DUALITY OF TENSOR PRODUCTS OF CONVERGE-FREE SPACES

by

GOTTFRIED KÖTHE

ABSTRACT:

Let λ,μ be two perfect convergence-free spaces. We prove the duality theorem

$$(\lambda \tilde{\otimes}_n \mu)'_n \cong \lambda^x \tilde{\otimes}_\epsilon \mu^x \text{ and } (\lambda \tilde{\otimes}_\epsilon \mu)'_n \cong \lambda^x \tilde{\otimes}_n \mu^x$$

where n denotes the normal topology T_n which coincides which Grothendieck's inductive topology on $\lambda \otimes \mu$.

1. INTRODUCTION

For two (F)-spaces E and F one has the duality

$$(1)\ (E \tilde{\otimes}_\pi F)'_c \cong E'_c \,\epsilon\, F'_c \quad \text{and} \quad (E'_c \epsilon F'_c)'_c \cong E \tilde{\otimes}_\pi F.$$

The sign $\cong$ means a topological isomorphism. In this form it is recorded in [5] § 45, 3. (1). This is a slight generalitzation of a theorem of Buchwalter. Defant and Floret investigated in [1] for which classes of quasi-complete locally convex spaces one or both of these relations are true.

I was lately interested in the class of convergence-free spaces and studied also completed tensor products of these spaces in [8] but I did not look for a duality theorem for these tensor products. This I will do now and I will prove in section 4 that a very satisfactory duality exists but it is different from the duality (1).

2. PRELIMINARIES

Since my theory of convergence-free spaces is widely unknown I will repeat the necessary definitions and state the theorems which I will need later.

Let $\mathfrak{W}$ be a class of subsets of the set $\mathbb{N}$ of natural numbers with the properties: a) All finite subsets of $\mathbb{N}$ are in $\mathfrak{W}$, b) with W every subset of W is in $\mathfrak{W}$, c) $W_1 \cup W_2 \in \mathfrak{W}$ if W_1 and W_2 are in $\mathfrak{W}$. Then $\lambda_{\mathfrak{W}}$ is the space of all sequences $x = (x_i), i = 1,2, \ldots$, of real or complex numbers, whose non vanishing coordinates have indices which form a set $W \in \mathfrak{W}$. W is called the support of x. These spaces $\lambda_{\mathfrak{W}}$ are the convergence-free spaces and $\mathfrak{W}$ is the class of W-sets of λ.

An $x \neq 0$ is called positive if all non vanishing coordinates are positive.

The α-dual $(\lambda_{\mathfrak{W}})^\times$ consists of all $u = (u_i)$ with a support F for which $F \cap W$ is finite for every $W \in \mathfrak{W}$. These sets F satisfy a), b) and c) so they constitute a class $\mathfrak{W}$ and we have $(\lambda_{\mathfrak{W}})^\times = \lambda_{\mathfrak{W}^\times}$. The elements of $\mathfrak{W}^\times$ are also called the F-sets of $\lambda_{\mathfrak{W}}$. If $x \in \lambda_{\mathfrak{W}}$, $u \in \lambda_{\mathfrak{W}^\times}$, then we have the scalar product $\langle u,x\rangle = ux = \sum_{i=1}^{\infty} u_i x_i$ which is always a finite sum.

We recall that the α-dual of $\lambda_{\mathfrak{W}^\times}$ is $\lambda_{\mathfrak{W}^{\times\times}}$ and $\lambda_{\mathfrak{W}}$ is called perfect if $\mathfrak{W} = \mathfrak{W}^{\times\times}$. That $\lambda_{\mathfrak{W}}$ is perfect means also that $\lambda_{\mathfrak{W}}$ is complete for the Mackey topology $T_k(\lambda_{\mathfrak{W}^\times})$ (cf. [4] § 30, 5. (9)). Other topologies on $\lambda_{\mathfrak{W}}$: The weak topology $T_s(\lambda_{\mathfrak{W}^\times})$, the normal topology $T_n(\lambda_{\mathfrak{W}^\times})$ with the seminorms $p_u(x) = \sum_{i=1}^{\infty} |u_i|\,|x_i|$, $u \in \lambda_{\mathfrak{W}^\times}$, the strong topology $T_b(\lambda_{\mathfrak{W}^\times})$ and the topology $T_c(\lambda_{\mathfrak{W}^\times})$, the topology of uniform convergence on all absolutely convex compact sets of $\lambda_{\mathfrak{W}^\times}$.

In the following λ, μ will denote convergence-free spaces which will be defined by families $\mathfrak{W}$ which we will omit in the notation.

If M is a subset of $\mathbb{N}$, then λ_M will consist of all $x_M = (x_j)$, $j \in M$, $x \in \lambda$. λ_M is called a sectional subspace of λ and we have $(\lambda_M)^\times = (\lambda^\times)_M$. If M is a W-set of λ, then λ_M can be identified with ω, if M is an F-set of λ, then $\lambda_M = \varphi$.

The normal cover P^n of a subset P of a sequence space E consists of all sequences y such that $|y_i| \leq |x_i|$, $i = 1,2, \ldots$, for some $x \in P$. So the normal topology $T_n(E^\times)$ on a sequence space E is the topology of the uniform convergence on the normal covers $\{u\}^n$ of the elements $u \in E^\times$.

(1) *A bounded subset* B *of a perfect convergence-free space* λ *is contained in the normal cover of an element* x *of* λ *and is therefore contained in an absolutely convex compact subset of* λ.

Proof: The first statement follows from [9], p. 221, Satz 3. Hence $B \subset \{x\}^n \subset \lambda_W$, W the support of x. Now λ_W is ω and a bounded subset of ω is contained in an absolutely compact subset. Since $\lambda^\times$ is perfect (1) implies

(2) *On a convergence-free space* λ *the topologies* $T_c(\lambda^\times)$, $T_n(\lambda^\times)$, $T_k(\lambda^\times)$ *and* $T_b(\lambda^\times)$ *coincide.*

In the following we will tacitly assume that λ is equipped with this topology. (2) implies that every convergence-free space is barrelled. It is also ultrabornological (therefore bornological) (cf. [7], p. 158 (2)) and nuclear by [6], p. 128, (11). If λ is perfect it is also reflexive ([6], p. 128 (9)).

No doubt, the convergence-free spaces are a very well behaved class of sequence spaces (cf. [3], [6] and [7], where a more detailed exposition of their properties is given).

I add some remarks to enrich the picture. We will use in the following the composition $\lambda\mu$ of two convergence-free spaces. If $x = (x_i)\epsilon\lambda$ we replace $x_i \neq 0$ by an element $y^{(i)}\epsilon\mu$, and $x_i = 0$ by $y^{(i)} = (0,0,\ldots)$. The space of all these double sequences $z = (y^{(1)}, y^{(2)}, \ldots)$ is $\lambda\mu$. It is again a sequence space if we rearrange all the double sequences in the same way in sequences. In the following we will write the elements z of $\lambda\mu$ as infinite matrices with the $y^{(i)}$ as columns.

$\lambda\mu$ is again convergence-free and is perfect if and only if λ and μ are perfect. One has $(\lambda\mu)v = \lambda(\mu v)$ and $(\lambda\mu)^x = \lambda^x\mu^x$. This is easy to see (cf. [6], p. 129).

It follows from (1) that every convergence-free space is "locally complete" in the sense that every bounded subset lies in a complete sectional subspace. It seems at the first moment difficult to find a convergence-free space which is not complete.

I gave only one example on p. 126 of [6] which is rather pathological. A natural way to construct noncomplete convergence-free spaces is the following:

On [5], p. 411 it is shown that the sum $\varphi\omega + \omega\varphi$, both spaces considered as subspaces of $\omega\omega$ is convergence-free. $(\varphi\omega + \omega\varphi)^x = \varphi\varphi$, but $\varphi\omega + \omega\varphi$ is a proper subspace of $(\varphi\varphi)^x = \omega\omega$, hence $\varphi\omega + \omega\varphi$ is not complete.

We generalize: For a convergence-free $\lambda \subset \omega$ one has $(\lambda + \lambda^x)^x = \lambda^x \cap \lambda = \varphi$, hence $(\lambda + \lambda^x)^{xx} = \omega$. Hence if $\lambda + \lambda^x \neq \omega$ then $\lambda + \lambda^x$ is not complete.

Further for any convergence-free space $\mu \subset \omega$ which has a sectional subspace $\mu_M = \lambda$ for which $\lambda + \lambda^x$ is incomplete, the sum $\mu + \mu^x$ is not complete.

3. Tensor products of convergence-free spaces

Let λ, μ be convergence-free then $\lambda\otimes\mu$ consists of all elements $a = \sum_{p=1}^{m} \sum_{q=1}^{m} x^{(p)} \otimes y^{(q)}$, $x^{(p)}\epsilon\lambda$, $y^{(q)}\epsilon\mu$. Then $Aa = \sum_{p=1}^{m} \sum_{q=1}^{m} ((x^{(p)}u)y^{(q)}$, $u\epsilon\lambda^x$, is a continuous mapping of λ^x into μ.

The correspondence $a \to A$ is an algebraic isomorphism of $\lambda\otimes\mu$ onto the subspace $F(\lambda^x, \mu)$ of all linear maps of finite rank of $L(\lambda^x, \mu)$ the space of all linear continuous mappings of λ^x in μ (cf. [5], § 41, 3. (7)).

If $x = (x_k) \in \lambda$, $y = (y_i) \in \mu$, then $x\otimes y$ is represented by the matrix $(y_i\, x_k)$ and $\lambda\otimes\mu$ can be identified with the linear span of all these matrices. Hence $\lambda\otimes\mu$ is again a sequence space with double indices (i,k), i,k = 1,2, . . .

$\lambda \otimes \mu$ *is in general not convergence-free:* $\omega \otimes \omega = F(\varphi,\omega)$ and $\mathbb{N} \times \mathbb{N}$ is the W-set of the matrix $e \otimes e$, $e = (1,1,\ldots)$, but $F(\varphi,\omega)$ does not contain all matrices with this W-set. The smallest convergence-free space containing $\omega \otimes \omega$ is $L(\varphi, \omega) = \omega\omega$.

We have in general

(1) *The normal cover* $(\lambda \otimes \mu)^n$ *of* $\lambda \otimes \mu$ *is the smallest convergence-free space containing* $\lambda \otimes \mu$.

Proof: If $x \in \lambda$ has the support M, $y \in \mu$ the support N then $x \otimes y$ has the support $M \times N$. Using property c) of the W-sets one sees that the support of any element of $\lambda \otimes \mu$ is contained in a set $W_1 \times W_2$, W_1 a W-set of λ, W_2 a W-set of μ. This implies (1) if we recall [6], p. 132 (7), which says that to a positive matrix (p_{ik}), $i \in M$, $k \in N$, there exists always a positive matrix $(y_i\, x_k)$, $i \in M$, $k \in N$, such that $p_{ik} \leqslant y_i\, x_k$.

(2) *If* λ,μ *are perfect covergence-free, then* $(\lambda \otimes \mu)^n$ *is also perfect.*

The α-dual $[(\lambda \otimes \mu)^n]^x$ consists if all matrices $U = (u_{ik})$ such that $\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} |a_{ik}| \cdot |u_{ik}| < \infty$ for all $(a_{ik}) \in (\lambda \otimes \mu)^n$. Since $(\lambda \otimes \mu)^n$ is convergence-free these double sums are always finite.

If W_1, W_2 are W-sets of λ, μ and F_1, F_2 are F-sets of λ, μ, then $W_1 \times W_2 \cap F_1 \times F_2$ is finite, so $F_1 \times F_2$ is a W-set of $[(\lambda \otimes \mu)^n]^x$. A W-set M of $[(\lambda \otimes \mu)^n]^{xx}$ must therefore have a finite intersection with every $F_1 \times F_2$. Let $M = \{ (j,l) \}$ be an infinite set of pairs of indices. The set M_1 of all j must have a finite intersection with every F_1 and is therefore a W-set of λ, since λ is perfect. Similarly $M_2 = \{ l \}$ is a W-set of μ and M therefore a W-set of $(\lambda \otimes \mu)^n$. Hence $[(\lambda \otimes \mu)^n]^{xx} = (\lambda \otimes \mu)^n$.

We write this result in a different way, we look at $\lambda \otimes \mu$ as a sequence space and as such he has the natural topology T_n, the normal topology. Hence we write $\lambda \otimes_n \mu$ for $\lambda \otimes \mu$ equipped with T_n. Then (2) can be written as.

(2′) *Let* λ,μ *be perfect convergence-free. The completion* $\lambda \tilde{\otimes}_n \mu$ of $\lambda \otimes_n \mu$ *is the normal cover of* $\lambda \otimes \mu$.

Our next step will be the concrete determination of the α-dual of $\lambda \otimes \mu$. We obtain for perfect λ, μ.

(3) $(\lambda \otimes \mu)^x = [(\lambda \otimes \mu)^n]^x = (\lambda \tilde{\otimes}_n \mu)^x = L(\lambda, \mu^x)$

Proof: In [6], p. 131 (4) I proved for perfect convergence-free λ,μ:
(*) $L(\lambda,\mu)$ consists of all matrices A whose sections $A_{M \times N}$ are finite (contain only finitely many coordinates $\neq 0$), M any W-set of μ^x, N any W-set of λ.

The covergence-free space of all matrices U for which all these sets M x N are W-sets is obviously $(\lambda \otimes \mu^x)^n$ and (*) is equivalent to $L(\lambda,\mu) = [(\lambda \otimes \mu^x)^n]^x$.

Replacing μ^x by μ and using reflexivity we obtain (3).

Two remarks. 1) (*) was proved in 1934 for $\lambda = \mu$ in [9]. Ruckle proved in [10], p. 151, that $(\lambda \otimes \mu)^x = L(\lambda, \mu^x)$ even for all sequence spaces equipped with the normal topology.
2) Theorem (4) in [6], p. 131 says more than (*). It states also that $L(\lambda, \mu)^x = (\lambda \otimes \mu^x)^n$. There is no proof for this in [6], but taking the α-dual of (3) and using (2) settles the proof.

We will need in the next section the following result 2. (1) from [8]:

(4) *Let λ,μ be perfect convergence-free. Then $\lambda \otimes_\epsilon \mu$ can be identified with* $L_b(\lambda^x, \mu)$.

4. THE DUAITY THEOREM

So far we used on $\lambda \otimes \mu$ only the normal topology T_n. The classical duality 1. (1) uses the π- and the ϵ-topology for spaces which have the approximation property. Since convergence-free spaces are nuclear, they have the approximation property and the π-and the ϵ-topology on $\lambda \otimes \mu$ coincide. If 1. (1) would be true for perfect convergence-free spaces it would take the form

$$1.(1) \quad (\lambda \tilde{\otimes}_\epsilon \mu)'_n \cong \lambda^x \tilde{\otimes}_\epsilon \mu^x \text{ and } \quad (\lambda^x \tilde{\otimes}_\epsilon \mu^x)'_n \cong \lambda \tilde{\otimes}_\epsilon \mu,$$

where we used $T_c = T_n$ (cf. 2.(2)).
(1) is in general not true as we will see later. Instead of (1) we have the following duality theorem :

(2) *If λ,μ are perfect convergence-free, we have the following duality relations*

$$(\lambda \tilde{\otimes}_\epsilon \mu)'_n \cong \lambda^x \tilde{\otimes}_n \mu^x \quad \textit{and} \quad (\lambda \tilde{\otimes}_n \mu)'_n \cong \lambda^x \tilde{\otimes}_\epsilon \mu^x.$$

Proof: a) We prove the second isomorphism. By 3. (3) we have $(\lambda \tilde{\otimes}_n \mu)' = L(\lambda,\mu^x)$. This is a topological isomorphism if we equip both sides with the topology $T_n(\lambda \tilde{\otimes}_n \mu) = T_n((\lambda \otimes \mu)^n)$. The o neighbourhoods of this topology on $L(\lambda,\mu^x)$ are of the form $\left\{ A \in L(\lambda,\mu^x), \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} y_i \, |a_{ik}| \, x_k \leq 1 \right\}$, where $x \in \lambda$, $y \in \mu$ are positive.

We show that this topology coincides with the topology of $L_b(\lambda,\mu^x)$ whose o-neighbourhoods are of the form $U(B,V) = \{ A \in L(\lambda,\mu^x), A(B) \subset V$, B bounded in λ, V a o-neighbourhood in $\mu^x\}$. A neighbourhood V is of the form $\left\{ z \in \mu^x, \sum_{i=1}^{\infty} y_i \, |z_i| \leqslant 1 \right\}$, y a positive element of μ. Since B is of the form $\{ x \}^n$, x positive, we have $U(B,V) = \left\{ A \in L(\lambda,\mu^x), \sum_{i=1}^{\infty} y_i \sup_{t \in \{x\}^n} | \sum_{k=1}^{\infty} a_{ik} t_k | = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} y_i \, |a_{ik}| \, x_k \leqslant 1 \right\}$. Hence we have $(\lambda \tilde{\otimes}_n \mu)'_n \cong L_b(\lambda,\mu^x)$. By 3.(4) $L_b(\lambda, \mu^x)$ can be identified with $\lambda^x \tilde{\otimes}_\epsilon \mu^x$, so the second statement of (2) is proved. b) The first isomorphism of (2) follows now immediately from the second by taking on both sides the α-dual.

To get a better understanding of (2) we look at Grothendieck's theory of tensor products and try to identify the topology T_n on $\lambda \otimes \mu$ with one of the topologies compatible with the tensor product.

We recall some of the basic definitions (cf. [5] p. 154). If B(x,y) is a separately continuous bilinear form on E x F, both locally convex, one has

$$B(x,y) = (\tilde{B}x)y = (\tilde{\tilde{B}}y)x, \text{ where } \tilde{B} \in L(E,F'_s),\ \tilde{\tilde{B}} \in L(F,E'_s).$$

The correspondences $B \to \tilde{B} \to \tilde{\tilde{B}}$ generate algebraic isomorphisms

$$B(E \times F) \cong L(E,F'_s) \cong L(F,E'_s),$$

where B(ExF) is the espace of all separately continuous bilinear forms.

If the topologies on E and F are the Mackey topologies T_k, then we have

$$(3) \qquad B(E \times F) \cong L(E,F'_k) \cong L(F,E'_k).$$

This is easy: $A \in L(E,F'_s)$ is also weakly continuous and therefor T_k-continuous, hence lies in $L(E,F'_k)$. Conversely an $A \in L(E,F'_k)$ is weakly continuous and therefore in $L(E,F'_s)$.

(3) is true for barrelled spaces. For barrelled spaces one knows (cf. [5], 159 (5)) that every separately continuous bilinear form is hypocontinuous and that every separately equicontinuous subset H of B(ExF) is equihypocontinuous.

This means that if H(ExF) is the space of hypocontinuous bilinear forms we have.

(4) $B(E \times F) = H(E \times F) = L(E,F'_k) = L(F,E'_k)$ algebraically for barrelled spaces E,F.

Let us recall that a subset H of H(E x F) is equihypocontinuous if to every bounded subset M of E there exists a o-neighbourhood $V \subset F$ such that

$|B(M,V)| \leqslant 1$ for all $B \in H$ and similarly that for every bounded $N \in F$ there exists a o-neighbourhood $U \subset E$ such that $|B(U,N)| \leqslant 1$ for all $B \in H$. Using (4) one sees that H is equihypocontinuous for barrelled E,F if $\widetilde{H}$ is equicontinuous in $L(E,F'_k)$ and $\widetilde{\widetilde{H}}$ equicontinuous in $L(F,E'_k)$.

Following Grothendieck the finest locally convex topology on $E \otimes F$ compatible with $E \otimes F$ is the inductive topology T_{in} of uniform convergence on all separately equicontinuous subsets of $B(E \times F)$ (see [5], § 44, 2.).

Recalling the above remarks on barrelled spaces E,F the topology T_{in} is the topology T_{eh} of uniform convergence on all equihypocontinuous subsets of $H(E \times F)$.

We are now able to identify T_n on $\lambda \otimes \mu$:

(5) *Let λ,μ be perfect convergence-free. The normal topology* T_n *on* $\lambda \otimes \mu$ *is the topology* $T_{in} = T_{eh}$.

Proof T_n on $\lambda \otimes \mu$ is defined by the polars of the normal covers $\widetilde{M}$ of positive matrices $A \in L(\lambda,\mu^x)$. We show that $\widetilde{M}$ is equicontinuous in $L(\lambda,\mu^x)$. For a bounded subset B in μ we take the normal cover of a positive y in μ. The polar B° defines a o-neighbourhood V in μ^x. Now $A'y \in \lambda^x$ and the polar of $\{A'y\}^n$ defines a o-neighbourhood $U \subset \lambda$ and we have

$$\sup_{z\in B,\, C\in\widetilde{M},\, x\in U} |z(Cx)| \leqslant \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} y_i a_{ik} |x_k| \leqslant 1$$

or $C(U) \subset V$ for all $C \in \widetilde{M}$, so $\widetilde{M}$ is equicontinuous.

One proves similarly that $\widetilde{\widetilde{M}}$ is equicontinuous in $L(\mu,\lambda^x)$.

Now the converse. If $\widetilde{M} \in L(\lambda,\mu^x)$ is equicontinuous it follows for every positive $x \in \lambda$ and every positive $y \in \mu$ that for $z \in \{y\}^n$, $t \in \{x\}^n$ one has

$$\sup_{C\in\widetilde{M}} |z(Ct)| = \sup \left|\sum_i \sum_k y_i x_k c_{ik}\right| < \infty.$$

This means that $\widetilde{M}$ is $T_s(\lambda \widetilde{\otimes}_n \mu)$-bounded in $L(\lambda,\mu^x)$ and § 2 (1) says that $\widetilde{M}$ is a bounded set and therefore contained in the normal cover a positive $A \in L(\lambda,\mu^x)$.

5. Supplementary results

We investigate now when our duality theorem 4.(2) differs from the classical theorem 4.(1).

For perfect covergence-free spaces we have always

$$(1)\quad \lambda \widetilde{\otimes}_n \mu = (\lambda \otimes \mu)^n \subset \lambda\mu \subset \lambda \widetilde{\otimes}_\epsilon \mu.$$

We explain the exaxt meaning of (1). If $x = (x_i) \in \lambda$, $y = (y_i) \in \mu$, then following the remarks in section 3 we write $x \otimes y$ as the matrix $(y_i x_k)$ and $\lambda \otimes \mu$

consists of the finite sums of all these matrices. The elements of $\lambda\mu$ are written as matrices with columns $y^{(1)}, y^{(2)}, \ldots$, with $y^{(i)} \in \mu$ and $y^{(i)} \neq 0$ only for the i of some W-set M of λ. Obviously $\lambda \otimes \mu \subset \lambda\mu$ and since $\lambda\mu$ is perfect and convergence-free we have also $(\lambda \otimes \mu)^n \subset \lambda\mu$.

Every matrix of $\lambda\mu$ is obviously an element of $L(\lambda^x,\mu)$ and by 3.(4) an element of $\lambda \tilde{\otimes}_\epsilon \mu$.

When is $(\lambda \otimes \mu)^n$ a strict subspace of $\lambda\mu$? We have

(2) $(\lambda \otimes \mu)^n = \lambda \tilde{\otimes}_n \mu$ *is a strict subspace of* $\lambda\mu$ *if and only if* $\lambda \neq \varphi$ *and* $\mu \neq \omega$.

a) Obviously $\varphi \otimes \mu = \varphi\mu = (\varphi \otimes \mu)^n$. Similarly we have $(\lambda \otimes \omega)^n = \lambda\omega$. To see this remember that $\omega\omega$ is the normal cover of $\omega \otimes \omega$. Hence since for a W-set N of λ we have $\lambda_N = \omega$, we have $(\lambda_N \otimes \omega)^n = \lambda_N\omega$ and $\lambda\omega$ is the union of all $\lambda_N\omega$.

b) $(\omega \otimes \varphi)^n$ *is a strict subspace of* $\omega\varphi$: The elements of $\omega \otimes \varphi$ are represented by the adjoints of the matrices representing $\varphi\omega$. Since $(\varphi\omega)^n = \varphi\omega$, $(\omega \otimes \varphi)^n$ consists of all matrices with a finite number of rows different from o, but $\omega\varphi$ consists of all matrices having finite columns and so $(\omega \otimes \varphi)^n$ is a strict subspace of $\omega\varphi$.

c) Assume now $\lambda \neq \varphi$, and $\mu \neq \omega$. Then λ has a sectional subspace $\lambda_M = \omega$ and μ has a sectional subspace $\mu_N = \varphi$. Then by b) $(\lambda \otimes \mu)^n_{M\times N}$ is a strict subspace of $\lambda_M\mu_N = (\lambda\mu)_{M\times N}$. This implies that $(\lambda \otimes \mu)^n$ is a strict subspace of $\lambda\mu$.

For the second inequality in (1) we have.

(3) $\lambda\mu$ *is a strict subspace of* $\lambda \tilde{\otimes}_\epsilon \mu$ *if and only if* $\lambda \neq \omega$ *and* $\mu \neq \varphi$.

Proof: a) One checks easily that the matrices representing the elements of $\lambda\varphi$ are exactly the matrices of $L(\lambda^x,\varphi) = \lambda \tilde{\otimes}_\epsilon \varphi$. Hence $\lambda\varphi = \lambda \tilde{\otimes}_\epsilon \varphi$.

Similarly $\omega\mu$ can be identified with $L(\varphi,\mu) = \omega \tilde{\otimes}_\epsilon \mu$ (cf. [6] p. 133).

b) If $\lambda \neq \omega$ and $\mu \neq \varphi$ there exist sectional subspaces $\lambda_M = \varphi$ and $\mu_N = \omega$. Then $\lambda\mu$ has the sectional subspace $\lambda\mu_{M\times N} = \lambda_M\mu_N = \varphi\omega$. On the other hand the sectional subspace $L(\lambda^x,\mu)_{M\times N} = L(\lambda^x_M,\mu_N) = L(\omega,\omega)$ and this is the space of all row-finite matrices which is strictly larger tha $\varphi\omega$. It follows that $\lambda\mu$ is a strict subspace of $L(\lambda^x,\mu)$ which can be identified with $\lambda^x \tilde{\otimes}_\epsilon \mu^x$.

From (2) and (3) follows immediatly

(4) *Let* λ,μ *be perfect convergence-free. Then* $\lambda \tilde{\otimes}_n \mu$ *is a strict subspace of* $\lambda \tilde{\otimes}_\epsilon \mu$ *except in the cases* $\lambda = \varphi = \mu$ *and* $\lambda = \omega = \mu$. *With these exceptions the topology* T_n *on* $\lambda \otimes \mu$ *is strictly finer than* $T_\epsilon = T_\pi$.

This shows that the duality 1.(1) is true only in the cases $\lambda = \varphi = \mu$ and $\lambda = \omega = \mu$.

We give an application of (4). Hollstein proved in [2].

(5) *$\varphi \otimes_\epsilon \omega$ is not barrelled.*

We give a simpler proof using his main idea: $\varphi \otimes \omega = \varphi\omega$ is a Pták space (see [5] p. 31). The identity map I of $\varphi \otimes_n \omega$ onto $\varphi \otimes_\epsilon \omega$ is continuous and if $\varphi \otimes_\epsilon \omega$ were barrelled, I would be an isomorphism ([5] p. 27 (3)) which contradicts (4).

More general:

(6) *Two perfect convergence-free spaces λ, μ are barrelled and even ultrabornological. With the exception of* $\lambda = \varphi = \mu$ and $\lambda = \omega = \mu$ *the tensor product* $\lambda \otimes_\epsilon \mu$ *is not even barrelled.*

Look again at he identity map I of $\lambda \otimes_n \mu$ onto $\lambda \otimes_\epsilon \mu$. We have a sectional subspace $\lambda_M \otimes \mu_N = \varphi \otimes \omega$ or $\omega \otimes \varphi$, which is isomorphic to $\varphi \otimes \omega$, and I restricted to $\lambda_M \otimes \mu_N$ is the identity map of $\varphi \otimes_n \omega$ onto $\varphi \otimes_\epsilon \omega$. Since by (5) $\lambda_M \otimes_\epsilon \mu_N$ is not barrelled, $\lambda \otimes_\epsilon \mu$ can not be barrelled.

BIBLIOGRAPHY

[1] Defant, A. and Floret, K.: Localization and duality of topological tensor products. Collectanea Math. 35 (1984), 43-61.

[2] Hollstein, R.: Über die Tonneliertheit von lokalkonvexen Tensorprodukten. Manuscripta math. 22 (1937), 7-12.

[3] Köthe, G.: Die konvergenzfreien Räume abzählbarer Stufe. Math. Annalen 111 (1935), 229-258.

[4] Köthe, G.: Topological vector spaces I, Springer 1969.

[5] Köthe, G.: Topological vector spaces II, Springer 1979.

[6] Köthe, G.: On a class of nuclear spaces I. Portug. Math. 41 (1982), 125-138.

[7] Köthe, G.: On a class of nuclear spaces II. Math. Nachr. 119 (1984), 157-164.

[8] Köthe, G.: Tensor products of convergence-free spaces. J.A. Barroso, editor, Aspects of Mathematics and its applications, 485-494, Elsevier 1986.

[9] Köthe, G. and Toeplitz, O.: Lineare Räume und Ringe unendlicher Matrizen. J. reine angew. Math. 171 (1934), 193-226.

[10] Ruckle, W.H.: Sequence spaces. Research Notes in Mathematics 49, Pitman 1981.

Prof. Dr. Gottfried Köthe
Parkstr. 14
6000 Frankfurt
Bundesrepublik Deutschland