ON H-SPACES OVER A BASE SPACE

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ABSTRACT:

Working in the category of k-spaces we study the question when the group of vertical homotopy classes $\pi_0(\mathrm{SEC}(B,E))$ of sections of a group-like space $E \to B$ over B is nilpotent. As an application we obtain e.g. that the group of homotopy classes of fibre homotopy equivalences of a fibration $X \to B$ inducing the identity on $H_*(X_b; \mathbb{Z})$ is nilpotent, if B is a connected finite-dimensional and the fibre X_b is a connected nilpotent finite-dimensional CW-complex.

KEY WORDS: H-spaces over a base space, fibre homotopy equivalence, spaces of sections.

SUBJECT CODE CLASSIFICATIONS: 55P45, 55S37.

0. INTRODUCTION

A look through the recent book [6] motivated me to reconsider the theory of H-spaces over a base space. In particular, in section 1 we shall reformulate theorem (7.41) of [6] in this setting. In section 2 we shall discuss the existence of homotopy inverses; this section will be partly illustrative, partly will it prepare for section 3. There we will obtain the most interesting application: we will combine the theory with results of [4] to study more closely the nilpotency of certain groups of fibre homotopy classes of fibre homotopy equivalences. In section 4 we will reformulate theorem (7.43) of [6] by introducing ideals of H-spaces.

We will work within the category of k-spaces (see [2], section 7.2). But all the base spaces B occurring are supposed to be hausdorff. We denote by T_B the category of spaces over B ([6], chap.3) and by T_B^B the category of sectioned spaces over B ([6], chap.3). The corresponding morphism sets are denoted by $\mathrm{MAP}_B(X,Y)$ (resp. $\mathrm{MAP}_B^B(X,Y)$).

1. H-SPACES OVER A BASE SPACE

1.1. ***Definition:*** An H-space in T_B is a fibration π: $E \to B$ together with a section σ: $B \to E$ and a multiplication m: $E \times_B E \to E$ over B which has σ as a homotopy unit over B, i.e. $(m/E \times_B \sigma(B) \cup \sigma(B) \times_B E)$ is homotopic over B to the folding map $E \times_B \sigma(B) \cup \sigma(B) \times_B E \to E$ given by $(e,\sigma(b)) \to e$ and $(\sigma(b),e) \to e$.

The H-space is called "homotopy-associative", if m is homotopy-associative over B; it is called "group-like", if it is homotopy-associative and admits a homotopy inverse over B.

Similarly we define an H-space (resp. group-like space) in T_B^B (comp. [6], (5.41) ff.); i.e. all maps and homotopies also have to respect the section σ.

For $b \in B$ we denote by E_b the fibre of $E \to B$ over b.

Note that the multiplication of E restricts to a multiplication on each fibre E_b such that E_b is an H-space in T_* (resp. T_*^*); in both cases we will refer to E_b as an H-space.

Let SEC(B,E) be the set of sections of π and let $\pi_0(\mathrm{SEC}(B,E))$ be the set of vertical homotopy classes of sections. Two sections s,t: $B \to E$ can be multiplied in the obvious way, i.e. st: $= m(s \times t)\Delta$, where Δ: $B \to B \times B$ is the diagonal, $s \times t$: $B \times B \to E \times_B E$ is the cartesian product and m is the multiplication. This multiplication induces a multiplication on $\pi_0(\mathrm{SEC}(B,E))$.

1.2. ***Proposition:*** Let π: $E \to B$ be an H-space over B such that $\pi_0(\mathrm{SEC}(B,E))$ is a group. Let $\mathrm{SEC}_1(B,E)$ be the set of sections s such that for all $b \in B$ the points $\sigma(b)$ and s(b) lie in the same path component of E_b. Let B admit a numerable covering $V_1 \cup \ldots \cup V_n$ by categorical subsets V_i (i.e. the V_i are contractible in B).

Then $\pi_0(\mathrm{SEC}_1(B,E))$ is nilpotent of class less than n.

Proof: The argument in the proof of (7.41) of [6] can be adapted in the obvious way. But since the formulas have to be changed slightly, we will give it here.

We may assume that $\{V_1, \ldots, V_n\}$ is numerically defined, i.e. there is a partition $\{\alpha_1, \ldots, \alpha_n\}$ of unity with V_i: $= \alpha^{-1}(0,1]$, $i = 1, \ldots, n$. Define U_k: $= V_1 \cup \ldots \cup V_k$ and let $\Gamma_k \subset \pi_0(\mathrm{SEC}_1(B,E))$ be the subgroup of homotopy classes represented by sections which are homotopic to σ over U_{k+1}. Then $\Gamma_0 = \pi_0(\mathrm{SEC}_1(B,E))$ by (6.58) of [6]; we have to show that the commutator of $\phi \in \Gamma_{k-1}$ and $\psi \in \Gamma_0$ is an element of Γ_k.

Let $\{\alpha,\beta\}$ be a numeration of the covering $\{U_k, V_{k+1}\}$ of U_{k+1}. Let G_t be a vertical homotopy between $\phi \mid U_k$ and $\sigma \mid U_k$, let H_t be a vertical homotopy between $\psi \mid V_{k+1}$ and $\sigma \mid V_{k+1}$. Define families of sections

$$K,L\colon I \times I \times (U_k \cap V_{k+1}) \to E \mid U_k \cap V_{k+1}$$

by $K(s,t,b)\colon = m(H_s(b), G_t(b))$, $L(s,t,b)\colon = m(G_t(b), H_s(b))$.

Let us first assume that σ is a strict homotopy unit over B. Then a vertical homotopy between $\phi\psi \mid U_{k+1}$ and $\psi\phi \mid U_{k+1}$ is given by the formulas

$$h_t(b) := \begin{cases} L(2t, 4t\alpha(b), b) & t \leqslant \frac{1}{2} \\ K(2\text{-}2t, 4\alpha(b)-4t\alpha(b), b) & t \geqslant \frac{1}{2}, \end{cases} \quad \alpha(b) \leqslant \frac{1}{2}, \ b \in U_k \cap V_{k+1},$$

$$h_t(b) := \begin{cases} L(4t\beta(b), 2t, b) & t \leqslant \frac{1}{2} \\ K(4\beta(b) - 4t\beta(b), 2-2t, b) & t \geqslant \frac{1}{2}, \end{cases} \quad \alpha(b) \geqslant \frac{1}{2}, \ b \in U_k \cap V_{k+1},$$

$$h_t(b) := \begin{cases} m(\phi, H_{2t}(b)) & t \leqslant \frac{1}{2} \\ m(H_{2-2t}(b), \phi(b)) & t \geqslant \frac{1}{2}, \end{cases} \quad b \in V_{k+1} \backslash U_k$$

$$h_t(b) := \begin{cases} m(G_{2t}(b), \psi(b)) & t \leqslant \frac{1}{2} \\ m(\psi(b), G_{2\text{-}2t}(b)) & t \geqslant \frac{1}{2}, \end{cases} \quad b \in U_k \backslash V_{k+1}.$$

If σ is only a unit up to homotopy we have to "insert" another homotopy between the two parts h_t^1 with $t \leqslant \frac{1}{2}$ and h_t^2 with $t \geqslant \frac{1}{2}$ of h_t defined as above. Let M_t be a homotopy between $m \mid E \times_B \sigma(B) \cup \sigma(B) \times_B E$ and the folding map. Then a required homotopy $\tilde{h}_t$ is obtained as follows:

$$\tilde{h}_t(b) := \begin{cases} h_{2t}^1(b) & \text{for } 0 \leqslant t \leqslant \frac{1}{4}, \\ M_{4t\text{-}1}(h_{1/2}^1(b)) & \text{for } \frac{1}{4} \leqslant t \leqslant \frac{1}{2}, \\ M_{3\text{-}4t}(h_{1/2}^2(b)) & \text{for } \frac{1}{2} \leqslant t \leqslant \frac{3}{4}, \\ h_{2t\text{-}1}^2(b) & \text{for } \frac{3}{4} \leqslant t \leqslant 1. \end{cases}$$

1.3. The relation to theorem (7.41) of [6].

Let ρ: $X \to B$ and η: $Y \to B$ be spaces over B. Let $map_B(X,Y)$ be the set over B with fibres $MAP(X_b,Y_b)$ topologized in the category of k-spaces as in [2], section 7.

Then $MAP_B(X,Y)$ and $SEC(B,map(X,Y))$ are homeomorphic by [2], section 7. If ρ,η are fibrations, then $map_B(X,Y) \to B$ is a fibration by [3], proposition 6.

In particular, if ρ is a fibration, then $map_B(X,X) \to B$ is a fibration; the fibrewise composition of maps defines a mapping (see [1], cor.1.3.) $\mathrm{map}_B(X,X) \times_B \mathrm{map}_B(X,X) \to \mathrm{map}_B(X,X)$ such that $\mathrm{map}_B(X,X)$ is an H-space in T_B^B.

Moreover, $\mathrm{map}_B(X,X)$ contains $\mathrm{aut}_B(X,X)$ with fibres $(\mathrm{aut}_B(X,X))_b$ the spaces of homotopy equivalences $X_b \to X_b$. The map $\mathrm{aut}_B(X,X) \to B$ is a fibration by [3], corollary 7.

Now, fibre homotopy classes of fibre homotopy equivalences of ρ correspond to vertical homotopy classes of sections of $\mathrm{aut}_B(X,X) \to B$ (see [3]). Thus, if B has numerable category n, the group of homotopy classes of those fibre homotopy equivalences of ρ which on each fibre are homotopic to the identity of the fibre, is nilpotent of class less than n. Hence we recover theorem (7.41) of [6] in case ρ: $X \to B$ is a fibration in the category of k-spaces.

*1.4. **Remark:*** Let B be paracompact and let ρ: $X \to B$ be a sectioned fibration which is locally trivial as sectioned space. Let $\mathrm{aut}_B^B(X,X) \to B$ be the space over B with fibre over $b \in B$ the set of pointed homotopy equivalences $X_b \to X_b$. Because of the local triviality of $\mathrm{aut}_B^B(X,X) \to B$ and the paracompactness of B the map $\mathrm{aut}_B^B(X,X) \to B$ is a fibration ([6], (7.48)). Therefore the discussion of 1.3 also applies to fibre homotopy equivalences of the sectioned space ρ: $X \to B$ (Comp. [7], [10]).

Note that it would be useful to know when $\mathrm{aut}_B^B(X,X) \to B$ is a fibration, if ρ is just a fibration.

1.5. Application to groups of homotopy classes of maps into group-like spaces over a base space.

Let ρ: $X \to B$ be a fibration (resp. a sectioned fibration over a paracompact B which is locally trivial as sectioned space). Let π: $E \to B$ be a group-like space in T_B (resp. a group-like space in T_B^B which is locally trivial as sectioned space).

The multiplication of E obviously induces a group structure on the set $\pi_0(\mathrm{MAP}_B(X,E))$ of homotopy classes of maps over B (resp. on the set $\pi_0(\mathrm{MAP}_B^B(X,E))$ of homotopy classes of maps over and under B).

Proposition: If B has numerable category n, the subgroup of $\pi_0(\mathrm{MAP}_B(X,E))$ (resp. of $\pi_0(\mathrm{MAP}_B^B(C,E))$ consisting of homotopy classes of maps which on each fibre X_b are homotopic to the constant map onto $\sigma(b)$ is nilpotent of class less than n.

Proof: It suffices to identify $\mathrm{MAP}_B(X,E)$ (resp. $\mathrm{MAP}_B^B(X,E)$) with $\mathrm{SEC}(\mathrm{map}_B(X,E))$ (resp. $\mathrm{SEC}(\mathrm{map}_B^B(X,E))$) and to note that $\mathrm{map}_B(X,E) \to B$ (resp. $\mathrm{map}_B^B(X,E) \to B$) are H-spaces over B. E.g. the multiplication $\mathrm{map}_B(X,E) \times_B \mathrm{map}_B(X,E) \to \mathrm{map}_B(X,E)$ is defined as the composition of

$map_B(X,E) \times_B map_B(X,E) \to map_B(X \times_B X, E \times_B E)$ given by the fibrewise cartesian product with Δ^* and m_* where $\Delta: X \to X \times_B X$ is the diagonal and $m: E \times_B E \to E$ the multiplication.

1.6. The question arises what can be said about $\pi_0(SEC(B,E))$ *more generally.*

Proposition: Assume that B has finite numerable category. Let $\pi: E \to B$ be a group-like space over B such that $\pi_0(E_b)$ is solvable for all $b \in B$. Then $\pi_0(SEC(B,E))$ is solvable.

Proof: Let $\pi_0(SEC(B,E))^{(k)}$ be the derived series of $\pi_0(SEC(B,E))$. For any categorical subset $U \subset B$ one has $\pi_0(SEC(U, E \mid U)) \cong \pi_0(E_b)$ by (6.58) of [6]. Hence some member $\pi_0(SEC(B,E))^{(j)}$ of the derived series is contained in $\pi_0(SEC_1(B,E))$ which is nilpotent by 1.2 proposition.

Remark: Let $\pi: E \to B$ be an H-space over a pathwise connected base B. Then all the H-spaces E_b, $b \in B$, are H-equivalent. In particular, if π is group-like, all the groups $\pi_0(E_b)$ are isomorphic.

Remark: If in addition to the assumptions of the proposition one assumes that $\pi_0(E_b)$ is nilpotent for all $b \in B$, it does in general not follow that $\pi_0(SEC(B,E))$ is nilpotent. An example is found in [11], p.72.

In section 3 we will discuss this matter further.

2. EXISTENCE OF HOMOTOPY INVERSES.

*2.1. **Lemma:*** Let $\pi: E \to B$ be a homotopy associative H-space in T_B such that E_b is group-like for all $b \in B$. Assume that B has a numerable covering by categorical sets.

Then π is group-like in T_B.

Proof: The shearing map $E \times_B E \to E \times_B E$, $(x,y) \to (x,m(x,y))$, is a homotopy equivalence on each fibre, hence it is a homotopy equivalence over B by [6], (7.59). One can now argue as in the absolute (but unpointed) case (see e.g. [14], chap. III, (4.16) to obtain a homotopy inverse over B.

*2.2. **Remark:*** In [5] a criterion is given when a fibre homotopy equivalence of sectioned spaces is actually an equivalence of sectioned spaces. Using this one may obtain a similar result in the case of T_B^B.

*2.3. **Definition:*** An H-space X with multiplication $m: X \times X \to X$ is called "weakly regular", if all left and right translations $L_x: X \to X$, $y \to m(x,y)$, and $R_x: X \to X$, $y \to m(y,x)$, are homotopy equivalences.

*2.4. **Lemma:*** Let $\pi: E \to B$ be a homotopy associative H-space over the CW-complex B. Let each fibre E_b be weakly regular. Then $\pi_0(SEC(B,E))$ is a group.

Proof: By [9] (where a weakly regular H-space is called an "H'-space") the shearing map $\chi: E \times_B E \to E \times_B E$, $(x,y) \to (x,m(x,y))$ is a weak homotopy equivalence on each fibre. It follows from the exactness of the homotopy sequence of the fibration $E \times_B E \to B$ that χ is a weak homotopy equivalence.

Let now $s: B \to E$ be a section, let $s': = s \times \sigma: B \to E \times_B E$, $b \to (s(b), \sigma(b))$, then there exists a map $\tau: B \to E \times_B E$ with $\chi\tau$ homotopic to s'. Let $\tilde{\pi}$ be the projection $E \times_B E \to B$; then $\tilde{\pi}\tau = \tilde{\pi}\chi\tau \sim \tilde{\pi}s' = id_B$; therefore there is τ' such that $\tau \sim \tau'$ and $\tilde{\pi}\tau' = id_B$. But the two sections $\chi\tau'$, s' being homotopic are vertically homotopic ([6], (6.45)).

It follows that $\pi_2\tau'$ is a homotopy inverse of s over B, where $\pi_2: E \times_B E \to E$ is the projection onto the second factor.

3. Weakly nilpotent H-spaces. Applications.

*3.1. **Lemma:*** Let X be a weakly regular homotopy-associative H-space. Then the group $\pi_0(X)$ operates canonically on the groups $\pi_n(X,*)$ where $*$ is an element of the identity component of $\pi_0(X)$.

Proof: This has been shown in [9].

Let us recall the way the action is defined.

Let X_* be the path component of X containing $*$. The canonical map $a: \pi_n(X,*) \to \pi_0(MAP(S^n,X_*))$ is bijective. For $[x] \in \pi_0(X)$ choose $y \in X$ with $[x][y] = [x]$. Identifying a class $[f] \in \pi_n(X,*)$ with its image $a([f])$ define $[x] . [f]: = [L_x R_y f]$.

Then $\pi_0(MAP(S^n,X))$ is isomorphic to the semi-direct product $\pi_n(X,*) \times_s \pi_0(X)$.

*3.2. **Definition:*** A weakly regular homotopy associative H-space X is called "weakly nilpotent", if $\pi_0(X)$ is nilpotent and operates nilpotently on all groups $\pi_n(X,*)$ for $n \geq 1$. (Compare [11], [12]).

*3.3. **Remark:*** This condition is equivalent to the statement that $\pi_0(MAP(S^n,X))$

is nilpotent for all $n \geqslant 1$. For $\pi_0(\mathrm{MAP}(S^n, X))$ is isomorphic to the semi-direct product $\pi_n(X,*) \times_s \pi_0(X)$.

3.4. Remark: Let X be a weakly nilpotent H-space, let B be a finite dimensional CW-complex. Then the H-space MAP(B,X) is weakly nilpotent.

Proof: Since $\pi_0(\mathrm{MAP}(B,X))$ is a group by [9], MAP(B,X) is weakly regular; it is also homotopy associative, hence it remains to show that $\pi_0(\mathrm{MAP}(S^n,\mathrm{MAP}(B,X)))$ $\cong \pi_0(\mathrm{MAP}(S^n \times B,X))$ is nilpotent. This follows from [12], Satz 1 (Note that it is not necessary to assume that E is of the homotopy type of a CW-complex as it was done there).

3.5. Proposition: Let B be of the homotopy type of a connected finite dimensional CW-complex and let π: $E \to B$ be a homotopy-associative H-space in T_B such that one fibre E_b is (and hence all fibres are) weakly nilpotent.

Then $\pi_0(\mathrm{SEC}(B,E))$ is nilpotent.

Proof: The proof of 3.Satz in [13] works also under these slightly weaker assumptions. (Note also that 3.Lemma of [13] which is trivially false is not needed in that proof).

Similarly to 3.4 remark one obtains.

3.6. Corollary: The H-space SEC(B,E) is weakly nilpotent.

3.7. Application to groups of homotopy classes of fibre homotopy equivalences.

Let B be of the homotopy type of a connected finite dimensional CW-complex. Let ρ: $X \to B$ be a fibration with connected fibre.

Proposition: (a) Assume that the fibre of ρ is of the homotopy type of either a finite dimensional CW-complex or of a CW-complex whose homotopy groups vanish above a certain degree. Let ρ be sectioned and locally trivial as sectioned space and let B be paracompact. Let G^π be the group of homotopy classes of homotopy equivalences of the sectioned space ρ: $X \to B$ which induce the identity on $\pi_*(X_b,*)$ for one fibre (and hence all fibres) X_b.

Then G^π is nilpotent.

(b) Let the fibre of ρ be nilpotent and of the homotopy type of a finite dimensional CW-complex. Let G^H be the group of homotopy classes of fibre homotopy equivalences of ρ inducing the identity of $H_*(X_b; \mathbb{Z})$ for one fibre (and hence all fibres) X_b.

Then G^H is nilpotent.

Proof: Let $\mathrm{aut}^{\pi}(X) \to B$ (resp. $\mathrm{aut}^{H}(X) \to B$) be the maps with fibres $\mathrm{aut}^{\pi}(X)_b = \mathrm{AUT}^{\pi}(X_b)$ (resp. $\mathrm{AUT}^{H}(X_b)$) where $\mathrm{AUT}^{\pi}(X_b)$ (resp. $\mathrm{AUT}^{H}(X_b)$) consists of those pointed homotopy equivalences of X_b which induce the identity on $\pi_*(X_b,*)$ (resp. of the homotopy equivalences of X_b inducing the identity on $H_*(X_b;\mathbb{Z})$). Both maps are fibrations; the first one, because it is locally trivial and B paracompact; it follows from the proof of [3], corollary 7, that the second one is a fibration. Hence $\mathrm{aut}^{\pi}(X) \to B$ and $\mathrm{aut}^{H}(X) \to B$ are homotopy associative H-spaces over B with weakly regular fibres. It has been shown in [4] that the classifying spaces of $\mathrm{AUT}^{\pi}(X_b)$ and $\mathrm{AUT}^{H}(X_b)$ are nilpotent spaces. This is equivalent to $\mathrm{AUT}^{\pi}(X_b)$ and $\mathrm{AUT}^{H}(X_b)$ being weakly nilpotent H-spaces. Note that in [4] the compact-open topology is used; but if an H-space is weakly nilpotent, it is also weakly nilpotent with its topology changed into the k-topology. The result now follows from 3.5 by the isomorphisms $G^{\pi} \cong \pi_0(\mathrm{SEC}(B, \mathrm{aut}^{\pi}(X)))$ and $G^{H} \cong \pi_0(\mathrm{SEC}(B, \mathrm{aut}^{H}(X)))$.

Remark: In view of other results of [4] there are some possible variations of the proposition.

4. Ideals of H-spaces.

4.1. Definition: Let $E \to B$ be an H-space. Let $C \to B$ be a fibration such that C is a subset of E and the inclusion $C \to E$ (over B) is continuous. Then $C \to B$ is called an "ideal of $E \to B$", if the multiplication m of E induces maps $E \times_B C \to C$ and $C \times_B E \to C$.

The ideal is called "trivial on the right", if for any $b \in B$ and each $x \in C_b$ the point $m(x,y)$ does not depend on $y \in E_b$.

4.2. Example: Let X be a space, then MAP(X,X) is an H-space with multiplication the composition of maps which contains X as the subspace of constant maps. Obviously X is an ideal in MAP(X,X) which is trivial on the right.

The subspace of MAP(X,X) consisting of the maps inducing the zero map on reduced homology is an ideal.

4.3. Proposition: Let $\pi: E \to B$ be an H-space over B with ideal $\pi_C: C \to B$ which is trivial on the right. Let $s_1, \ldots, s_n \in \mathrm{SEC}(B,E)$ such that for all $b \in B$ and $i = 1, \ldots, n$ there is a path from $s_i(b)$ to C_b within E_b.

Then, if n is the numerable category of B, the section $(\ldots(s_1 s_2) \ldots s_n)$ is homotopic to a section of π_C.

Proof: The proof of theorem (7.43) of [6] is easily adapted. This time we only indicate the necessary changes and begin by giving the corresponding version of lemma (7.42) of [6].

Lemma: Let $B = U \cup V$ with $\{U,V\}$ an open numerable covering. Let θ,ϕ be sections of π such that $\theta \mid U$ is vertically homotopic to a section $\overline{\theta}$ of $C \mid U$ by a homotopy K_t and such that $\phi \mid V$ is vertically homotopic to a section $\overline{\phi}$ of $C \mid V$ by a homotopy L_t.

Then $m(\phi \times \theta)$ is homotopic to a section $s\colon B \to C$.

Proof: Let $\{\eta,\rho\}$ be a numeration of $\{U,V\}$. Define s by the formula

$$s(b) := \begin{cases} m(L_{\rho(b)}(b), \overline{\theta}(b)) & \text{for } b \in U \cap V, \\ m(\phi(b), \overline{\theta}(b)) & \text{for } b \in U \setminus V, \\ m(\overline{\phi}(b), \theta(b)) & \text{for } b \in V \setminus U. \end{cases}$$

The remaining task, i.e. to show that s is vertically homotopic to $m(\phi \times \theta)$, can now be accomplished using the procedure of (7.42) of [6].

Similarly, the proof of the proposition now follows closely the proof of (7.43) of [6].

4.4. Relation to theorem (7.43) of [6].

Let $\rho\colon X \to B$ be a fibration (resp. a locally trivial sectioned fibration over paracompact B). Let $E \to B$ be the H-space $\mathrm{map}_B(X,X) \to B$ (resp. $\mathrm{map}_B^B(X,X) \to B$) over B. Let C be the subspace of E with fibres C_b the set of constant maps $X_b \to X_b$ (resp. the constant map $X_b \to \{\sigma(b)\} \subset X_b$). Then C is an ideal of E which is trivial on the right.

Let now $s_1, \ldots, s_n \in \mathrm{MAP}_B(X,X)$ (resp. $\mathrm{MAP}_B^B(X,X)$) such that $s_i \mid X_b\colon X_b \to X_b$ is nulhomotopic for each b and $i = 1, \ldots, n$; then the $s_1, \ldots, s_n$ correspond to sections of $E \to B$ to which the proposition may be applied. Hence $(\ldots(s_1 s_2)\ldots s_n)$ is nulhomotopic over B (Comp. [8], [10]).

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