ON THE THREE-SPACE PROPERTY FOR LOCALLY CONVEX SPACES

by

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ABSTRACT

The purpose of this paper is to prove that some familiar classes of locally convex spaces and, in particular, of Fréchet spaces, have the so-called three-space property, and also to give counterexamples showing that some important classes, which are often encountered in the applications, do not enjoy the above property.

We use standard notions and results from the theory of locally convex spaces, for which we refer to [4] and [3]. We also make reference to [7] for what concerns operator ideals on Banach spaces, ideals of locally convex spaces (space ideals) and Grothendieck space ideals. Here we confine ourselves to recalling that space ideals with the three-space property are termed *three-space ideals.* This means that a class $\mathcal{C}$ of locally convex spaces is a three-space ideal if and only if the following conditions are satisfied:

(i) The finite-dimensional spaces belong to $\mathcal{C}$;
(ii) $\mathcal{C}$ is stable under isomorphisms;
(iii) If $E\in\mathcal{C}$ and F is a complemented subspace of E, then $F\in\mathcal{C}$;
(iv) *Three-space property:* If F is a subspace of E such that $F, E/F\in\mathcal{C}$, then $E\in\mathcal{C}$.

1. GENERAL LOCALLY CONVEX CLASSES

In this section we shall show that some general classes of locally convex spaces which are often encountered in the applications have the three-space property. The results are meant to supplement those in [8].

We shall consider the following classes of locally convex spaces

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$N\omega$ = the class of spaces which do not contain a copy of ω,
$N\varphi<$ = the class of spaces which do not contain a complemented copy of φ,
CON = the class of spaces with continuous norm,
Groth($\mathcal{A}$) = the Grothendieck space ideal generated by the operator ideal $\mathcal{A}$ (cf. [7]).

*1.1. **Proposition.*** $N\omega$ *is a three-space ideal.*

Proof. Let E be a locally convex space and let F be a closed subspace of E such that F and $E/F \in N\omega$. Suppose that E contains ω as a subspace. Then $F\cap\omega$ can be at most finite-dimensional, since $F \not\supset \omega$, and we may assume that $F \cap \omega = \{0\}$. If $q: E \to E/F$ is the quotient map, then its restriction q_ω to ω is a continuous bijection onto $q(\omega)$. By a well-known property of ω, q_ω must be also open, hence an isomorphism and, therefore, $E/F \supset q(\omega) \simeq \omega$. But this is a contradiction and we conclude that $E \in N\omega$.

*1.2. **Proposition.*** $N\varphi<$ *is a three-space ideal.*

Proof. Let E be such that F and $E/F \in N\varphi<$ for some closed subspace F of E. Supposing that $\varphi < E$, we denote by $p: E \to \varphi$ a continuous projection and let p_F be its restriction to F. If $\dim p(F) = \infty$, then $p(F) \simeq \varphi$. In this case $p_F: F \to p(F)$ is surjective and also open, because φ has the finest locally convex topology. But then $\varphi \simeq p(F)$ is a quotient of F and hence $\varphi < F$, as well known. Since this contradicts our hypothesis on F, we must have $\dim p(F) < \infty$. Without loss of generality we may then suppose that $p(F) = 0$. If $q: E \to E/F$ is the quotient map, a map $r: E/F \to \varphi$ may then be defined by $rq = p$. Note that r is well defined, since $\ker q \subset \ker p$, and is also continuous. Since r is surjective and automatically open, we obtain that $\varphi < E/F$ and again arrive at a contradiction. Thus $E \in N\varphi<$.

*1.3. **Remark.*** In the above proof we have used the fact that $N\varphi< = N\varphi Q$ (= the class of spaces which do not have φ as a quotient). Thus, since $N\omega = N\omega<$ (because ω is injective), we see that Propositions 1.1 and 1.2 are dual to each other.

*1.4. **Proposition.*** CON *is a three-space ideal.*

Proof. Suppose that the locally convex space E contains a subspace F for which F and $E/F \in$ CON and assume that $E \notin$ CON. By [9], Proposition 2, E contains a non-trivial, very strongly convergent net (x_α) (i.e., the net $(\xi_\alpha x_\alpha)$ converges to 0 for every net (ξ_α) of scalars). Now observe that the subnet $(x_\alpha) \cap F$ of

(x_α) cannot be cofinal for (x_α), otherwise it would be non-trivial and very strongly convergent in F, implying, again by Proposition 2 of [9], that F $\notin$ CON. But then, if $(x_\alpha) \cap F$ is not cofinal, its complement $(x_\alpha) \setminus (x_\alpha) \cap F$ must be cofinal for (x_α) and hence non-trivial and very strongly convergent. As a consequence, its image in E/F is non-trivial and very strongly convergent. But this is a contradiction, because it implies that E/F $\notin$ CON by the same Proposition 2 of [9].

1.5. Remark. Note that CON $\subsetneqq$ Nω (cf., e.g., [9], Example 1).

1.6. Remark. The ideal Nφ< is injective and surjective, while Nω and CON are injective but not surjective.

Now let E be a locally convex space and let U be an absolutely convex 0-neighbourhood in E. If p_U is the associated seminorm, we denote, as usual, by E_U the completion of the space E/ker p_U normed by the norm induced by p_U.

1.7. Remark. Let F be a closed subspace of E, let V = V $\cap$ F and let q : E $\to$ E/F be the quotient map. Then $F_V = F \cap E_U$ and $(E/F)_{q(U)} = E_U/F_V$, i.e. F_V is a closed subspace of E_U and $(E/F)_{q(U)}$ is a quotient of E_U.

Following [7] we denote by $\mathcal{H}$ the ideal of operators between Banach spaces that factor through a Hilbert space. Recall that Groth($\mathcal{H}$) is the Grothendieck ideal of *hilbertian* locally convex spaces. The following result implies that most Grothendieck ideals have the three-space property.

1.8. Theorem. Let $\mathcal{A}$ *be an arbitrary operator ideal on Banach spaces and let* E $\in$ Groth($\mathcal{H}$). *If* F *is a closed subspace of* E *such that* F *and* E/F $\in$ Groth($\mathcal{A}$), *then* E $\in$ Groth($\mathcal{A}$).

Proof. With reference to Remark 1.7 we observe that, if U_0 is an absolutely convex 0-neighbourhood in E contained in U, then the following self-explanatory, diagrams commute:

$$\begin{tikzcd} E_{U_0} \arrow[r, "\phi_{U_0U}"] & E_U \\ F_{V_0} \arrow[u, "i_0"] \arrow[r, "\phi_{V_0V}"'] & F_V \arrow[u, "i"'] \end{tikzcd} \qquad \begin{tikzcd} E_{U_0} \arrow[r, "\phi_{U_0U}"] \arrow[d, "q_0"'] & E_U \arrow[d, "q"] \\ (E/F)_{q_0(U_0)} \arrow[r, "\phi_{q_0(U_0),q(U)}"'] & (E/F)_{q(U)} \end{tikzcd}$$

where i_0, i are the canonical embeddings and q_0, q the quotient maps. Now, since $E \in \mathrm{Groth}(\mathcal{H})$, we may assume that both E_U and E_{U_0} are Hilbert spaces. Then

$$E_U = F_V \oplus (E/F)_{q(U)} \text{ and } E_{U_0} = F_{V_0} \oplus (E/F)_{q_0(U_0)}.$$

Hence, if U and U_0 are chosen so that ϕ_{V_0V} and $\phi_{q_0(U_0),q(U)} \in \mathcal{A}$, we have that also $\phi_{U_0U} \in \mathcal{A}$ and the assertion follows.

*1.9. **Corollary.** Let $\mathcal{A}$ be an operator ideal on Banach spaces. If* Groth($\mathcal{A}$) *is contained in the ideal of nuclear spaces, then* Groth($\mathcal{A}$) *is a three-space ideal.*

Proof. It is known that the nuclear spaces form a three-space ideal (cf. [8], Proposition 3.8). Thus if F and $E/F \in \mathrm{Groth}(\mathcal{A})$, then E is nuclear, hence hilbertian and Theorem 1.8 applies.

*1.10. **Corollary.** The λ-nuclear spaces (as defined in* [5]*) and, in particular, the ϕ-nuclear spaces and the strongly-nuclear spaces form a three-space ideal.*

1.11. ***Remark.*** We note that also the ideal Groth($\mathcal{W}$) ($\mathcal{W}$ = the weakly compact operators) of infra-Schwartz spaces is a three-space ideal, since it is well known that reflexive Banach spaces enjoy the three-space property. This is also true of the ideal Groth($\mathcal{K}$) ($\mathcal{K}$ = the compact operators) of Schwartz spaces (cf. [8], Proposition 3.7).

1.12. ***Remark.*** In general, if $\mathcal{A}$ is an operator ideal on Banach spaces, the locally convex space ideal Space ($\mathcal{A}$) is defined as follows: $E \in$ Space ($\mathcal{A}$) if (and only if) E has a basis $\mathcal{U}$ of absolutely convex 0-neighbourhoods such that $I_U \in \mathcal{A}$ for each $U \in \mathcal{U}$, where I_U is the identity of the Banach space E_U associated to U. In many cases Space($\mathcal{A}$) is a three-space ideal (cf. [2]). Note that Groth($\mathcal{W}$) = Space($\mathcal{W}$).

1.13. ***Remark.*** Finally, we observe that the assumption $E \in \mathrm{Groth}(\mathcal{H})$ in Theorem 8 is necessary, since Groth($\mathcal{H}$) itself is not a three-space ideal (cf. [1] and also Corollaries 2.4, 2.5 and Theorem 2.7 below).

We conclude this section with the following curious application:

1.14. ***Proposition.*** *Let* F *be a Schwartz (in particular, nuclear) subspace of* $(\ell^2)^{\mathbb{N}}$. *Then* $(\ell^2)^{\mathbb{N}}/F \simeq (\ell^2)^{\mathbb{N}}$.

Proof. It is easy to see that any quotient of $(\ell^2)^{\mathbb{N}}$ is isomorphic to ω, ℓ^2, $\omega \times \ell^2$ or $(\ell^2)^{\mathbb{N}}$.

We consider the various possibilities.

(a) $(\ell^2)^{\mathbb{N}}/F \simeq \omega$.

This case is impossible because, as remarked in 1.11, Schwartz spaces are a three-space ideal.

(b) $(\ell^2)^{\mathbb{N}}/F \simeq \ell^2$.

Put $X_n = \ell^2$ for all n and write $(\ell^2)^{\mathbb{N}} = \prod_n X_n$. Because F is Schwartz, $\dim(X_n/X_n \cap F) = \infty$ for all n and hence we can choose elements $x_n \in X_n \backslash X_n \cap F$ for each n. Denoting by H the closed linear span of the sequence (x_n) in $(\ell^2)^{\mathbb{N}}$, it is clear that $H \simeq \omega$ and that $\dim(H/H \cap F) = \infty$. It follows that, if G is any topological complement of $H \cap F$ in H, then $G \simeq \omega$ and $G \cap F = \{0\}$. If q: $(\ell^2)^{\mathbb{N}} \to (\ell^2)^{\mathbb{N}}/F$ is the quotient map, then its restriction to G is a continous bijection onto q(G) and hence an isomorphism. Thus $(\ell^2)^{\mathbb{N}}/F \supset q(G) \simeq \omega$ and we get a contradiction.

(c) $(\ell^2)^{\mathbb{N}}/F \simeq \omega \times \ell^2$.

With q as above, we see that $q^{-1}(\omega) = F \oplus G$ with $G \simeq \omega$, hence $q^{-1}(\omega)$ is a Schwartz space such that $(\ell^2)^{\mathbb{N}}/q^{-1}(\omega) \cong \ell^2$, which is impossible by case (b).

The only possible case left is $(\ell^2)^{\mathbb{N}}/F \simeq (\ell^2)^{\mathbb{N}}$, as claimed.

2. Special classes

The classes considered here will be the following ones:

QUO = the class of quojections,
PRO = the class of countable products of Banach spaces,
SUM = the class of countable direct sums of Banach spaces,
Groth($\mathcal{H} \circ \mathcal{K}$) = the Schwartz-Hilbert class, i.e. the class of locally convex spaces that are both Schwartz and hilbertian,

and we shall show that of the above classes, only QUO is a three-spaceideal. Recall that a *quojection* is the projective limit of a sequence of Banach spaces with respect to surjective maps and that, by [6], PRO $\subsetneq$ QUO. A space $E \in$ QUO\PRO is called *twisted.*

2.1. Proposition. QUO *is a three-space ideal.*

Proof. Let E be a locally convex space having a subspace F such that F and $E/F \in$ QUO. By Propositions 1.3 and 2.1 of [8] E is Fréchet. Let (p_n) be an increasing sequence of seminorms defining the topology of E. If $E_n = E/\ker p_n$, then E_n is a Fréchet space when endowed with the quotient topology from E. Moreover, the E_n have continuous norms and the canonical maps $\emptyset_n : E_{n+1} \to E_n$ are surjective.

Put $F_n = F/(F \cap \ker p_n)$; then F_n is a subspace of E_n on which the induced topology coincides with the quotient topology from F. Now F_n has a continuous norm under the former topology, while it is a quojection (for so is F by assumption) under the latter. Therefore, each F_n is Banach.

Next, denote by G the quotient E/F and by (q_n) the sequence of seminorms on G that are quotients of the p_n. Putting $G_n = G/\ker q_n$, one has that $G_n = E_n/F_n$ algebraically and the closed graph theorem shows that on G_n the quotient topology from G is identical to the quotient topology from E_n. Then, again, each G_n is a quojection with a continuous norm and hence is Banach. Thus each E_n has a Banach subspace F_n such that E_n/F_n is Banach: it follows from [8], Theorem 3.2, that the E_n are Banach and, therefore, $E \in$ QUO.

2.2. Theorem. PRO *does not have the three-space property.*

Proof. We construct a counterexample based on the method of [6]. Let $X \not\simeq \ell^2$ be a Banach space and let Y be a non-complemented subspace of X. Put

$$E_1 = (X/Y \oplus X/Y \oplus \ldots)_2$$

and, for each $n > 1$,

$$E_n = (\underbrace{X \oplus \ldots \oplus X}_{n-1} \oplus X/Y \oplus X/Y \oplus \ldots)_2 .$$

Clearly the quotient map $q : X \to X/Y$ induces, together with the identity of X/Y, obvious quotient maps $Q_n : E_{n+1} \to E_n$, so that the projective limit $E = \mathrm{proj}_n(E_n, Q_n)$ is a quojection. As in [6] E is twisted, hence $E \notin$ PRO. Now put

$$F_1 = (\{0\} \oplus \{0\} \oplus \ldots)_2$$

and, for each $n > 1$,

$$F_n = (\underbrace{Y \oplus \ldots \oplus Y}_{n\text{-}1} \oplus \{0\} \oplus \{0\} \oplus \ldots)_2 .$$

Clearly each F_n is a closed subspace of E_n and the restriction R_n of Q_n to F_{n+1} is onto F_n. It follows that $F = \text{proj}_n(F_n, R_n)$ is a closed subspace of E and that $F \simeq Y^{\mathbb{N}}$, so that $F \in$ PRO.

Next, observe that

$$E_n/F_n = (X/Y \oplus X/Y \oplus \ldots)_2$$

for all n, from which

$$E/F = \text{proj}_n(E_n/F_n) = (X/Y \oplus X/Y) \oplus \ldots)_2 ,$$

i.e., E/F is Banach and hence belongs to PRO. However, $E \notin$ PRO, as we have seen above.

*2.3. **Corollary**. There exists a twisted quojection* E *containing a subspace* $F \in$ PRO *and such that* E/F *is Banach.*

*2.4. **Corollary**. There exists a twisted quojection* E *containing a subspace* F *such that* $F \simeq (\ell^2)^{\mathbb{N}}$ *and* $E/F \simeq \ell^2$. *Moreover,* $E \notin \text{Groth}(\mathcal{H})$.

Proof. Let X be the Banach space Z in [1] having a subspace $Y \simeq \ell^2$ such that $X/Y \simeq \ell^2$, but $X \not\simeq \ell^2$. Clearly Y cannot be complemented in X and we obtain the desired result by taking this X and this Y in the proof of Theorem 2.2.

Now dualizing Corollary 2.4 we get

*2.5. **Corollary**. There exists a strict* (LB)*-space* $E \notin$ SUM *containing a subspace* G *such that* $G \simeq \ell^2$ *and* $E/G \simeq (\ell^2)^{(\mathbb{N})}$ (= *the direct sum of countably many copies of* ℓ^2). *Moreover,* $E \notin \text{Groth}(\mathcal{H})$.

*2.6. **Remark**.* By [6] the space E in Corollary 2.4 (resp., in Corollary 2.5) cannot have an unconditional basis, although, of course, both F and E/F (resp., G and E/G) have unconditional bases.

To conclude the paper, we recall that the Schwartz spaces form a three-space ideal, while the hilbertian ones do not. What happens for the Schwartz-Hilbert class, which is intermediate between the two? The answer is provided by the following

2.7. Theorem. The Schwartz-Hilbert ideal Groth($\mathcal{H} \circ \mathcal{K}$) *does not have the three-space property.*

Proof. The counterexample will be based on the Enflo-Lindenstrauss-Pisier space Z [1] already used in the proof of Corollary 2.4. For each n let $Z_n = \ell^2_n \oplus \ell^2_{n^2}$ with the norm $|||\ \ |||$ as in [1], p. 209. Put $Y_n = \ell^2_{n^2}$; then Y_n, under the norm induced by Z_n, is isometric to a Hilbert space and the same is true of Z_n/Y_n. Furthermore, for any projection P of $(Z_n, |||\ \ |||)$ onto $(Y_n, |||\ \ |||)$ one has

$$(*) \qquad \|P\| \geq c\,(\log n)^{1/2}$$

for a suitable $c > 0$. Let $Z = (\underset{n}{\oplus} Z_n)_2$; Z contains $Y = (\underset{n}{\oplus} Y_n)_2$ which is isometric to ℓ^2. Moreover, Z/Y is isometric to $(\underset{n}{\oplus} \ell^2_n)_2$, which is also isometric to ℓ^2, but Z is not even isomoprhic to ℓ^2 since Y is not complemented in Z.

Now for each n let I_n be the identity of Z_n; obviously $I_{n|Y_n}$ is the identity of Y_n. Choose a sequence of real numbers $\alpha_n > 0$ such that

$$(**) \qquad \lim_n \alpha_n = \infty \quad \text{and} \quad \lim_n \frac{(\alpha_n)^k}{\log n} = 0 \text{ for all k,}$$

and define the operator $T: Z \to Z$ by

$$T = \sum_n (\alpha_n)^{-1} I_n .$$

Clearly T is compact and has a dense range. Moreover, T maps Y into Y with a dense range and also induces a compact map $T_0 : Z/Y \to Z/Y$ which has a dense range too. Let E be the projective limit with respect to the powers T^k of T, i.e. $E = \mathrm{proj}_k(Z,T)$. If $F = \mathrm{proj}_k(Y,T)$, then F is a closed subspace of E and $E/F = \mathrm{proj}_k(Z/Y,T_0)$ by construction. Further, E,F and E/F are Schwartz, because T and T_0 are compact, while, clearly, F and E/F are also hilbertian. Thus, F and E/F $\in$ Groth($\mathcal{H} \circ \mathcal{K}$) and it remains to show that E $\notin$ Groth($\mathcal{H}$). This will be achieved by the following

2.8. Lemma. For any k,T^k *does not factor through* ℓ^2.

Proof. Supposing the contrary, for some k we have the diagram

$$\begin{tikzcd} Z \arrow[rr, "T^k"] \arrow[dr, "S"'] & & Z \\ & \ell^2 \arrow[ur, "R"'] & \end{tikzcd}$$

i.e. $T^k = RS$ and we may assume $\| S \| \leqslant 1$. Then, putting $R_n = R_{|\overline{S(Z_n)}}$ and $S_n = S_{|Z_n}$, from $T^k{}_{|Z_n} = (\alpha_n)^{-k} I_n$ we have the diagram

$$\begin{tikzcd} Z_n \arrow[rr, "(\alpha_n)^{-k} I_n"] \arrow[dr, "S_n"'] & & Z_n \\ & \ell^2 \arrow[ur, "R_n"'] & \end{tikzcd}$$

i.e. $(\alpha_n)^{-k} I_n = R_n S_n$, where we have set $R_n = 0$ on the orthogonal complement of $S(Z_n)$ in ℓ^2. Putting $A = \|R\| \geqslant \|R_n\|$, one has, for each $x \in Z_n$,

$$A^{-1}(\alpha_n)^{-k} \|x\| \leqslant |||\, x \,|||_{Z_n} \leqslant A(\alpha_n)^k \|x\|,$$

where $\| \;\|$ is the hilbertian norm on Z_n. But then, since there is a norm-one projection from $(Z_n, \| \;\|)$ onto $(Y_n, \| \;\|)$, there must be a projection P_n: $(Z_n, |||\; |||) \to (Y_n, |||\; |||)$ such that $|||P_n||| \leqslant A^2(\alpha_n)^{2k}$. However, this is absurd, by (*) and (**).

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