ELEMENTS OF FINITE ORDER IN THE FUNDAMENTAL GROUP OF A BRANCHED CYCLIC COVERING

por

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To the memory of A. Reyes

ABSTRACT

In this paper we establish relations between the order of a non-free automorphism of a P.L. manifold and its fundamental group. Similar relations can be found in the literature: [B] and [C].

1. PRELIMINARIES

Let M be a connected q-dimensional P.L. manifold, and $\tilde{L} \subset \mathring{M}$ be a locally flat, codimension two, submanifold. Suppose that ϕ is a periodic automorphism of M of order h, whose fixedpoint set is $\tilde{L}$. Let $M/\phi = N$ be the quotient manifold then $p : M \to N$ is an h-fold cyclic covering branched over L with $L \supset p(\tilde{L})$.

We choose $o \in N\text{-}L$ as base point. Assume that there is a fixed triangulation in M. We shall call *meridian* m of L every element of π_1 (N-L,o) which has a representative bcb^{-1} where b is a path connecting o with the star of a vertex v of L, St(v), and c is a representative of a generator of π_1 (St(v)-L).

Let i: N-L $\to$ N be the inclusion map. We can consider a presentation of π_1 (N-L,o) as follows:

$$< m_i, n_s \mid r_t >_{i \in I,\, s \in S,\, t \in T}$$

where the m_i are meridians of L, I is the set of vertices of L, and $s \in S$ are elements such that $i_* (n_s) \neq 1$ in $\pi_1(N)$.

We call $\{ o_1, \ldots, o_h \} = p^{-1}(o)$, $o_n = \phi^{n-1}(o_1)$. Let $\omega: \pi_1(N\text{-}L,o) \to S_h$ be the representation corresponding to the covering $p: M \to N$. Note that there is at least a meridian which is sent to an h-cycle; call this meridian m_1 and suppose

$\omega(m_1) = (1 \ldots h)$. In [F] R.H. Fox describes a process for obtaining a presentation of π_1 (M). We shall apply this process to our case.

Let o_1 be the base point of M. We shall call m_{ij} the lift of the meridian m_i origin in o_j. We define the *connecting tree* by $A = m_{11} \cup m_{12} \cup \ldots \cup m_{1,h-1}$. Then π_1 (M, A) ($\simeq \pi_1$ (M,o_1) because A is contractible) is generated by $(m_{ij})_{\substack{i \in I \\ j=1,\ldots,h}}$ and $(n_{sj})_{\substack{s \in S \\ j=1,\ldots,h}}$ the lifts of n_s. We have the relations:

1. Lifting of the relations r_t, $t \in T$: $(r_{tj})_{\substack{t \in T \\ j=1,\ldots,h}}$
2. Branch relations: If $\omega(m_i) = \ldots(a_1 \ldots a_\ell)\ldots$, $\ell \mid h$, then $m_{ia_1} \ldots m_{ia_\ell} = 1$.
3. Connecting relations: $m_{11} = 1, \ldots, m_{1,h-1} = 1$.

Note that $m_{1h} = 1$ beacause of the branch relation $m_{11}m_{12} \ldots m_{1h} = 1$ and the connecting relations $m_{11} = 1, \ldots, m_{1,h-1} = 1$. If we call $B = A \cup m_{1h}$ then π_1 (M,A) $\simeq \pi_1$ (M,B). It is very important to note that ϕ (B) = B, then we can consider $\phi_*: \pi_1$ (M,B) $\to \pi_1$ (M,B), the automorphism of π_1 (M,B) induced by ϕ. Now we have that $\phi_*(m_{ij})$ is represented by the lifting of m_i with origin in $\phi(o_j) = o_{j+1}$; thus $\phi_*(m_{ij}) = m_{i,j+1}$. Also $\phi_*(n_{sj}) = n_{sj+1}$.

2. The Main Result

Theorem. Let M *be a connected* q*-dimensional* P.L. *manifold, and* $\tilde{L} \subset \mathring{M}$ *a locally flat, codimension two, submanifold. Suppose that* ϕ *is a periodic automorphism of* M *of order* h, *whose fixed-point set is* $\tilde{L}$. *Assume that:*

(1) ϕ *induces the identity on* π_1 (M)

(2) π_1 (M) $\not\cong \pi_1$ (M/ϕ).

Then there is in π_1 (M) *a non-trivial element whose order divides* h.

Proof.

If $m_{ij} = 1$ for every $i \in I$, $j = 1, \ldots, h$ then since $\phi_* = \mathrm{id}$ we have

$$\pi_1(M) \cong \langle m_i, n_j \mid r_t, m_i \rangle \cong \pi_1(N)$$

Assume that there is $m_{ij} \in \pi_1$ (M,A) with $m_{ij} \neq 1$. Suppose that $\omega(m_i) = \ldots(ja_2 \ldots a_\ell)\ldots$ then $\ell \mid h$. But $m_{ij} = \phi_* m_{ij} = m_{i,j+1}$ so $m_{ij} = m_{ip}$, $p = a_2, \ldots, a_\ell$. We have the branch relation $m_{ij}m_{ia_2} \ldots m_{ia_\ell} = 1$, that is to say, $m_{ij}^\ell = 1$; hence the order of m_{ij} divides ℓ (and h). □

Scholia:

(1) *Under the conditions of the theorem we can say that* π_1 (M) *is generated by elements whose orders divide* h *and by elements whose orders are the orders of some elements of* π_1 (M/ϕ).

(2) *Let all the elements of* π_1 (M) *except the unity have infinite order and let* ψ *be a periodic automorphism of* M *of order* h, *isotopic to the identity, whose fixed-point set is* $\widetilde{L}$. *If* $\pi_1 (M) \not\cong \pi_1 (M/\psi)$ *there exists* $x \in \pi_1$ (M) *such that* $x \notin C(\pi_1(M))$ *(center of* π_1 (M)) *and* $x^h \in C(\pi_1(M))$.

Proof. All the elements of π_1 (M) have infinite order and $\pi_1 (M) \not\cong \pi_1 (M/\psi)$ then, by the theorem, ψ_* is not the identity. Since the automorphism ψ is isotopic to the identity, ψ_* is an interior automorphism of order h; hence there exists $x \in \pi_1$ (M), $x \notin C(\pi_1(M))$ and $x^h \in C(\pi_1(M))$.

Corollaries:

(1) *Let* p: $M \to S^n$ *be an* h*-fold cyclic covering such that for an* $x \in S^n$, $p^{-1}(x)$ *consists of just one point (in particular, this is the case if* h *is prime). If* $\pi_1 (M) \neq 1$ *and the covering transformation induces the identity on* π_1 (M), *then there exists* $x \in \pi_1$ (M), $x \neq 1$, *such that the order of* x *divides* h.

(2) *Suppose that* M *is a manifold which has infinitely many periodic automorphisms* $\{\phi_i\}_{i \in I}$ *of prime different periods and* π_1 (M) *is finite. Assume that the fixed point set of* ϕ_i *is a submanifold of codimension* 2 *of* M, *for each* $i \in I$. *Then there exists a finite subset* $F \subset I$ *such that* $\pi_1 (M) = \pi_1 (M/\phi_i)$ *for* $i \in I\text{-}F$.

3. EXAMPLE

Suppose that π_1 (M) is the binary icosahedral group, I^*, and ϕ is an automorphism of M with non-empty fixed-point set such that $M/\phi = S^n$ then $|\phi|$ is not coprime with 2, 3 or 5 (note that Aut (I^*) is S_5, see [M-B-D]). For example, the homology sphere of Poincaré [P] is not an h-fold cyclic covering with h prime different of 2,3 or 5.

On the other hand, we take $M \times S^2$. We call $\phi\alpha = id \times r_\alpha$, where r_α is the rotation of angle α of S^2. Then $\phi \frac{2\pi}{n}$, $n \in N$, provide us with infinitely many periodic automorphisms of different periods, but it is clear that in this case $\pi_1 (M \times S^2) = \pi_1 (M \times S^2/\phi_\alpha)$.

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