SOME RESULTS ABOUT THE SIZE OF THE EXCEPTIONAL SET IN NEVANLINNA'S SECOND FUNDAMENTAL THEOREM

by

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ABSTRACT

Let F be a meromorphic function in the plane. Some conditions are given on the size of the set of positive real numbers, outside which the term S(r,F) which arises in the logarithmic derivative Lemma is small compared with the characteristic function T(r,F) of F.

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1. INTRODUCTION

Let F(z) be a meromorphic function in the plane. We shall use the usual notation of Nevanlinna theory. For any complex value a we define

$$m(r,a) = \frac{1}{2\P} \int_0^{2\P} \log^+ \frac{1}{|F(re^{i\theta}) - a|} \, d\theta \ ,$$

$$N(r,a) = \int_0^r \frac{n(t,a) - n(0,a)}{t} \, dt + n(0,a) \log r \ ,$$

where n(t,a) denotes the number of roots according with their multiplicities of the equation F(z) = a in $|z| \leqslant t$.

Similarly we define

$$m(r,\infty) = \frac{1}{2\P} \int_0^{2\P} \log^+ |F(re^{i\theta})| \, d\theta$$

$$N(r,\infty) = \int_0^r \frac{n(t,\infty) - n(0,\infty)}{t} + n(0,\infty)\log r.$$

The function

$$T(r,F) = m(r,\infty) + N(r,\infty),$$

is called the characteristic function of the meromorphic function F.

Next we state the second fundamental theorem of Nevanlinna.

Theorem A.– *Let* F(z) *be a meromorphic function in the plane. Let* r *be a positive real number,* $0 \leqslant r < \infty$, *and* $a_1, a_2, \ldots, a_q$, *are* $q > 2$ *distinct values of the extended complex plane such that* $|a_\mu - a_\nu| \geqslant \delta$, $1 \leqslant \mu < \nu \leqslant q$, *for a certain* $\delta > 0$. *Then*

$$(q-2)\,T(r,F) < N(r,a_1) + N(r,a_2) + \ldots + N(r,a_q) - N_1(r) + S(r), \quad (1.1)$$

where $N_1(r)$ *is a positive term given by*

$$N_1(r) = N(r,\frac{1}{F'}) + 2N(r,F) - N(r,F'),$$

and

$$S(r,F) = m(r,\frac{F'}{F}) + m(r, \sum_{\nu=1}^{q} \frac{F'}{F-a_\nu}) + q\log^+ \frac{3q}{\delta} + \log 2$$

$$+ \log \frac{1}{F'(0)}, \quad (1.2)$$

with modifications if $F(0) = \infty$ *or* $F'(0) = 0$.

The quantity S(r,F) will in general be negligible with respect T(r,F) and the combination of Theorem A and the estimation for S(r,F) constitute Nevanlinna's second fundamental theorem.

The following theorem due to R. Nevanlinna gives an estimation for S(r,F),

Theorem B.– *Suppose that* F *is a meromorphic function in the plane,* S(r,F) *is defined by* (1.2) *and* λ *is a positive fixed number, then we have*

$$S(r,F) = O(\log T(r,F)) + O(\log r), \quad (1.3)$$

as $r \to \infty$ *through all values if* $F(z)$ *has finite order and otherwise as* $r \to \infty$ *outside a set* E_λ *satisfying*

$$\int_{E_\lambda} r^\lambda dr < \infty . \tag{1.4}$$

In this paper we show that if we consider, instead of (1.3), the weaker condition,

$$S(r,F) = o(T(r,F)), \tag{1.5}$$

we obtain an stronger conclusion than (1.4). We also give a new condition of a different type that (1.4) on the size of the exceptional set, outside which, (1.5) holds. These conditions turn out to be sharp as is proved in [1].

2. STATEMENT OF THE RESULTS

Theorem 1. – *The error term* $S(r,F)$ *in Nevanlinna's second fundamental theorem satisfies* (1.5), *i.e.*

$$S(r,F) = o(T(r,F)),$$

as $r \to \infty$ *outside a set* E, *independent of* λ, *such that*

$$\int_E r^\lambda dr < \infty \tag{2.1}$$

for every $\lambda > 0$.

Theorem 2. – *The error term* $S(r,F)$ *satisfies* (1.5), *as* $r \to \infty$ *outside a set* E, *which can be contained in a sequence of intervals* $[r_n, r_n + \delta_n]$, *such that*

$$\delta_n < \frac{1}{\Psi(n)^2} \quad \textit{where } \Psi(1) = 1, \ \Psi(n) = e^{\Psi(n-1)}. \tag{2.2}$$

Both conditions (2.1) and (2.2) imply that the exceptional set has finite measure but they are differente in character, i.e. neither implies the other for $\lambda \geqslant 0$.

The condition (2.2) gives a limitation only on the size of the intervals $[r_n, r_n + \delta_n]$ and (2.1) takes into account the position of the exceptional set.

3. SOME PRELIMINARY RESULTS OF THEOREM 1

In the proof of Theorem 1 we shall use the following result which implies in particular Theorem B.

Theorem C.– *Suppose that* F *is a meromorphic function in the plane and* $\Phi(r)$ *an increasing function for which there exists a constant* C *such that*

$$\Phi(r+1) \leqslant C\ \Phi(r),$$

then

$$S(r,F) \leqslant 20 \log^{+} T(r,F) + 12 \log^{+} \Phi(r) + 10 \log^{+} r + \text{constant} \qquad (3.1)$$

outside a set E_Φ satisfying

$$\int_{E_\Phi} \Phi(r)\, dr < \infty$$

In the proof of Theorem C we shall use the following lemmas,

Lemma 3.1. (Logarithmic derivative lemma).– *Let* F(z) *be meromorphic in the plane. For* $0 \leqslant r \leqslant R$, *we have*

$$m(r, \frac{F'}{F}) < 4 \log^{+} T(R,F) + 4 \log^{+} \log^{+} \frac{1}{|F(0)|} +$$

$$+ 5 \log^{+} R + 6 \log^{+} \frac{1}{R-r} + \log^{+} \frac{1}{r} + 14$$

Lemma 3.2.– *Suppose* T(r) *continuous, increasing and* $T(r) \geqslant 1$ *for* $r_0 \leqslant r < +\infty$ *and* $\Phi(r)$ *increasing for* $r_0 \leqslant r < +\infty$ *such that*

$$\Phi(r+1) \leqslant C\ \Phi(r),\ r \geqslant r_0$$

for a certain constant C. *Then we have*

$$T\left(r + \frac{1}{\Phi(r)\, T(r)}\right) < 2\, T(r),$$

outside a set E_Φ *satisfying*

$$\int_{E_\Phi} \Phi(r)\, dr < \infty$$

This is Borel's lemma amplified. The proof is the same as the one given in [2] pag. 36.

4. Proof of theorem C

S(r,F) was defined in (1.2) and it can be written as

$$S(r,F) = m(r, \frac{F'}{F}) + m(r, \frac{G'}{G}) + \text{constant}, \tag{4.1}$$

where

$$G(z) = \prod_{\nu=1}^{q} (F(z) - a_\nu),$$

By Lemma 3.1 we have for $0 \leqslant r < R$

$$m(r, \frac{G'}{G}) < 4 \log^+ T(R,G) + 5 \log^+ R + 6 \log^+ \frac{1}{R-r} + \text{constant}$$

for r bigger than a certain $r_0 > 0$.

We take $R = r + \dfrac{1}{\Phi(r)\, T(r,F)}$. Then for r large we obtain

$$5 \log^+ R < 5 \log^+ r + \text{constant},$$

$$6 \log^+ \frac{1}{R-r} = 6 \log^+ \Phi(r) + 6 \log^+ T(r,F)$$

and by Lemma 3.2 we have

$$4 \log^+ T(R,G) \leqslant 4 \log^+ (qT(R,F) + \text{constant} \leqslant 4 \log^+ T(r,F) + \text{constant}.$$

outside a set E_Φ satisfying $\displaystyle\int_{E_\Phi} \Phi(r)\, dr < \infty$.

Thus

$$m(r, \frac{G'}{G}) \leqslant 10 \log^+ T(r,F) + 6 \log^+ \Phi(r) + 5 \log^+ r + \text{constant} \tag{4.2}$$

and in particular

$$m(r,\frac{F'}{F}) \leqslant 10 \log^+ T(r,F) + 6 \log \Phi(r) + 5 \log^+ r + \text{constant} \quad (4.3)$$

With (4.1), (4.2) and (4.3) we conclude

$$S(r,F) \leqslant 20 \log^+ T(r,f) + 12 \log^+ \Phi(r) + 10 \log^+ r + \text{constant},$$

which is (3.1).

5. PROOF OF THEOREM 1

We may assume that F is transcendental, since otherwise there is no exceptional set. Then.

$$\frac{\log r}{T(r,F)} \to 0 \text{ as } r \to \infty .$$

Let a(r) be an increasing function such that

$$a(r) \to \infty , \frac{a(r) \log r}{T(r)} \to 0 , r \to \infty$$

$$(a(r+1) - a(r)) \log r \leqslant C_1 , \frac{a(r)}{r} \leqslant C_2 , \forall r \geqslant r_0 ,$$

and set

$$\Phi(r) = r^{a(r)}$$

Then

$$\frac{\Phi(r+1)}{\Phi(r)} \leqslant C$$

so that by Theorem C

$$S(r,F) \leqslant 20 \log^+ T(r,F) + 12 \log^+ \Phi(r) + 10 \log^+ r + \text{constant} =$$

$$= 20 \log^+ T(r,F) + 12 \max(0, a(r) \log r) + 10 \log^+ r + \text{constant} =$$

$$= 0(T(r)),$$

outside a set E_Φ with

$$\int_{E_\Phi} \Phi(r)\, dr < \infty .$$

Since $\Phi(r) \geqslant r^\lambda$, for $r \geqslant r_\lambda$, for every $\lambda \geqslant 0$, we conclude $\int_{E_\Phi} r^\lambda dr < \infty$. and the proof of Theorem 1 is complete.

6. AN AUXILIARY LEMMA TO THEOREM 2

In the proof of Theorem 2 we shall use the following lemma.

Lemma 6.1.– Suppose that $T(r)$ *is a continuous, increasing real-valued function for* $r_0 \leqslant r < \infty$ *, and that* $T(r) \geqslant 1$ *there. Then we have*

$$T(r + \frac{1}{T(r)}) < e^{2T(r)^{1/2}} \tag{6.1}$$

outside an exceptional set contained in a union of intervals $\bigcup_n [r_n, r_n + \delta_n]$, *such that* δ_n *satisfies*

$$\delta_n < \frac{1}{\Psi(n)^2} \quad \textit{where } \Psi(1) = 1,\ \Psi(n) = e^{\Psi(n-1)},\ \textit{i.e.}\ (2.2)$$

We set $t(r) = T(r)^{1/2}$, then (6.1) becomes

$$t(r + \frac{1}{t(r)^2}) < e^{t(r)}. \tag{6.2}$$

Let r_1 be the lower bound of all $r \geqslant r_0$ such that (6.2) is false or equivalently the first value for which

$$t(r_1 + \frac{1}{t(r_1)^2}) \geqslant e^{t(r_1)}.$$

We write $r_1' = r_1 + t(r_1)^{-2}$ and let r_2 be the lower bound of all $r \geqslant r_1'$ such that (6.2) is false. We can define in this way a sequence r_n writing

$$r_{n-1}' = r_{n-1} + \frac{1}{t(r_{n-1})^2},$$

and defining r_n as the lower bound of all $r \geqslant r'_{n-1}$ such that

$$t(r + \frac{1}{t(r)^2}) \geqslant e^{t(r)}.$$

The excepcional set is contained in the union

$$\bigcup_n [r_n, r'_n] = \bigcup_n [r_n, r_n + \frac{1}{t(r_n)^2}].$$

We write $\delta_n = t(r_n)^{-2}$ and since

$$t(r_n) \geqslant t(r'_n) \geqslant e^{t(r_{n-1})} \text{ and } t(r_0) \geqslant 1,$$

we obtain by induction

$$t(r_n) \geqslant \Psi(n+1) \geqslant \Psi(n),$$

and then we conclude

$$\delta_n = \frac{1}{t(r_n)^2} \leqslant \frac{1}{\Psi(n)^2}.$$

This completes the proof of Lemma 6.1.

7. PROOF OF THEOREM 2

Again we write S(r,F) in the form

$$S(r,F) = m(r,\frac{F'}{F}) + m(r,\frac{G'}{G}) + \text{constant}, \tag{7.1}$$

where

$$G(z) = \prod_{\nu=1}^{q} (F(z) - a_\nu).$$

By the logarithmic derivative Lemma, we have for $0 \leqslant r < R$

$$m(r,\frac{G'}{G}) \leqslant 4 \log^+ T(R,G) + 5 \log^+ R + 6 \log \frac{1}{R-r} + \log^+ \frac{1}{r} + 14.$$

Now, we take

$$R = r + \frac{1}{T(r,F)},$$

then with the same argument of Theorem C but using Lemma 6.1 instead of Lemma 3.2 we obtain that

$$S(r,F) \leqslant 16\, T(r,F)^{1/2} + 12 \log^{+} T(r,F) + 10 \log^{+} r + \text{constant} =$$

$$= o\,(T(r,F)),$$

outside a set, which can be contained in a sequence of intervals $[r_n, r_n + \delta_n]$ with δ_n satisfying (2.2) and the proof of Theorem 2 is finished.

BIBLIOGRAPHY

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