NEW CHARACTERIZATIONS OF T_0-SPACES

JOSEP GUIA, RAFAEL LLEDO, ISABEL SEGURA

ABSTRACT

Making use of the essential derived operator, we present some new characterizations of T_0-spaces in the same way that the already known for T_D and ET_D-spaces. It is also proved that an analogous treatment for T_1, $ET_1 = R_0$, T_{UD} and ET_{UD} spaces is not valid.

1. T_α-SPACES ($\alpha = 1$, D, UD, O)

(1.1) DEFINITION. In a topological space $(X, \mathcal{T})$, we call *essential derived* set of a subset A of X, the set $DA = \overline{A} \setminus \cup \{ <x> / x \in A \setminus dA \}$, where $\overline{A}$ is the closure of A, dA is the derived set of A and $<x>$ is *the covering of x*, that is $<x> = \{\overline{x}\} \cap (\cap \{ 0/0 \in \mathcal{T}, x \in 0 \})$.

(1.2) PROPOSITION. In a topological space $(X, \mathcal{T})$, if A is subset of X, then $DA = \cup \{ F/F \text{ closed}, F \subset dA \}$.

PROOF. Let x be a point of $\cup \{ F/F \text{ closed}, F \subset dA \}$, then x belongs to $\overline{A}$ and $<x> \subset \{\overline{x}\} \subset F \subset dA$ for a closed set F included in dA. On the other hand, $<x> \neq <y>$ for every point y of $dA \setminus A$ and $<x> \cap (\cup \{ <y> / y \in A \setminus dA \}) = \emptyset$, hence $x \in \overline{A} \setminus \cup \{ <y> / y \in A \setminus dA \}$.

Conversely, let us assume that there is a point x in DA such that $\{\overline{x}\} \not\subset dA$, then $\{\overline{x}\} \cap (\overline{A} \setminus dA)$ is not empty and for every point y in $\{\overline{x}\} \cap (\overline{A} \setminus dA)$, x belongs to every open set containing y and there exists $O_y \in \mathcal{T}$ such that $O_y \cap A = \{y\}$. On the other hand, y belongs to $<x>$, otherwise there would exist an open set O_x with $x \in O_x$ and $y \notin O_x$ and thus, $O_x \cap O_y$ would be an open set containing x for which $O_x \cap O_y \cap A = \emptyset$, against the fact that $x \in dA \subset \overline{A}$. Therefore $\{\overline{x}\} \subset DA$ for every $x \in DA$ and consequently $DA \subset \cup \{ F/F \text{ closed}, F \subset dA \}$. #

(1.3) DEFINITION. [1]. A topological space (X, $\mathcal{T}$) is a T_1, T_D, T_{UD} or T_0-space if, for every point x of X, the correspondent following assertion holds:

(T_1)	d { x }	is empty.
(T_D)	d { x }	is closed.
(T_{UD})	d { x }	is a union of disjoint closed sets.
(T_0)	d { x }	is a union of closed sets.

(1.4) PROPOSITION. [1]. A topological space (X, $\mathcal{T}$) is T_D iff, for every subset A of X, dA is closed.

(1.5) PROPOSITION. A topological space (X, $\mathcal{T}$) is T_0 iff one of the following conditions holds:

1) For every point x of X, d { x } = D { x }
2) For every subset A of X, dA = DA
3) For every subset A of X, dA is a union of clossed sets.

PROOF. From (1.2) it is immediate that d { x } = D { x } for every point x of X, iff (X, $\mathcal{T}$) is T_0. If (X, $\mathcal{T}$) is T_0, that is $< x > = \{x\}$ for every point x of X, then DA = $\bar{A}\setminus$ {x/x $\in$ A\dA } = dA for each subset A of X. If dA = DA, A $\subset$ X, it follows from (1.2) that dA is a union of closed sets. Finally, the last statement implies trivially that (X, $\mathcal{T}$) is T_0. #

The third characterization of T_0-spaces given in (1.5) may be obtained without taking into account the essential derived operator, as we prove in the following proposition.

(1.6) PROPOSITION. A topological space (X, $\mathcal{T}$) is T_0 iff, for every subset A of X, dA is a union of closed sets.

PROOF. If A is a subset of the T_0-space (X, $\mathcal{T}$) such that dA is not a union of closed sets, there exists a point x $\in$ dA for which $\{\bar{x}\} \cap (\bar{A}\setminus dA)$ is not empty. The same reasoning that the one followed in the proof of (1.2) would prove the existence of a point y in $< x >$ different from x, against the fact that (X, $\mathcal{T}$) is a T_0-space.

The inverse is immediate. #

(1.7) REMARK. The statement "for every subset A of X, dA is a union of disjoint closed sets" is strictly stronger than the T_{UD}-axiom, as it is shown in the following example. The spaces defined by the above statement will be called T_{UD}^A-spaces.

(1.8) EXAMPLE. Let X be the set of real numbers and let the closed sets be X, $\emptyset$, $\{x\}$ $(0 < x \leqq 1)$, $\{-x,x\}$ $(0 < x \leqq 1)$, $]x,1]$ $(x < -1)$, $[x,1]$ $(x \leqq -1)$, $]0,x[$ $(x > 1)$, $]0,x]$ $(x \geqq 1)$ and their finite unions. This space is a T_{UD}-space but, for $A = \{0\} \cup [1,2]$, dA is not a union of disjoint closed sets.

(1.9) REMARK. The statement "for every subset A of X, dA is empty" is strictly stronger than the T_1-axiom. In fact, it characterizes the discrete spaces [3].

2. ET_α-SPACES ($\alpha = 1$, D, UD, 0)

(2.1) DEFINITION. A topological space (X, $\mathcal{T}$) is a ET_1 ("essentially T_1"), ET_D, ET_{UD} or ET_0-space if its T_0-identification space [5] is a T_1, T_D, T_{UD} or T_0-space, respectively.

(2.2) REMARK. It is known that ET_1-spaces and ET_0-spaces are respectively the classes of R_0-spaces [3] and all topological spaces [4].

(2.3) PROPOSITION [2]. A topological space (X, $\mathcal{T}$) is a ET_1, ET_D, ET_{UD} or ET_0-space iff, for every point x of X, the correspondent following assertion holds:

(ET_1)	$D\{x\}$	is empty.
(ET_D)	$D\{x\}$	is closed.
(ET_{UD})	$D\{x\}$	is a union of disjoint closed sets.
(ET_0)	$D\{x\}$	is a union of closed sets.

(2.4) PROPOSITION. A topological space (X, $\mathcal{T}$) is either ET_D or ET_0 iff, for every subset A of X, the respective following assertion holds:

(ET_D)	DA	is closed.
(ET_0)	DA	is a union of closed sets.

PROOF. The first assertion is stated in [2]. The other one follows from (2.1). #

(2.5) REMARK. The statement "for every subset A of X, DA is a union of disjoint closed sets" is strictly stronger than the ET_{UD}-axiom, as it is shown in the following example. The spaces defined by the above statement will be called ET^A_{UD}-spaces.

(2.6) EXAMPLE. Let X be the set of real numbers and let the closed sets be X, $\emptyset$, $\{x\}$ $(0 < x \leqq 1)$, $\{-x, x\}$ $(0 < x \leqq 1)$, $[-1, 1]$, $]-x, -1[\cup]0, x[$ $(x > 1)$,

$[-x, -1[\cup]0, x]$ $(x \geqq 1)$ and their finite unions. This space is a ET_{UD}-space but not T_{UD} and, for $A = \{0\} \cup [1,2]$, DA is not a union of disjoint closed sets.

(2.7) ***REMARK.*** The statement "for every subset A of X, DA is empty" is istrictly stronger than the ET_1-axiom. In fact, it characterizes the E-discrete spaces [3].

In the following diagram aie related all axions mentioned in the present paper:

$$\begin{array}{ccccccccccc}
\text{discrete spaces} & \longrightarrow & T_1 & \longrightarrow & T_D & \longrightarrow & T^A_{UD} & \longrightarrow & T_{UD} & \longrightarrow & T_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{E-discrete spaces} & \longrightarrow & ET_1 = R_0 & \longrightarrow & ET_D & \longrightarrow & ET^A_{UD} & \longrightarrow & ET_{UD} & \longrightarrow & ET_0 = \text{arbitrary spaces}
\end{array}$$

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Department of Algebra,
University of Valencia,
Burjassot (Valencia), Spain

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