PROPERTIES OF INTERSECTION IN THE GEOMETRY OF SEQUENCES IN BANACH SPACES

by

FELICISIMO GARCIA-CASTELLON AND ESTEBAN INDURAIN

ABSTRACT:

We consider properties of sequences in a Banach space through the geometrical behaviour of the closed linear spans of their subsequences. We study new families of intersection properties, generalizing the results of Courage-Davis, Plans, and Reyes.

We end with a table showing the structure of a sequence by means of its intersection properties. This table, in our opinion, closes the study of "finite" intersection properties of a sequence.

INTRODUCTION

Let B denote an infinite-dimensional Banach space, N the set of natural numbers, and [–] "closed linear span".

In the sequel we shall consider *complete* ([$\int$] = B) sequences in B.

Definition: A sequence $\int = (a_n)_{n \in N}$ in a Banach space B verifies the *intersection property* relative to a certain condition C, if $[a_s; s \in S] \cap [a_t; t \in T] = [a_h; h \in S \cap T]$, provided that S,T $\subset$ N are restricted to the fixed condition C. Denote it by i. p. (C).

The idea of obtaining properties of a sequence by means of intersection properties of the closed linear spans of its subsequences began with Plans |P|$_1$ and Courage-Davis |C-D| who independently characterized M-bases by means of an i. p.

Later on, Plans and Reyes |P-R|, |R| gave a classification of sequences through intersection properties.

We consider new conditions on the pair (S,T) examining the sets S,T, S$\cup$T, S$\cap$T, S–T ant T–S, and conclude with a table of characterizations, that, we think, gives a complete outlook of them and closes the problem of finite i.p.

Remark: – By "finite" i.p. we understand those expressed by $[a_s; s\epsilon S_1 \cap \ldots \cap S_k] = [a_s; s\epsilon S_1] \cap \ldots \cap [a_s; s\epsilon S_k]$, with the family $(S_i)_{i=1}^k$ restricted to some fixed condition C.

(It is easy to relate them with properties where only two indices appear). The "infinite" intersection properties are quite different (see | I |$_1$).

1. DEFINITIONS AND PREVIOUS RESULTS.

Let $\int = (a_n)_{n\epsilon N}$ be a complete sequence in B.
Given $S \subset N$, call

$$W_S = [a_s; s\epsilon S] \quad (W_\phi = \{0\})$$

and

$$W_S^* = \bigcap_{k \notin S} W_{N-\{k\}} .$$

We say that $S \epsilon \sigma_1$ if S is finite, $S \epsilon \sigma_2$ if N–S is finite, and $S \epsilon \sigma_3$ if S and N–S are both infinite.

We associate to $\int$, in a natural way, the following closed subspaces:
Kernel of $\int$:

$$K(\int) = \bigcap_{n\epsilon N} [a_n, a_{n+1}, a_{n+2}, \ldots],$$

equivalently, $K(\int) = \bigcap_{S \epsilon \sigma_2} W_S$.

Strict kernel of $\int$:

$$K_s(\int) = \bigcap_{S \epsilon \sigma_3} W_S .$$

Let $M_\int = \{ n\epsilon N; a_n \epsilon W_{N-\{n\}} \}$.

Definition: $\int$ has *absorbent kernel* (respectively *absorbent strict kernel*) if $W_{M_\int} \subset K(\int)$ (respectively $W_{M_\int} \subset K_s(\int)$).

Theorem 1.1: (See | P-R | and | R |)
(i) $\int$ has absorbent kernel if and only if it verifies $W_S \cap W_T = W_{S\cap T}$ for every $S,T\epsilon\sigma_2$.
(ii) $\int$ has absorbent strict kernel if and only if it verifies $W_S \cap W_T = W_{S\cap T}$ for every $S\epsilon\sigma_2$ and $T\epsilon\,\sigma_3$.

Definition: $\int$ is *minimal* if it satisfies the following equivalent conditions:
(m_1) $M_{\int} = \phi$,
(m_2) $W_S \cap W_T = W_{S \cap T}$ for every $S \epsilon \sigma_1$ and $T \epsilon \sigma_2$,
(m_3) There exists a sequence $(a_n^*)_{n \epsilon N}$ in the Banach space B*(*dual* of B), such that $a_n^*(a_m) = \delta_{nm}$ (Kronecker delta of n,m).

Definition: $\int$ is an *M-basis* (Markusevich basis) of B, if it is mimimal and $K(\int) = \{0\}$, equivalently (see |C-D| and $|P|_1$) if $W_S \cap W_T = \{0\}$ whenever S and T are disjoint.

Definition: $\int$ is *a strong M-basis* if $W_S^* = W_S$, for every $S \subset N$.

Theorem 1.2:

The following statements are equivalent for a sequence $\int$:
(i) $W_S \cap W_T = W_{S \cap T}$ for every $S,T \subset N$,
(ii) $W_S \cap W_T = W_{S \cap T}$ for every $S,T \epsilon \sigma_3$,
(iii) $\int$ is a strong M-basis.

Proof: See |P-R| and |R|.

Remark: It is clear that Strong M-basis ==> M-basis ==> minimal ==> Absorbent strict kernel ==> Absorbent kernel.

The converse implications are not true. (See |S|).

Definiton: $\int$ is called σ_1*-sequence* (respectively σ_2*-sequence*, σ_3*-sequence*) if it verifies $W_S^* = W_S$ for every $S \epsilon \sigma_1$ (r. $S \epsilon \sigma_2$, $S \epsilon \sigma_3$).

Propositon 1.3:
(i) $\int$ is a σ_1-sequence if and only if it is an M-basis.
(ii) $\int$ is a σ_2-sequence if and only if it has absorbent kernel.

Proof:
(i) Suppose that $\int$ is a σ_1-sequence. Let $S \epsilon \sigma_1$ and $T \epsilon \sigma_2$. Then $W_S \cap W_T = W_S^* \cap W_T \subset W_S^* \cap W_T^* = W_{S \cap T}^* = W_{S \cap T}$, so $\int$ is minimal.
Since $K(\int) \subset W_\phi^* = W_\phi = \{0\}$, $\int$ is an M-basis.
The converse is obvious.
(ii) Suppose that $\int$ is a σ_2-sequence. Let $S,T \epsilon \sigma_2$. It follows that $W_S \cap W_T = W_S^* \cap W_T^* = W_{S \cap T}^* = W_{S \cap T}$, so, by 1.1, $\int$ has absorbent kernel.

For the converse, let $S \epsilon \sigma_2$, $N - S = \{q_1, \ldots, q_k\}$.

So $W_S^* = \bigcap_{i=1,\ldots,k} W_{N-\{q_i\}} = W_S$, by 1.1. Therefore $\mathcal{f}$ is a σ_2-sequence.□

2. CONDITIONS ON $S \cup T$.

Theorem 2.1:

For $\mathcal{f} = (a_n)_{n \epsilon N}$ in B:

(i) $\mathcal{f}$ verifies $W_S \cap W_T = W_{S \cap T}$ whenever $S \cup T \epsilon \sigma_1$ if and only if $\mathcal{f}$ is a linearly independent set of vectors.

(ii) $\mathcal{f}$ verifies $W_S \cap W_T = W_{S \cap T}$ whenever $S \cup T \epsilon \sigma_2$ if and only if $\mathcal{f}$ is a strong M-basis.

Proof:

(i) It is obvious.

(ii) Suppose $W_S \cap W_T = W_{S \cap T}$ whenever $S \cup T \epsilon \sigma_2$.

Let $P, R \subset N$ and consider P_1, $R_1 \subset N$ such that $P_1 \cup R_1 \epsilon \sigma_2$ and $P_1 \cap R_1 = P \cap R$. We have

$W_{P \cap R} \subset W_P \cap W_R \subset W_{P_1} \cap W_{R_1} = W_{P_1 \cap R_1} = W_{P \cap R}$, so $\mathcal{f}$ is a strong M-basis.

The converse is obvious. □

Definition: Given a property H on sequences, we say that a sequence $\mathcal{f} = (a_n)_{n \epsilon N}$ is *almost-H* if every subsequence $(a_s)_{s \epsilon S}$, with $S \epsilon \sigma_3$, verifies H, in the Banach subspace $[a_s; s \epsilon S]$.

Remark: Almost properties on a sequence $\mathcal{f}$ have been studied in $|I|_2$, and the structure induced in the sequence $\mathcal{f}$ is given in terms of "unitarian position", improving the result in Theorem 4.1 of $|R|$.

Theorem 2.2:

$\mathcal{f}$ verifies $W_S \cap W_T = W_{S \cap T}$ whenever $S \cup T \epsilon \sigma_3$ if and only if it is almost-strong M-basis.

Proof: It is analogous to 2.1 (ii).

3. CONDITIONS ON $S \cap T$.

Let $\mathcal{f} = (a_n)_{n \epsilon N}$ be a complete sequence in B.

Theorem 3.1:

(i) $\int$ verifies $W_S \cap W_T = W_{S\cap T}$ whenever $S\cap T\epsilon\sigma_1$ if and only if $\int$ is an M-basis.

(ii) $\int$ verifies $W_S \cap W_T = W_{S\cap T}$ whenever $S\cap T\epsilon\sigma_2$ if and only if $\int$ has absorbent kernel.

Proof:

(i) It is analogous to the characterization of M-bases given in $|P|_1$ or |C-D|.

(ii) If $S\cap T\epsilon\sigma_2$, then $S,T\epsilon\sigma_2$ and conversely. Apply now 1.1. □

Lemma 3.2:

Let $\int$ be a sequence having absorbent kernel. Then, for every $R\epsilon\sigma_1 \cup \sigma_3$ there exist R_1, $R_2 \epsilon\sigma_3$, $R_1 \cap R_2 = R$, such that $W_R^* = W_{R_1} \cap W_{R_2}$.

Proof: It is a strengthening of Proposition 5.1 in |R|.

Theorem 3.3:

$\int = (a_n)_{n\epsilon N}$ verifies $W_S \cap W_T = W_{S\cap T}$ whenever $S\cap T\epsilon\sigma_3$ if and only if it is a σ_3–sequence.

Proof:

===>) If $\int$ verifies $W_S \cap W_T = W_{S\cap T}$ whenever $S\cap T\epsilon\sigma_3$, by 1.1 it has absorbent strict kernel, and therefore, absorbent kernel.

Let $R\epsilon\sigma_3$. By 3.2 take R_1 and R_2 such that $W_R^* = W_{R_1} \cap W_{R_2}$.

By hypothesis, $W_{R_1} \cap W_{R_2} = W_{R_1 \cap R_2} = W_R$. Thus, $\int$ is a σ_3–sequence.

<===) Given $S\cap T\epsilon\sigma_3$, it follows that

$$W_S \cap W_T \subset W_S^* \cap W_T^* = W_{S\cap T}^*.$$

But, by hypothesis, $W_{S\cap T}^* = W_{S\cap T}$. □

4. CONDITIONS ON S–T AND T–S.

Consider now sequences $\int = (a_n)_{n\epsilon N}$ verifying intersection properties according to the cardinality (in terms of σ_1, σ_2, σ_3) of the sets S, T, $S\cap T$, S–T and T–S.

There are four main cases. (The remaining cases can be dealt with by means of these, or using the scheme in |R|).

Case 1: $W_S \cap W_T = W_{S\cap T}$ for every $S,T\epsilon\sigma_3$ such that S–T, $T-S\epsilon\sigma_3$ and $S\cap T\epsilon\sigma_1$.

Case 2: $W_S \cap W_T = W_{S\cap T}$ for every $S,T\epsilon\sigma_3$ such that S–T, $T-S\epsilon\sigma_1$ and $S\cap T\epsilon\sigma_3$.

Case 3: $W_S \cap W_T = W_{S\cap T}$ for every $S,T\epsilon\sigma_3$ such that $S-T\epsilon\sigma_1$ and T–S, $S\cap T\epsilon\sigma_3$.

Case 4: $W_S \cap W_T = W_{S\cap T}$ for every $S,T\epsilon\sigma_3$ such that S–T, T–S and $S\cap T\epsilon\sigma_3$.

Definition: $\int$ is called *M-basoidic* (see $|P|_2$ and $|T|_1$) if it can be represented by $\int = \{b_1, \ldots, b_k\} \cup \int_1$, where $\int_1 = (b_{k+n})_{n \in N}$ is an M-basis, $[b_1, \ldots, b_k] \subset [\int_1]$, and $[b_1, \ldots, b_k] \cap (W_S + W_{N-(S \cup \{1, \ldots k\})}) = \{0\}$, for every $S \subset N - \{1, \ldots k\}$, $S \epsilon \sigma_3$.

It is not difficult to see that the intersection property in Case 1 is equivalent to the following: $W_S \cap W_T = W_{S \cap T}$, for every disjoint S, $T \epsilon \sigma_3$, but this property characterizes the M-basoidic sequences (see $|P|_2$ and $|T|_1$).

With respect to cases 2 and 3 it is straightforward to see that they characterize, respectively, the sequences almost–"with absorbent kernel" and almost–"with absorbent strict kernel".

We study now Case 4.

Lemma 4.1:

Let $\int = (a_n)_{n \in N}$ be a complete sequence in B, and $x \epsilon B$.

Then, there exist $S, T \epsilon \sigma_3$ (which depend on x) such that $x = x_1 + x_2$, with $x_1 \epsilon W_S$ and $x_2 \epsilon W_T$.

Proof: See $|T|_2$.

Lemma 4.2:

Let $\int = (a_n)_{n \in N}$ be a sequence verifying $W_S \cap W_T = W_{S \cap T}$, for every $S, T \epsilon \sigma_3$ such that S–T, T–S and $S \cap T \epsilon \sigma_3$. (I)

Then, for every $S \epsilon \sigma_3$, $W_S \cap W_{N-S} \subset K_s(\int)$.

Proof:

Let $S \epsilon \sigma_3$ and $x \epsilon W_S \cap W_{N-S}$. Fix $R \epsilon \sigma_3$. We can find $R_1 \epsilon \sigma_3$ such that $S \cup R_1$ and $(N-S) \cup R_1 \epsilon \sigma_3$, with $R_1 \subset R$. So, we have $x \epsilon W_{S \cup R_1} \cap W_{(N-S) \cup R_1} = W_{R_1} \subset W_R$ (for every $R \epsilon \sigma_3$)

Therefore $x \epsilon K_s(\int)$. □

Lemma 4.3:

If $\int = (a_n)_{n \in N}$ verifies (I), then it has absorbent strict kernel.

Proof:

Suppose, for instance, $a_1 \epsilon W_{N-\{1\}}$. By 4.1 we can find $S, T \epsilon \sigma_3$, $S, T \subset N - \{1\}$ such that $a_1 = u + v$, with $u \epsilon W_S$, $v \epsilon W_T$.

The following possibilities arise:

(a) $S \cap T$, S–T and $T - S \epsilon \sigma_3$.

Then $a_1 - u \epsilon W_{S \cup \{1\}} \cap W_T = W_{S \cap T}$, and thus $a_1 \epsilon W_S$.

On the other hand, since $1 \notin S$, $a_1 \epsilon W_{N-S}$. So, by 4.2, $a_1 \epsilon K_s(\int)$.

(b) $S-T\epsilon\sigma_1$, $S\cap T$ and $T-S\epsilon\sigma_3$.
Then $S\cup T\epsilon\sigma_3$ and $a_1 \epsilon W_{S\cup T}$.
Since $1\notin S\cup T$, it follows that $a_1 \epsilon W_{N-(S\cup T)}$, so by 4.2, $a_1 \epsilon K_s(\int)$.

(c) $S-T$, $T-S\epsilon\sigma_1$; $S\cap T\epsilon\sigma_3$.
It is similar to (b).

(d) $S-T$, $T-S\epsilon\sigma_3$; $S\cap T\epsilon\sigma_1$.
Then $W_S = W_{S-T} + W_{S\cap T}$ and $W_T = W_{T-S} + W_{S\cap T}$
Take $a \epsilon W_{S-T}$, $b \epsilon W_{S\cap T}$ and $c \epsilon W_{T-S}$ such that $a_1 = a + b + c$.
We have $a_1 - a \epsilon W_T \cap W_{\{1\}\cup(S-T)}$. So, by 4.2, $a_1 - a \epsilon K_s(\int) \subset W_{S-T}$.
Therefore $a_1 \epsilon W_{S-T}$.
On the other hand $a_1 \epsilon W_{\{1\}\cup(T-S)}$, so we conclude that $a_1 \epsilon K_s(\int)$.
Consequently, $\int$ has absorbent strict kernel. □

Theorem 4.4:
A sequence $\int = (a_n)_{n\epsilon N}$ verifies (I) if and only if it is a σ_3–sequence.

Proof: It follows from 1.1, 3.3 and 4.3.

5. STRUCTURE OF σ_3–SEQUENCES.

We consider now some properties of the kernel and strict kernel of a complete sequence $\int$, which characterize $\int$ as a σ_3–sequence.

Lemma 5.1:
For a sequence $\int = (a_n)_{n\epsilon N}$,

$$K(\int) \subset \bigcup_{F\epsilon\sigma_3} (W_F \cap W_{N-F})$$

Proof: See |P-R|.

Corollary 5.2:
Let $\int = (a_n)_{n\epsilon N}$ be a σ_3–sequence. Then $K(\int) = K_s(\int)$.

Proof: It is a consequence of 4.2 and 5.1.

Lemma 5.3:
An M-basis $\int = (a_n)_{n\epsilon N}$ is a strong M-basis if and only if it verifies (I).

Proof: It follows from 1.2, 3.1, 3.3 and 4.4.

Proposition 5.4:

Let $\int = (a_n)_{n\epsilon N}$ be a σ_3–sequence. Consider, in the Banach space $B/K(\int)$, the set $\int_{K(\int)} = \{a_k + K(\int); a_k \notin K(\int)\}_{k\epsilon N}$. Then we have

(i) If $\int_{K(\int)}$ is finite, it is a linearly independent set in $B/K(\int)$,

(ii) If $\int_{K(\int)}$ is infinite, it is a strong M-basis in $B/K(\int)$.

Proof:

(i) It follows from the fact of $\int$ having absorbent kernel.

(ii) Since $\int$ has absorbent kernel, $\int - K(\int)$ is a minimal sequence, and $\int_{K(\int)}$ is an M-basis.

Let $\int_{K(\int)} = \{a_{q_n} + K(\int)\}_{n\epsilon N}$ and take $S,T\epsilon\sigma_3$ such that $S-T$, $T-S$ and $S\cap T\epsilon\sigma_3$.

Let $x\epsilon B$, we have, applying (I)

$$x + K(\int) \epsilon [a_{q_s} + K(\int); s\epsilon S] \cap [a_{q_t} + K(\int); t\epsilon T] <==>$$

$$<==> x \epsilon [\{a_{q_s}; s\epsilon S\}, K(\int)] \cap [\{a_{q_t}; t\epsilon T\}, K(\int)] = (*)$$

$$= [a_{q_h}; h\epsilon S\cap T] <==> x + K(\int) \epsilon [a_{q_h} + K(\int); h\epsilon S\cap T]. \ (**)$$

So $\int_{K(\int)}$ is also strong. □

Theorem 5.5:

$\int = (a_n)_{n\epsilon N}$ is a σ_3–sequence if and only if it verifies the conditions

(i) $K(\int) = K_s(\int)$ and

(ii) The set $\int_{K(\int)}$ is a) linearly independent, if finite,

b) A strong M-basis in $B/K(\int)$, if infinite.

Proof:

$===>$) It follows from 5.2 and 5.4.

$<===$) Let $S,T\epsilon\sigma_3$ such that $S-T$, $T-S$ and $S\cap T\epsilon\sigma_3$, let $x\epsilon B$.

$$x\epsilon [a_s; s\epsilon S] \cap [a_t; t\epsilon T] <==> x\epsilon [\{a_s; s\epsilon S\}, K_s(\int)] \cap$$

$$\cap [\{a_t; t\epsilon T\}, K_s(\int)] \underset{(i)}{<==>} x\epsilon [\{a_s; s\epsilon S\}, K(\int)] \cap$$

$$\cap [\{a_t. t\epsilon T\}, K(\int)] <==> x + K(\int) \epsilon [a_s + K(\int); s\epsilon S] \cap$$

$$\cap [a_t + K(\int); t\epsilon T] \underset{(ii)}{<===>} x + K(\int) \epsilon [a_h + K(\int); h\epsilon S\cap T] <==>$$

$$<==> x\epsilon [\{a_h; h\epsilon S\cap T\}, K(\int)] \underset{(i)}{=} [a_h; h\epsilon S\cap T]. \ \square$$

(*) Apply here that $K(\int) = K_s(\int)$.

(**) Observe that [–] stands here for closed linear span in $B/K(\int)$ and also in B.

6. TABLE.

We conclude with the following scheme for intersection properties:

$W_S \cap W_T = W_{S \cap T}$

S	T	S∪T	S∩T	S--T	T–S	KIND OF SEQUENCE
–	–	–	–	–	–	strong M-basis
σ_1	–	–	–	–	–	minimal
σ_2	–	–	–	–	–	minimal
σ_3	–	–	–	–	–	strong M-basis
–	–	σ_1	–	–	–	linearly independent
–	–	σ_2	–	–	–	strong M-basis
–	–	σ_3	–	–	–	almost strong M-basis
–	–	–	σ_1	–	–	M-basis
–	–	–	σ_2	–	–	absorbent kernel
–	–	–	σ_3	–	–	σ_3 –sequence
σ_1	σ_1	–	–	–	–	linearly independent
σ_1	σ_2	–	–	–	–	minimal
σ_1	σ_3	–	–	–	–	almost minimal
σ_2	σ_2	–	–	–	–	absorbent kernel
σ_2	σ_3	–	–	–	–	absorbent strict kernel
σ_3	σ_3	–	–	–	–	strong M-basis
σ_3	σ_3	–	σ_1	–	–	M-basoidic
σ_3	σ_3	–	σ_3	σ_1	σ_1	almost absorbent kernel
σ_3	σ_3	–	σ_3	σ_1	σ_3	almost absorbent strict kernel
σ_3	σ_3	–	σ_3	σ_3	σ_3	σ_3–sequence
–	–	σ_3	σ_1	–	–	almost M-basis

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Departamento de Geometría y Topología.
Facultad de Ciencias.
50009 - ZARAGOZA (SPAIN).