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DISCRETE SUBSETS IN HAUSDORFF TOPOLOGIES

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ABSTRACT. It is shown that the problem of recovering a Hausdorff topology from the family of its discrete subspaces has a negative answer in general, and a sufficient condition in terms of neighborhood bases is given.

The following notation is used. If T is a Hausdorff topology in X, a *discrete* or T–*discrete* subset is a $D \subset X$ such that the relative topology $T_{|D}$ is discrete, and a *totally discrete* or T–*totally discrete* subset is a $D \subset X$ such that there is an open disjoint family $\{U_a; a \in D\}$ with $a \in U_a$ for all $a \in D$. We denote $\Delta = \Delta(T)$ (resp. $\Theta = \Theta(T)$) the family of discrete (resp. totally discrete) subsets; clearly $\Theta \subset \Delta$. Also, T_0 will denote the topology whose open sets are the differences $U - Z$ with U open in T and Z countable. Since T_0 is finer than T, it is also Hausdorff. Finally, denote by $X_1 = X_1(T) \subset X$ the (possibly empty) subset of points $x \in X$ such that no T–neighborhood of x is countable and by $G = G(T) = \{g\}$ the group of bijections $g : X \to X$ such that the non-fixed points of g form a countable subset of X_1; $g(T)$ will denote the image topology (i.e., $g(U)$ is $g(T)$–open if and only if U is T–open). Observe that G is not reduced to the identity when X_1 contains more than one point.

1. Proposition. *Assume that T is Hausdorff and hereditarily Lindelöf. Then:*

(a) *for each $g \in G(T)$, the discrete subsets for $g(T_0)$ are exactly the countable subsets of X.*

(b) *if $g_1 \neq g_2$ then $g_1(T_0)$ and $g_2(T_0)$ are incomparable.*

Proof. Clearly a countable subset of X is T_0–discrete. Conversely, let $D \subset X$ be T_0–discrete. Then there is a T–open covering $\{U\}$ of D such that $U - D$ is countable for each U. Let $\{V\} \subset \{U\}$ be a countable subcovering. Then $D = \cup\{V \cap D\}$ is countable, which proves (a) for T_0. Since g is a homeomorphism between T_0 and $g(T_0)$, (a) follows for all $g(T_0)$.

To prove (b) it suffices to show that if g is not the identity, then $g(T_0)$ is not finer than T_0. Choose $x_1 \in X_1$ such that $g(x_1) \neq x_1$ and let U, V be disjoint T-neighborhoods of x_1 and $g(x_1)$. An arbitrary $g(T_0)$-neighborhood W of $g(x_1)$ is of the form $W = g(U_1)$ with U_1 a T_0-neighborhood of x_1, whence $W = (Z_1 \cup U_1) - Z_2$ for suitable countable sets Z_1, Z_2. Since $x_1 \in X_1$, we have $W \cap U \supset (U_1 \cap U) - Z_2 \neq \emptyset$ and therefore $W \not\subset V$. Hence V is not a $g(T_0)$-neighborhood of $g(x_1)$ and (b) follows.

When X is the real line and T the ordinary topology, we have $X_1 = X$ and G is uncountable. Thus:

Corollary. *There exist uncountably many pairwise incomparable Hausdorff topologies on the real line with the same discrete subsets.*

We turn now to conditions insuring that T can be recovered from $\Delta = \Delta(T)$. To be precise, we say that $\Delta(T)$ *determines* T if for any other Hausdorff topology T_1 the condition $\Delta(T) = \Delta(T_1)$ implies $T = T_1$. First observe that the closure $\bar{D}$ of a discrete set is completely determined by Δ alone, since

$$\bar{D} = D \cup \{x;\ \{x\} \cup D \notin \Delta\}.$$

Hence for any $A \subset X$, the set

$$A^* = \bigcup \{\bar{D};\ D \subset A \text{ and } D \in \Delta\}$$

is also determined by Δ alone, and therefore:

2. *if $A^* = \bar{A}$ for each $A \subset X$, then T is determined by $\Delta(T)$.*

A closely related construction is the following:

$$\tilde{A} = \bigcup \{\bar{D};\ D \subset A \text{ and } D \in \Theta\},$$

obtained from Δ and Θ rather than Δ. Since

$$D_1 = D \cup \{x;\ \{x\} \cup D \notin \Theta\}$$

may be different from $\bar{D}$ for $D \in \Theta$ (see example 6 below) we can not say that $\tilde{A}$ is determined by $\Theta(T)$ alone. However $A \to \tilde{A}$ is better behaved than $A \to A^*$ as the following shows.

3. Proposition. *The mapping $A \to \tilde{A}$ is a closure operator, the resulting topology tT is finer than T, and $t(tT) = tT$.*

Proof. The properties $\tilde{\emptyset} = \emptyset$ and $A \subset \tilde{A}$ are obvious. To show that $\tilde{\tilde{A}} \subset \tilde{A}$ let $x \in \bar{D}$ for $D \in \Theta$ and $D \subset \tilde{A}$. For $d \in D$, let $D_d \in \Theta$ satisfy $d \in \bar{D}_d$ and $D_d \subset A$, and choose an open disjoint family $\{U_d;\ d \in D\}$ with $d \in U_d$. Clearly

$$E = \bigcup\{D_d \cap U_d;\ d \in D\}$$

is totally discrete, $E \subset A$, and $x \in \bar{E}$ so that $x \in \tilde{A}$, as desired. Now if $A, B \subset X$ and $x \in (A \cup B)^{\sim}$ there exists $D \in \Theta$ with $D \subset A \cup B$ and $x \in \bar{D}$. Consider $D' = D \cap A$, $D'' = D \cap B$; clearly $x \in \bar{D}'$ or $x \in \bar{D}''$ and therefore $x \in \tilde{A} \cup \tilde{B}$. Hence $\tilde{A}$ is a closure, and since $A \subset \tilde{S} \subset \bar{A}$, the topology tT is finer than T.

Finally, denote temporarily A^+ the $t(tT)$–closure of A. By the first part of the proof $A \subset A^+ \subset \tilde{A} \subset \bar{A}$. Using $\tilde{D} = \bar{D}$ for $D \in \Theta(T)$ and $\Theta(T) \subset \Theta(tT)$ we get

$$\begin{aligned} \tilde{A} &= \bigcup\{\bar{D};\ D \subset A \text{ and } d \in \Theta(T)\} \\ &\subset \bigcup\{\tilde{D};\ D \subset a \text{ and } D \in \Theta(tT)\} \\ &= A^+ \end{aligned}$$

Hence $\tilde{A} = A^+$ and tT and $t(tT)$ coincide, as claimed.

Corollary. *If a Hausdorff topology satisfies $tT = T$, then $\Delta(T)$ determines T.*

Proof. Since always $A \subset \tilde{A} \subset A^* \subset \bar{A}$, the hypothesis $tT = T$ implies (from Prop. 3) that $\tilde{A} = A^* = \bar{A}$, and using 2. the corollary follows.

In Prop. 1 we presented a method for describing many topologies *not* determined by their discrete subsets. Coupling the previous corollary and the following proposition, an equally vast class of topologies that *are* determined by their discrete subsets can be obtained.

4. Proposition. *If the Hausdorff topology T satisfies:*

(4.a) *every point has a totally ordered neighborhood base,*

then $tT = T$.

Corollary. *First countable and order Hausdorff topologies are determined by their discrete subsets.*

Proof of 4. Taking appropiate cofinal families we may actually assume that every point x has a well-ordered neighborhood base $B(x)$. We prove that if $x \in \bar{A} - A$, then $x \in \bar{D}$ for some totally discrete subset D of A, as follows. Consider the families $F \subset B(X)$ with the property: $V_1, V_2 \in F$ and $V_1 \not\subset V_2$ imply $(V_1 - V_2)^\circ \cap A \neq \emptyset$ (S° denotes the interior of a subset $S \subset X$). Define an order for these families by saying that F_2 follows F_1 if F_1 is a subfamily of F_2 and every $W \subset F_2 - F_1$ is contained in all $V \in F_1$; Zorn's lemma implies the existence of a maximal family F_m. Since T is Hausdorff and $\{x\}$ is not open, $F_m \subset B(X)$ is also a well ordered neighborhood base without last element. For every V in F_m pick x_V in $(V - V')^\circ \cap A$ where $V' \subset V$ is the succesor of V in F_m. The set $D = \{x_V\}$ is totally discrete and $x \in \bar{D}$, which completes the proof.

We close this note with remarks and examples (some suggested by M. Rajagopalan).

5. Discrete and totally discrete subsets need not coincide, even in a compact space: let ω (resp. Ω) be the first infinite (resp. uncountable) ordinal and let

$$X = [0,\omega] \times [0,\Omega],$$
$$S = \{\alpha + 1;\ 0 \leq \alpha < \Omega\},$$
$$D = (\{\omega\} \times S) \cup ([0,\omega) \times \{\Omega\}).$$

Then D is discrete but not totally discrete in X.

6. Denote $R(t)$ the ray $r\exp(it)$, $r > 0$, in the complex plane $\mathbb{C}$ $(= X)$, and define T by neighborhood bases $B(z) = \{V\}$ as follows: if $z \in R(t)$, take $V = R(t) \cap U$ where U ranges over the open discs centered at z, and for $z = 0$ take the sets V obtained by removing from $\mathbb{C}$ a finite number of complete rays $R(t_1, R(t_2), \ldots, R(t_n)$ together with finitely many points from of each of countably many additional rays $R(t_{n+1}), R(t_{n+2}), \ldots$. Then

$$D = \{\exp(i/n);\ n = 1,2,\ldots\}$$

is totally discrete and closed, while $\{0\} \cup D$ is discrete but not totally discrete.

7. There are spaces satisfying (4.a) which are not regular (see [1], Ch. 1, §9, Ex. 21, c)) and compact spaces not satisfying (4.a) (e.g., X in 5. above or any uncountable power of the discrete two-point space). It is easy to see, however, that in regular spaces $D^* = \bar{D}$ for all $d \in \Theta$ and therefore similar arguments show that for T regular $\Theta(T)$ determines T. It may happen though that for a regular (Hausdorff) T satisfying (4.a) finer topologies exist having the same totally discrete subsets: take T to be ordinary topology on the real line, and $T' = T_0$ as defined above.

8. If T is Hausdorff and hereditarily Lindelöf, then all discrete subsets of T_0 are (countable, hence) T_0–closed. Then $A = \tilde{A} = A^*$ and $t(T_0)$ is the discrete topology.

In particular for the real line we have: (a) $t(T_0) \neq (tT)_0$ ($= T_0$); (b) the inclusions $\Theta(T_0) \subset \Delta(T_0) \subset \Theta(t(T_0))$ are proper (see [2]).

9. The topology tT is determined from T using totally discrete subsets in a way that parallels the definition of the k–space kT associated to T ([Engelking], p. 204). The properties of $T \to tT$ (unlike $T \to kT$) are not clear.

References

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