ON THE MEAN VALUES OF AN ENTIRE FUNCTION REPRESENTED BY DIRICHLET SERIES OF SEVERAL COMPLEX VARIABLES

by

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ABSTRACT:

Let $f(s_1, s_2)$ be an entire function represented by Dirichlet series of two complex variables. We have defined for $0 < \delta < \infty$ and $k_1, k_2 > 0$ the functions $I_\delta(\sigma_1, \sigma_2)$, $A_\delta(\sigma_1, \sigma_2)$ and $G_{\delta, k_1, k_2}(\sigma_1, \sigma_2)$, then the Ritt-order ρ, the lower order λ and the orders ρ_1, ρ_2 with respect to the variables s_1, s_2 could be expressed in terms of these functions.

1. Consider a double* entire Dirichlet series.

(1.1.) $$f(s_1, s_2) = \sum_{m, n = 0}^{\infty} a_{m, n} \exp(\lambda_m s_1 + \mu_n s_2)$$

of complex variables s_1 and s_2, where the coefficients $a_{m, n}$ are complex numbers, $\lambda_0 = \mu_0 = 0$, $(\lambda_m)_{m \geqslant 1}$, $(\mu_n)_{n \geqslant 1}$ are two sequences of real increasing numbers whose limits are infinity, and further [2]

(1.2.) $$\limsup_{m+n \to \infty} \frac{\log(m+n)}{\lambda_m + \mu_n} = D < + \infty$$

(1.3.) $$\limsup_{m+n \to \infty} \frac{\log |a_{m,n}|}{\lambda_m + \mu_n} = -\infty$$

As usual, the symbole $M(\sigma_1, \sigma_2)$ and $\mu(\sigma_1, \sigma_2)$ denote the maximum modulus and the maximum term respectively for $f(s_1, s_2)$ defined as

(1.4.) $$M(\sigma_1, \sigma_2) = \sup_{-\infty < t_1, t_2 < \infty} |f(\sigma_1 + it_1, \sigma_2 + it_2)|$$

(1.5.) $$\mu(\sigma_1, \sigma_2) = \max_{m,n \geqslant 0} |a_{m,n}| \exp(\lambda_m \sigma_1 + \mu_n \sigma_2)$$

* In this note, we consider the two variables case for the sake of simplicity, although our results can easily be extended to any finite number of variables.

It is known that [2] for all $\sigma_1, \sigma_2 > 0$ and for some $\alpha > 0$

$$\mu(\sigma_1, \sigma_2) \leqslant M(\sigma_1, \sigma_2) \leqslant K\,\mu(\sigma_1 + \alpha, \sigma_2 + \alpha) \tag{1.6.}$$

where $K = K(\alpha)$

We define the Ritt-order ρ and the lower order λ of $f(s_1, s_2)$ as follows [3]

$$\lim_{\sigma_1, \sigma_2 \to \infty} \begin{Bmatrix} \sup \\ \\ \inf \end{Bmatrix} \frac{\log \log M(\sigma_1, \sigma_2)}{\sigma_1 + \sigma_2} = \begin{Bmatrix} \rho \\ \\ \lambda \end{Bmatrix} \tag{1.7.}$$

2. Let us define, for $0 < \delta < \infty$

$$I_\delta(\sigma_1, \sigma_2) = I_\delta(\sigma_1, \sigma_2, f) \tag{2.1.}$$

$$= \lim_{T \to \infty} \frac{1}{(2T)^2} \int_{-T}^{T}\!\!\int |f(\sigma_1 + it_1, \sigma_2 + it_2)|^\delta \, dt_1 \, dt_2 .$$

where the integral in (2.1.) exists on account of the absolute convergence of the series for $f(s_1, s_2)$.

Note that the mean value for $\delta = 2$ has been studied in [3], but the single variable case already dealth in [1], [4] and [5].

Theorem 1. If $f(s_1, s_2)$ defined by Dirichlet series (1.1.), be an entire function of finite Ritt-order ρ and lower order λ, then

$$\lim_{\sigma_1, \sigma_2 \to \infty} \begin{Bmatrix} \sup \\ \\ \inf \end{Bmatrix} \frac{\log \log I_\delta(\sigma_1, \sigma_2)}{\sigma_1 + \sigma_2} = \begin{Bmatrix} \rho \\ \\ \lambda \end{Bmatrix} \tag{2.2.}$$

The proof of this theorem is similar to deal with the proof of theorem 1 in [3].

Corollary 1. If $f(s_1, s_2)$ defined by Dirichlet series (1.1.), be an entire function of finite Ritt-order ρ and finite lower order λ, then

$$\lim_{\sigma_1, \sigma_2 \to \infty} \begin{Bmatrix} \sup \\ \\ \inf \end{Bmatrix} \frac{\log\log A_\delta(\sigma_1, \sigma_2)}{\sigma_1 + \sigma_2} = \begin{Bmatrix} \rho \\ \\ \lambda \end{Bmatrix} \tag{2.3.}$$

where

$$A_\delta(\sigma_1, \sigma_2) = A_\delta(\sigma_1, \sigma_2, f) = \left\{ I_\delta(\sigma_1, \sigma_2) \right\}^{\frac{1}{\delta}}$$

Let for $k_1, k_2 > 0$ (see [3], for $k_1 = k_2 = k$)

$$G_{\delta,k_1,k_2}(\sigma_1, \sigma_2) = G_{\delta,k_1,k_2}(\sigma_1, \sigma_2, f) \tag{2.4.}$$

$$= \exp\left\{ k_1 k_2 \exp - (\sigma_1 k_1 + \sigma_2 k_2) \int_0^{\sigma_1} \int_0^{\sigma_2} \operatorname{Log} A_\delta(x_1, x_2) \exp(x_1 k_1 + x_2 k_2)\, dx_1\, dx_2 \right\}$$

Lemma 1. Let $f(s_1, s_2)$ be an entire function defined by Dirichlet series (1.1.), then for $0 < \sigma_1 < \sigma_1^\circ$, $0 < \sigma_2 < \sigma_2^\circ$, we have

$$\operatorname{Log} G_{\delta,k_1,k_2}(\sigma_1, \sigma_2) \leqslant \log A_\delta(\sigma_1, \sigma_2) \tag{2.5.}$$

$$\leqslant \frac{\operatorname{Log} G_{\delta,k_1,k_2}(\sigma_1^\circ, \sigma_2^\circ)}{\left\{1 - \exp(\sigma_1 - \sigma_1^\circ) k_1\right\} \left\{1 - \exp(\sigma_2 - \sigma_2^\circ) k_2\right\}}$$

Proof: We have

$$\log G_{\delta,k_1,k_2}(\sigma_1, \sigma_2) = k_1 k_2 \exp - (\sigma_1 k_1 + \sigma_2 k_2) \int_0^{\sigma_1} \int_0^{\sigma_2} \log A_\delta(x_1, x_2) \exp(x_1 k_1 + x_2 k_2)\, dx_1\, dx_2 \tag{2.6.}$$

$$\leqslant \log A_\delta(\sigma_1, \sigma_2)$$

Further, for $0 < \sigma_1 < \sigma_1^\circ$ and $0 < \sigma_2 < \sigma_2^\circ$

(2.7.)

$$\log G_{\delta,k_1k_2}(\overset{\circ}{\sigma}_1, \overset{\circ}{\sigma}_2) \geqslant k_1k_2 \exp -(\overset{\circ}{\sigma}_1 k_1 + \overset{\circ}{\sigma}_2 k_2) \int_{\sigma_1}^{\overset{\circ}{\sigma}_1} \int_{\sigma_2}^{\overset{\circ}{\sigma}_2}$$

$$\log A_\delta (x_1, x_2) \exp (x_1 k_1 + x_2 k_2)\, dx_1\, dx_2$$

$$\geqslant \log A_\delta (\sigma_1, \sigma_2) \left\{1-\exp(\sigma_1 - \overset{\circ}{\sigma}_1) k_1\right\} \left\{1-\exp(\sigma_2 - \overset{\circ}{\sigma}_2) k_2\right\}$$

From (2.6.) and (2.7.) follows the lemma.

Theorem 2. Let $f(s_1, s_2)$ be an entire Dirichlet series of finite Ritt-order ρ and finite lower order λ, then

$$\text{(2.8.)} \qquad \lim_{\sigma_1, \sigma_2 \to \infty} \begin{Bmatrix} \sup \\ \\ \inf \end{Bmatrix} \frac{\log \log G_{\delta,k_1,k_2}(\sigma_1, \sigma_2)}{\sigma_1 + \sigma_2} = \begin{Bmatrix} \rho \\ \\ \lambda \end{Bmatrix}$$

Proof: If $\overset{\circ}{\sigma}_1 = \sigma_1 + h_1$, $\overset{\circ}{\sigma}_2 = \sigma_2 + h_2$ where $h_1, h_2 > 0$, then we have

$$\text{(2.9)} \qquad \log G_{\delta,k_1,k_2}(\sigma_1, \sigma_2) \leqslant \log A_\delta (\sigma_1, \sigma_2) \leqslant$$

$$\leqslant \frac{\log G_{\delta,k_1,k_2}(\sigma_1 + h_1, \sigma_2 + h_2)}{\left\{1 - \exp -(h_1 k_1)\right\} \left\{1 - \exp -(h_2 k_2)\right\}}$$

Theorem follows directly from this inequality and from the relation (2.3.).

3. Let us define the finite order (ρ_1, ρ_2), ρ_1 and ρ_2 with respect to variables s_1 and s_2 respectively as

$$\text{(3.1.)} \qquad \begin{aligned} \limsup_{\sigma_2 \to \infty} \left\{ \limsup_{\sigma_1 \to \infty} \frac{\log \log M(\sigma_1, \sigma_2)}{\sigma_1} \right\} &= \rho_1 \\ \limsup_{\sigma_1 \to \infty} \left\{ \limsup_{\sigma_2 \to \infty} \frac{\log \log M(\sigma_1, \sigma_2)}{\sigma_2} \right\} &= \rho_2 \end{aligned}$$

As a consequence of the above results, we state the following

Theorem 3. Let $f(s_1, s_2)$ be an entire Dirichlet series of finite order (ρ_1, ρ_2), ρ_1 and ρ_2 with respect to variables s_1 and s_2, then

$$\limsup_{\sigma_2\to\infty}\left\{\limsup_{\sigma_1\to\infty}\frac{\log\log I_\delta(\sigma_1,\sigma_2)}{\sigma_1}\right\}=\rho_1$$

(3.2.)

$$\limsup_{\sigma_1\to\infty}\left\{\limsup_{\sigma_2\to\infty}\frac{\log\log I_\delta(\sigma_1,\sigma_2)}{\sigma_2}\right\}=\rho_2$$

Corollary 2. With the same notation for $f(s_1, s_2)$ in theorem 3, we have

$$\limsup_{\sigma_2\to\infty}\left\{\limsup_{\sigma_1\to\infty}\frac{\log\log A_\delta(\sigma_1,\sigma_2)}{\sigma_1}\right\}=\rho_1$$

(3.3.)

$$\limsup_{\sigma_1\to\infty}\left\{\limsup_{\sigma_2\to\infty}\frac{\log\log A_\delta(\sigma_1,\sigma_2)}{\sigma_2}\right\}=\rho_2$$

Theorem 4. If $f(s_1, s_2)$ is an entire function given by Dirichlet series, of finite order (ρ_1, ρ_2), ρ_1 and ρ_2 with respect to variables s_1 and s_2 respectively, then

$$\limsup_{\sigma_2\to\infty}\left\{\limsup_{\sigma_1\to\infty}\frac{\log\log G_{\delta,k_1,k_2}(\sigma_1,\sigma_2)}{\sigma_1}\right\}=\rho_1$$

$$\limsup_{\sigma_1\to\infty}\left\{\limsup_{\sigma_2\to\infty}\frac{\log\log G_{\delta,k_1,k_2}(\sigma_1,\sigma_2)}{\sigma_2}\right\}=\rho_2$$

Lemma 2. If $f(s_1, s_2)$ is an entire function given by Dirichlet series and if

$$A_\delta^{(j)}(\sigma_1,\sigma_2)=A_\delta(\sigma_1,\sigma_2\ \frac{\partial f}{\partial s_j}),\ j=1,2.$$

then, we have

(3.5.)
$$A_\delta^{(j)}(\sigma_1,\sigma_2)\geqslant\frac{A_\delta(\sigma_1,\sigma_2)\log A_\delta(\sigma_1,\sigma_2)}{\sigma_j}\quad ,j=1,2$$

Proof. It is sufficient to prove the Lemma for j = 1, and the auther case j = 2 will follow similarly. We have

$$A_\delta^{(1)}(\sigma_1,\sigma_2)=\left\{\lim_{T\to\infty}\frac{1}{(2T)^2}\int_{-T}^{T}\!\!\int \left|\frac{\partial}{\partial s_1} f(\sigma_1+it_1,\sigma_2+it_2)\right|^\delta dt_1\,dt_2\right\}^{\frac{1}{\delta}}$$

$$=\left\{\lim_{T\to\infty}\frac{1}{(2T)^2}\int_{-T}^{T}\!\!\int\right.$$

$$\left.\lim_{\epsilon\to 0}\left|\frac{f(\sigma_1+it_1,\sigma_2+it_2)-f(\sigma_1(1-\epsilon)+it_1,\sigma_2+it_2)}{\epsilon\sigma_1}\right|^\delta dt_1\,dt_2\right\}^{\frac{1}{\delta}}$$

by using Minkowski's inequality see [6] page 62, we have

$$A_\delta^{(1)}(\sigma_1,\sigma_2)\geqslant\lim_{\epsilon\to 0}\frac{1}{\epsilon\sigma_1}\left\{A_\delta(\sigma_1,\sigma_2)-A_\delta(\sigma_1(1-\epsilon),\sigma_2)\right\}$$

$$=\lim_{\epsilon\to 0}\frac{A_\delta(\sigma_1(1-\epsilon),\sigma_2)}{\epsilon\sigma_1}\left\{\frac{A_\delta(\sigma_1,\sigma_2)}{A_\delta(\sigma_1(1-\epsilon),\sigma_2)}-1\right\}$$

Since, in view of [5], for a fixed $\sigma_2>0$, $\log A_\delta(\sigma_1,\sigma_2)$ is a convex function of σ_1

$$\log A_\delta(\sigma_1,\sigma_2)=\log A_\delta(\sigma_1(1-\epsilon),\sigma_2)+\int_{\sigma_1(1-\epsilon)}^{\sigma_1}\nu(x,\sigma_2)\,dx \tag{3.6.}$$

where $\nu(x,\sigma_2)$ is non decresing function which tends to infinity with x, then

$$\log A_\delta(\sigma_1,\sigma_2)\geqslant\log A_\delta(\sigma_1(1-\epsilon),\sigma_2)+\epsilon\sigma_1\,\nu(\sigma_1(1-\epsilon),\sigma_2)$$

we observe that

$$\frac{A_\delta(\sigma_1,\sigma_2)}{A_\delta(\sigma_1(1-\epsilon),\sigma_2)}\geqslant\exp\left\{\epsilon\sigma_1\,\nu(\sigma_1(1-\epsilon),\sigma_2)\right\} \tag{3.7.}$$

therefore

$$A_\delta^{(1)}(\sigma_1,\sigma_2)\geqslant\lim_{\epsilon\to 0}\frac{A_\delta(\sigma_1(1-\epsilon),\sigma_2)}{\epsilon\sigma_1}\left\{\epsilon\sigma_1\,\nu(\sigma_1(1-\epsilon),\sigma_2)+0(\epsilon^2)\right\}$$

$$= A_\delta(\sigma_1, \sigma_2)\,\nu(\sigma_1, \sigma_2)$$

On the other hand, let $\sigma_1^\circ > 0$ be real number, such that $0 < A_\delta(\sigma_1^\circ, \sigma_2) \leqslant 1$, then from (3.6.)

$$\nu(\sigma_1, \sigma_2) \geqslant \frac{\log A_\delta(\sigma_1, \sigma_2)}{\sigma_1 - \sigma_1^\circ} \geqslant \log A_\delta(\sigma_1, \sigma_2)$$

Now, if no σ_1° is possible as defined, we shall regard the right-hand side of (3.5.) as $(1 + o(1)) \dfrac{A_\delta(\sigma_1, \sigma_2) \log A_\delta(\sigma_1, \sigma_2)}{\sigma_J}$ where $o(1) \to 0$ as $\sigma_J \to \infty$, $J = 1, 2$.

Theorem 5. If $f(s_1, s_2)$ is an entire function defined by Dirichlet series of finite order (ρ_1, ρ_2), ρ_1 and ρ_2 with respect to variables s_1 and s_2 respectively, then

(3.8.)
$$\limsup_{\sigma_2 \to \infty} \left\{ \limsup_{\sigma_1 \to \infty} \frac{\log \left\{ \dfrac{A_\delta^{(1)}(\sigma_1, \sigma_2)}{A_\delta(\sigma_1, \sigma_2)} \right\}}{\sigma_1} \right\} = \rho_1$$

$$\limsup_{\sigma_1 \to \infty} \left\{ \limsup_{\sigma_2 \to \infty} \frac{\log \left\{ \dfrac{A_\delta^{(2)}(\sigma_1, \sigma_2)}{A_\delta(\sigma_1, \sigma_2)} \right\}}{\sigma_2} \right\} = \rho_2$$

Proof. It is sufficient to prove the first equation of (3.8.). From (3.5.), we have for $J = 1$.

$$\limsup_{\sigma_2 \to \infty} \left\{ \limsup_{\sigma_1 \to \infty} \frac{\log \left\{ \dfrac{A_\delta^{(1)}(\sigma_1, \sigma_2)}{A_\delta(\sigma_1, \sigma_2)} \right\}}{\sigma_1} \right\} \geqslant \rho_1 \qquad (3.9)$$

Further, for any $\sigma_2 > 0$, $\log A_\delta(\sigma_1, \sigma_2)$ is an increasing convex function for σ_1 [5], this enables us to write $\log A_\delta(\sigma_1, \sigma_2)$ in the following form:

$$\log A_\delta(\sigma_1 + \alpha, \sigma_2) = \log A_\delta(\sigma_1, \sigma_2) + \int_{\sigma_1}^{\sigma_1+\alpha} \frac{\frac{\partial}{\partial x} A_\delta(x, \sigma_2)}{A_\delta(x, \sigma_2)} dx, \alpha > 0$$

$$\geqslant \alpha \frac{A_\delta^{(1)}(\sigma_1, \sigma_2)}{A_\delta(\sigma_1, \sigma_2)}$$

then

$$\text{(3.10)} \qquad \limsup_{\sigma_2 \to \infty} \left\{ \limsup_{\sigma_1 \to \infty} \frac{\log \left\{ \frac{A_\delta^{(1)}(\sigma_1, \sigma_2)}{A_\delta(\sigma_1, \sigma_2)} \right\}}{\sigma_1} \right\} \leqslant \rho_1$$

From (3.9.) and (3.10.) follows the first equation of the theorem.

Theorem 6. If $f(s_1, s_2)$ is an entire function defined by Dirichlet series of finite order (ρ_1, ρ_2), ρ_1 and ρ_2 with respect to variables s_1 and s_2 respectively, then

$$\text{(3.11.)} \qquad \begin{aligned} \limsup_{\sigma_2 \to \infty} \left\{ \limsup_{\sigma_1 \to \infty} \frac{\log \left\{ \frac{G_{\delta,k_1,k_2}^{(1)}(\sigma_1, \sigma_2)}{G_{\delta,k_1,k_2}(\sigma_1, \sigma_2)} \right\}}{\sigma_1} \right\} &= \rho_1 \\ \limsup_{\sigma_1 \to \infty} \left\{ \limsup_{\sigma_2 \to \infty} \frac{\log \left\{ \frac{G_{\delta,k_1,k_2}^{(2)}(\sigma_1, \sigma_2)}{G_{\delta,k_1,k_2}(\sigma_1, \sigma_2)} \right\}}{\sigma_2} \right\} &= \rho_2 \end{aligned}$$

Proof. In view of (2.4) and for small $\epsilon > 0$, we have

$$\log G_{\delta,k_1,k_2}^{(1)}(\sigma_1, \sigma_2) = k_1 k_2 \exp - (\sigma_1 k_1 + \sigma_2 k_2) \int_0^{\sigma_1} \int_0^{\sigma_2}$$

$$\left\{ \log \left\{ \frac{A_\delta^{(1)}(x_1 x_2)}{A_\delta(x_1, x_2)} \right\} + \log A_\delta(x_1, x_2) \right\}$$

$$\exp(x_1 k_1 + x_2 k_2)\, dx_1\, dx_2$$

$$\geqslant (1-\exp - \epsilon k_1)(1-\exp - \epsilon k_2)\log\left\{\frac{\overset{(1)}{A}_\delta(\sigma_1-\epsilon,\ \sigma_2-\epsilon)}{A_\delta(\sigma_1-\epsilon,\ \sigma_2-\epsilon)}\right\} +$$

$$+ \log G_{\delta,k_1,k_2}(\sigma_1,\sigma_2)$$

then

$$(3.12.)\qquad \limsup_{\sigma_2\to\infty}\left\{\limsup_{\sigma_1\to\infty}\frac{\log\left\{\dfrac{G^{(1)}_{\delta,k_1,k_2}(\sigma_1,\sigma_2)}{G_{\delta,k_1,k_2}(\sigma_1,\sigma_2)}\right\}}{\sigma_1}\right\} \geqslant \rho_1$$

Further, for any $\sigma_2 > 0$, $\log G_{\delta,k_1,k_2}(\sigma_1,\sigma_2)$ is an increasing convex function for σ_1, then, we can write, $\log G_{\delta,k_1,k_2}(\sigma_1,\sigma_2)$ as

$$\log G_{\delta,k_1,k_2}(\sigma_1+\alpha,\sigma_2) = \log G_{\delta,k_1,k_2}(\sigma_1,\sigma_2) + \int_{\sigma_1}^{\sigma_1+\alpha}\frac{G^{(1)}_{\delta,k_1,k_2}(x,\sigma_2)}{G_{\delta,k_1,k_2}(x,\sigma_2)}\,dx$$

using the techniques similar as the end of the theorem 5, we find

$$(3.13.)\qquad \limsup_{\sigma_2\to\infty}\left\{\limsup_{\sigma_1\to\infty}\frac{\log\left\{\dfrac{G^{(1)}_{\delta,k_1,k_2}(\sigma_1,\sigma_2)}{G_{\delta,k_1,k_2}(\sigma_1,\sigma_2)}\right\}}{\sigma_1}\right\} \leqslant \rho_1$$

From (3.12.) and (3.13.), we prove the first equation of (3.11.). The second equation will follow similarly.

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