \frac{N(t_0, \eta, \alpha)}{a(\epsilon)}.$$

Then from (i) and (5) it follows that for $t \geq t_0 + \sigma$ the following inequalities h

$$\begin{aligned} a(d(x(t; t_0, x_0), M(t))) &\leq V(t, x(t; t_0, x_0)) \\ &\leq V(t_0 + 0, x_0) \exp[-c(t - t_0)] \\ &\leq a(\epsilon), \end{aligned}$$

which means that the set M is globally equi-attractive with respect to system (1

Finally we shall prove that the solutions of system (1) are equi-M-bounded

In fact, let $t_0 \in I$, $\alpha > 0$ and $\eta > 0$. From lemma 1 it follows that there exis number $K(t_0, \alpha, \eta) > 0$ such that if $x \in S_\alpha \cap D_{\eta, t_0}$, then $V(t_0 + 0, x) \leq K(t_0, \alpha$ From the condition $a(r) \to \infty$ for $r \to \infty$ it follows that there exists a nur $\beta = \beta(t_0, \alpha, \eta) > 0$ such that $a(\beta) > K(t_0, \alpha, \eta)$.

Let $x_0 \in S_\alpha \cap D_{\eta, t_0}$ and let $x(t) = x(t; t_0, x_0)$ be a solution of system (1). T from (i), (ii) and (2) we get

$$\begin{aligned} a(d(x(t), M(t))) &\leq V(t, x(t)) \\ &\leq V(t_0 + 0, x_0) \\ &\leq K(t_0, \alpha, \eta) \\ &< a(\beta) \end{aligned}$$

for each $t \in J^+(t_0, x_0)$. From condition (D) it follows that $J^+(t_0, x_0) = (t_0, \infty)$. shows that

$$d(x(t; t_0, x_0), M(t)) < \beta \qquad \text{for } t > t_0,$$

hence the solutions of system (1) are equi-M-bounded.

This completes the proof of theorem 1.

ıeorem 2. *Let conditions* (A), (B), (C_1), (C_3) *and* (D) *hold and functions* $V \in \mathcal{V}_0$ *d* $a, b, c \in \mathcal{K}$ *exist such that*

(i) $a(d(x, M(t))) \le V(t+0, x) \le b(d(x, M(t)))$, *for* $(t,x) \in I \times \mathbb{R}^n$ *where* $a(r) \to \infty$ *for* $r \to \infty$.

(ii) $\dot{V}_{(1)}(t,x) \le -c(d(x, M(t)))$, *for* $(t,x) \in G$.

Then the set M *is uniformally globally asymptotically stable with respect to* *stem* (1).

oof. Let $\epsilon > 0$ be given. Choose the number $\delta = \delta(\epsilon) > 0$ so that $b(\delta) < a(\epsilon)$. Then $\alpha > 0$, $t_0 \in I$ and $x_0 \in S_\alpha \cap M(t_0, \delta)$ using successively (i), (ii), (2) we obtain

$$\begin{aligned} a(d(x(t;t_0,x_0), M(t))) &\le V(t, x(t;t_0,x_0)) \\ &\le V(t_0+0, x_0) \\ &\le b(d(x_0, M(t_0))) \\ &\le b(\delta) \\ &< a(\epsilon) \end{aligned}$$

$t \in J^+(t_0, x_0)$. Hence $J^+(t_0,x_0) = (t_0, \infty)$ and $x(t;t_0,x_0) \in M(t,\epsilon)$ for any $t > t_0$ ıich means that the set M is uniformly stable with respect to system (1).

Let $\epsilon > 0$ be given. Choose the number $\delta = \delta(\epsilon) > 0$ so that $b(\delta) < a(\epsilon)$.

Let $\eta > 0$,

$$\sigma = \sigma(\eta, \epsilon) > \frac{b(\eta)}{c(\delta(\epsilon))},$$

> 0, $t_0 \in I$, $x_0 \in S_\alpha \cap D_{\eta, t_0}$ and $x(t) = x(t;t_0,x_0)$ be a solution of system (1). ssume that for any $t \in [t_0, t_0+\sigma]$ the inequality $d(x(t), M(t)) \ge \delta(\epsilon)$ holds. Then ɔm (ii) we obtain

$$\int_{t_0}^{t} \dot{V}_{(1)}(s, x(s))\, ds \le -c(\delta(\epsilon))\,[t - t_0]. \tag{6}$$

On the other hand, if $\{t_i\}_{i=1}^{\infty}$ are the moments at which the integral curve of e solution $x(t)$ meets respectively the hypersurfaces $\{\sigma_i\}$, then from (2) we obtain, r $t \in (t_k, t_{k+1}]$,

$$\begin{aligned} \int_{t_0}^{t} \dot{V}_{(1)}(s, x(s))\, ds &= \sum_{i=1}^{k} \int_{t_{i-1}+0}^{t_i - 0} \dot{V}_{(1)}(s, x(s))\, ds + \int_{t_k+0}^{t-0} \dot{V}_{(1)}(s, x(s))\, ds \\ &= \sum_{i=1}^{k} [V(t_i, x(t_i)) - V(t_{i-1}+0, x(t_i+0))] \\ &\quad + V(t, x(t)) - V(t_k+0, x(t_k+0)) \\ &\ge V(t, x(t)) - V(t_0+0, x_0). \end{aligned}$$

From this inequality and (6) we deduce

$$V(t,x(t)) \leq V(t_0+0,x_0) - c(\delta)\,[t-t_0] \qquad \text{for } t \in [t_0, t_0+\sigma],$$

whence for $t = t_0 + \sigma$ we have

$$V(t,x(t)) \leq V(t_0+0,x_0) - c(\delta)\,\sigma \leq b(\eta) - c(\delta)\,\sigma < 0,$$

which contradicts (i).

Hence there exists a number $t' \in [t_0, t_0+\sigma]$ such that

$$d(x(t'), M(t')) < \delta(\epsilon).$$

Then for $t \geq t'$, hence for any $t \geq t_0 + \sigma$ as well the following inequalities hold

$$\begin{aligned} a(d(x(t), M(t))) &\leq V(t,x(t)) \\ &\leq V(t', x(t')) \\ &\leq b(d(x(t'), M(t'))) \\ &\leq b(\delta) \\ &< a(\epsilon), \end{aligned}$$

i.e. $x(t;t_0,x_0) \in M(t,\epsilon)$ which shows that the set M is uniformly globally attract with respect to system (1).

Finally we shall prove that the solutions of system (1) are uniformly M-bound

Let $\eta > 0$ be given. Choose $\beta = \beta(\eta) > 0$ so that $a(\beta) > b(\eta)$. This is possi in view of the condition $a(r) \to \infty$ for $r \to \infty$.

Let $t_0 \in I$, $\alpha > 0$ and $x_0 \in S_\alpha \cap D_{\eta,t_0}$. Applying successively (i), (ii) and (2) obtain

$$\begin{aligned} a(d(x(t;t_0,x_0), M(t))) &\leq V(t, x(t;t_0,x_0)) \\ &\leq V(t_0+0, x_0) \\ &\leq b(d(x_0, M(t_0))) \\ &\leq b(\eta) \\ &< a(\beta) \end{aligned}$$

for $t \in J^+(t_0,x_0)$. Hence $J^+(t_0,\infty) = (t_0,\infty)$ and $x(t;t_0,x_0) \in M(t,\epsilon)$ for $t > t_0$.

This completes the proof of theorem 2.

Corollary 1. *Let the conditions of theorem 2 hold, condition* (ii) *being replaced the condition*

(iii) $\dot{V}_{(1)}(t,x) \le -c\,V(t,x)$, *for* $(t,x) \in G$,
where c *is a constant.*

Then the set M *is uniformly globally asymptotically stable with respect to sys-* *m* (1).

The assertion is an immediate consequence of theorem 2. But if we make use of

$$V(t, x(t;t_0,x_0)) \le V(t_0+0,x_0)\,\exp[-c\,(t-t_0)] \qquad \text{for } t > t_0,$$

st inequality following from (iii) and (2), then the proof can be carried out as the ·oof of theorem 1.

Analogous to the proof of the theorem 2 is the proof of the following two theorems:

heorem 3. *Let the conditions of theorem 2 hold, condition* (i) *being replaced by* *ιe condition*

(iv) $a(d(x,M(t))) \le V(t+0,x) \le b(d(x,M(t)),\|x\|)$, *for* $(t,x) \in I \times \mathbb{R}^n$,
where $a \in \mathcal{K}$, $a(r) \to \infty$ *for* $r \to \infty$ *and the function* $b : r \to b(r,s) \in \mathcal{K}$ *for any* $s \ge 0$ *fixed.*

Then the set M *is* t*–uniformly globally asymptotically stable with respect to* *ʳstem* (1).

heorem 4. *Let the conditions of theorem 2 hold, condition* (i) *being replaced by* *ıe condition*

(v) $a(d(x,M(t))) \le V(t+0,x) \le b(d(x,M(t)))$, *for* $(t,x) \in I \times \mathbb{R}^n$,
where $a \in \mathcal{K}$, $a(r) \to \infty$ *for* $r \to \infty$ *and the function* $b : r \to b(t,r) \in \mathcal{K}$ *for any* $t \in I$ *fixed.*

Then the set M *is* α*–uniformly globally asymptotically stable with respect to* *ʳstem* (1).

heorem 5. *Let conditions* (A), (B), (C_1), (C_3) *and* (D) *hold and a function* $V \in \mathcal{V}_0$ *xist such that*

(i) $d(x,M(t)) \le V(t,x) \le K(\eta,\alpha)\,d(x,M(t))$,
for $\alpha > 0$, $\eta > 0$, $t \in I$, $x \in S_\alpha \cap D_{\eta,t}$.

(ii) $\dot{V}_{(1)}(t,x) \le -c\,V(t,x)$ *for* $(t,x) \in G$ *where* $c > 0$ *is a constant.*

Then the set M *is exponentially globally asymptotically stable with respect to* *ʳstem* (1).

'roof. As in the proof of theorem 2 it can be proved that the solutions of system (1) re uniformly M–bounded. Hence each solution $x(t;t_0,x_0)$ is defined in the interval $_0,\infty)$, $t_0 \in I$.

Let $\alpha > 0$, $\eta > 0$, $t_0 \in I$, $x_0 \in S_\alpha \cap D_{\eta,t_0}$. From (ii) and (2) we obtain

$$V(t, x(t; t_0, x_0)) \leq V(t_0 + 0, x_0) \exp[-c(t - t_0)] \qquad \text{for } t > t_0.$$

Then from (i) it follows that, for $t > t_0$:

$$\begin{aligned} d(x(t; t_0, x_0), M(t)) &\leq V(t, x(t; t_0, x_0)) \\ &\leq V(t_0 + 0, x_0) \exp[-c(t - t_0)] \\ &\leq K(\eta, \alpha)\, d(x_0, M(t_0)) \exp[-c(t - t_0)]. \end{aligned}$$

Theorem 5 is proved.

4. Examples

Example 1. Consider the system of differential equations with impulse effect at fix moments

$$\frac{dx}{dt} = \begin{cases} [B(t) + A(t)]\, x & \text{for } x > 0,\ t \neq t_i \\ 0 & \text{for } x \leq 0,\ t \neq t_i \end{cases}$$
$$\Delta x_{|t=t_i} = \begin{cases} I_i(x) & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

(

where $x \in \mathbb{R}^n$, $B(t)$ and $A(t)$ are $(n \times n)$–matrices of continuous functions on I a $B(t)$ is diagonal and $A(t)$ is skew-symmetric, $I_i(x)$, $i = 1, 2, \ldots$, are continuous a such that $x + I_i(x) > 0$ and $\|x + I_i(x)\| \leq 0$ for $x > 0$ ($x > 0$ ($x \leq 0$) means tl $x_i > 0$ ($x_i \leq 0$) for $i = 1, 2, \ldots, n$, where x_i is the i–th component of the vect $x \in \mathbb{R}^n$).

The moments $\{t_i\}$ of impulse effect form a strictly increasing sequence, i.e.

$$0 < t_1 < t_2 < \cdots < t_i < \cdots$$

and

$$\lim_{i \to \infty} t_i = \infty.$$

Let $M = I \times \{x \in \mathbb{R}^n : x \leq 0\}$. Consider the function

$$V(t, x) = \begin{cases} \langle x, x \rangle & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

ıen

$$\dot V_{(1)}(t,x) = \begin{cases} 2\langle x, B(t)\,x\rangle & \text{for } x>0,\, t\neq t_i \\ 0 & \text{for } x\leq 0,\, t\neq t_i \end{cases}$$

$$V(t_i+0, x+I_i(x)) = \begin{cases} \|x+I_i(x)\|^2 & \text{for } x>0 \\ 0 & \text{for } x\leq 0 \end{cases}$$

nce

$$V(t_i+0, x+I_i(x)) \leq V(t_i, x).$$

Let $B(t) = \operatorname{diag}(b_1(t), b_2(t), \ldots, b_n(t))$ and $b_i(t) \leq -\gamma_i < 0$ for $i = 1,2,\ldots,n$ and $\in I$.

Since $d(x, M(t)) = \|x\|$ for $t \in I$ and $x > 0$, then the conditions of theorem 2 are tisfied.

Hence the set M is uniformly globally asymptotically stable with respect to stem (7).

xample 2. Consider the system

$$\begin{aligned} \frac{dx}{dt} &= a(t)\,y + b(t)\,x\,(x^2+y^2) \\ \frac{dy}{dt} &= -a(t)\,x + b(t)\,y\,(x^2+y^2), \qquad t\neq t_i \\ \Delta x_{|t=t_i} &= c_i\,x(t_i), \\ \Delta y_{|t=t_i} &= y(t_i), \end{aligned} \tag{8}$$

here the functions $a(t)$ and $b(t)$ are continuous on I, $b(t) \leq -\gamma < 0$, $-1 > c_i \leq 0$, $1 < d_i \leq 0$, for $i = 1,2,\ldots$, $0 < t_1 < t_2 < \ldots$, and $\lim_{i\to\infty} t_i = \infty$. Let $M = t,0,0) : t \in I\}$.

The function $V(t,x,y) = x^2 + y^2$ satisfies the conditions of theorem 2. In fact,

$$\dot V_{(8)}(t,x,y) = 2\,b(t)\,(x^2+y^2)^2 \leq -\gamma\,(x^2+y^2)^2 \qquad t\neq t_i$$

$$V(t_i+0, x+c_i\,x, y+d_i\,y) \leq (1+c_i)^2\,x^2 + (1+d_i)^2\,y^2 \leq V(t_i,x,y).$$

Hence the set M is uniformly globally asymptotically stable with respect to 'stem (8).

References

[1] S. LEELA, Stability of differential systems with impulsive perturbations in ter of two measures, *Nonlinear Analysis* **1** (1977), 667–677.

[2] T. PAVLIDIS, Stability of systems described by differential equations contain impulses, *IEEE Trans.* AC–**12** (1967), 43–45.

[3] S. G. PANDIT, On the stability of impulsively perturbed differential systei *Bull. Austr. Math. Soc.* **17** (1977), 423–432.

[4] M. RAMA MOHANA RAO AND V. SREE HARI RAO, Stability of impulsiv perturbed systems, *Bull. Austr. Math. Soc.* **16** (1977), 99–110.

[5] V. D. MIL'MAN AND A. D. MYSHKIS, On the stability of motion in the prese of impulses, *Sib. Math. J.* **1** (1960), 233–237 (in Russian).

[6] A. M. SAMOILENKO AND N. A. PERESTYUK, On the stability of the soluti of systems with impulse effect, *Diff. Uravn.* **11** (1981), 1995–2001 (in Russia

[7] P. S. SIMEONOV AND D. D. BAINOV, The second method of Liapunov systems with an impulse effect, *Tamkang J. Math.* **16** (1985), 19–40.

[8] P. S. SIMEONOV AND D. D. BAINOV, Stability with respect to part of variables in systems with impulse effect, *J. Math. Anal. and Appl.* **117** (19 247–263.

[9] A. B. DISHLIEV AND D. D. BAINOV, Sufficient conditions for abscense of "be ing" in systems of differential equations with impulses, *Appl. Anal.* **18** (19 67–73.

[10] A. B. DISHLIEV AND D. D. BAINOV, Conditions for the absence of the p nomenon "beating" for systems of impulse differential equations, *Bull. Inst. Ma Acad. Sinica* **13** (1985), 237–256.

[11] S. G. HRISTOVA AND D. D. BAINOV, Application of Liapunov's functions studying the boundedness of solutions of systems with impulses, *COMPE.* (1986), 23–40.

[12] T. YOSHIZAWA, Liapunov's function and boundedness of solutions, *Funkci. Ekvacioj* **2** (1959), 95–142.

[13] T. YOSHIZAWA, Asymptotic behaviour of solutions of non-autonomous syst near sets, *J. Math. Kyoto Univ.* **1** (1962), 303–323.

[14] S. LEFSCHETZ, Liapunov and stability in dynamical systems, *Bol. Soc. M Mexicana* **3** (1958), 25-39.

[15] J. K. HALE AND A,. P. STOKES, Behaviour of solutions near integral manifo

Arch. Rational Mech. Anal. **6** (1960), 133–170.

] N. ROUCHE, P. HABETS AND M. LALOY, *Stability Theory by Liapunov's Direct Method*, Springer, Berlin, 1977.

Received 25/FEB/88
Revised version received 28/APR/88

G. K. Kulev and D. D. Bainov
P. O. Box 45
1504 Sofia
BULGARIA