ON INCLUSION RELATIONS BETWEEN LORENTZ SEQUENCE SPACES AND INEQUALITIES OF LEWIS TYPE

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ABSTRACT:

Are well known the Lorentz sequence spaces $\ell_{p,q}$ and the relations $\ell_{p,q} \subset \ell_{p_1,q}$ if $1 \leqslant p < p_1 \leqslant \infty$ and $\ell_{p,q} \subset \ell_{p,q_1}$ if $1 \leqslant p \leqslant \infty$, $1 \leqslant q < q_1 < \infty$ [4], [5], [1]. In this paper a generalization of these spaces is considered using the symmetric norming functions of $\mathfrak{R}$. Schatten [2], [8], [9] and some inclusion relations are presented. In the second part of the paper these inclusion relations are utilised to establish inequalities of Lewis type [1], [7] for some operator ideals generated by an additive s-function (s-number) [5], [6], [11].

1. INCLUSION RELATIONS BETWEEN THE SPACES $\ell_{\phi,\psi,p,q}$

Let be C_o the space of all real sequences converging to 0 and $\hat{C}$ a subspace of C_o containing the sequences of finite rank, i.e $x \in \hat{C}$ if $x = (x_1, x_2, \ldots, x_\eta, o, \ldots)$, $\eta < \infty$.

A function $\phi : \hat{C} \to \mathfrak{R}$ is called symmetric norming function ($\phi \in \mathcal{N}$) [2], [8], [9] if the following properties are verified.

1. ϕ is a norm on $\hat{C}$.
2. $\phi(1,0,0,\ldots) = 1$
3. $\phi(x_1, x_2, \ldots, x_\eta, o, \ldots) = \phi(|x_{i_1}|, |x_{i_2}|, \ldots, |x_{i_\eta}|, o, \ldots)$, where $i_1, i_2, \ldots, i_\eta$ is a permutation of the set $1, 2, \ldots, n$.

If $x \in C_o$ but $x \notin \hat{C}$ then $\phi(x) := \sup_\eta \phi(x_1, x_2, \ldots, x_\eta, o, \ldots)$.

In the sequel we consider only the sequences $x \in \overline{C}_o$ i.e $x_o \in C_o$ and $x_i \geqslant o$ for all i. We consider also that the sequences are writen in nonincreasing rearrengement.

We denote $\phi(\eta) = \phi(\underbrace{1, 1, \ldots, 1}_{n}, 0, \ldots)$ and $\mathcal{N}^* = \{\phi \in \mathcal{N} : \phi(\eta) < \phi(\eta+1)\}$, $n = 1, 2, \ldots$

Let be $x \in \overline{C}_0$, $\phi \in \mathcal{N}$ and $\psi \in \mathcal{N}^*$, $1 \leqslant p,q \leqslant \infty$. We say that $x \in \ell_{\phi,\psi,p,q}$ if $\|x\|_{\phi,\psi,p,q} = \left\{\phi(\psi(\eta)^{\frac{q}{p}-1} x_\eta^q)\right\}^{\frac{1}{q}} < \infty$. ($\|x\|_{\phi,\psi,p,\infty} = \phi_\infty([\psi(\eta)^{\frac{1}{p}} x_\eta])^{*)}$).

In a simple way it results that $\ell_{\phi,\psi,p,q}$ is a quasinormed vector space. For the particular case when $\phi(x) = \psi(x) = \phi_1(x) = \Sigma\, |x_i|$ it results the spaces $\ell_{p,q}$. For $\phi = \psi \neq \phi_1$ we have the spaces $\ell_{\phi,p,q}$, $\phi \in \mathcal{N}^*$. ($\ell_{\phi,p,\infty} = \left\{(x_i) \in C_0 : \sup\, [\phi(\eta)\, x_\eta] < \infty\right\}$).

From the properties of the functions ϕ it result

Lemma 1: *For all* $\phi \in \mathcal{N}^*$ *and* $\alpha \in [1, \infty)$, $\eta = 1, 2, \ldots$ *the following relation holds*

$$\phi(\eta)^\alpha - \phi(\eta-1)^\alpha \leqslant \alpha\, \phi(\eta)^{\alpha-1}.$$

Proof: $\phi(\eta) \leqslant \phi(\eta-1) + \phi(1) = \phi(\eta-1) + 1$

Hence

$$\phi(\eta)^\alpha - \phi(\eta-1)^\alpha \leqslant \phi(\eta)^\alpha - [\phi(\eta) - 1]^\alpha \leqslant \alpha \cdot \phi(\eta)^\alpha.$$

Remark. For $\alpha \in (0, 1)$ it results $\phi(\eta)^\alpha - \phi^\alpha(\eta-1) \geqslant \alpha\, \phi^{\alpha-1}(\eta)$.

Lemma 2: *If* $1 \leqslant i \leqslant n$ *then there exists* c, $0 < c < 1$, *such that*

$$c \cdot \phi^\alpha(n) \leqslant \psi\,(\phi(i)^\alpha - \phi^\alpha(i-1)),\ \textit{for all}\ \phi \in \mathcal{N}^*,\ o < \alpha < \infty,\ \psi \sim \phi_1.$$

The proof is easy and we omite.

Recall also that, for all $\phi \in \mathcal{N}$, $\phi(x) \leqslant \phi(y)$ if $x_\eta \leqslant y_\eta$, $\eta = 1, 2, \ldots$, [2].

Lemma 3: *Let* $1 \leqslant p,q < \infty$ *and* $x \in \ell_{\phi,\psi,p,q}$. *Then for every i holds*

1) $x_i \leqslant c(p,q)\, \psi(i)^{-\frac{1}{p}} \cdot \|x\|_{\phi,\psi,p,q}$ if $1 \leqslant p < q < \infty$, $\phi \sim \phi_1$.

2) $x_i \leqslant \psi(i)^{-\frac{1}{p}} \|x\|_{\phi,\psi,p,q}$ if $1 \leqslant q < p < \infty$ and $\phi(\eta) \geqslant \psi(\eta)$, $\eta = 1, 2, \ldots$

*) $\phi_\infty(\{x_\eta\}) = \sup\limits_\eta x_\eta$ [2].

Proof. If $x \in \ell_{\phi,\psi,p,q}$ it results

$$\|x\|^q_{\phi,\psi,p,q} \geqslant \phi\left(\left\{\psi(K)^{\frac{q}{p}-1} x_K^q\right\}_{K=1}^{i}\right) \geqslant x_i^q \cdot \phi\left(\left\{\psi(K)^{\frac{q}{p}-1}\right\}_{K=1}^{i}\right)$$

From Lemma 1 it follows

$$\|x\|^q_{\phi,\psi,p,q} \geqslant \frac{p}{q} x_i^q \, \phi\left(\left\{\psi(K)^{\frac{q}{p}} - \psi(K-1)^{\frac{q}{p}}\right\}_{K=1}^{i}\right) \text{ if } 1 \leqslant p < q < \infty.$$

By Lemma 2 we obtain

$$\|x\|^q_{\phi,\psi,p,q} \geqslant c \ \frac{p}{q} x_i^q \cdot \psi(i)^{\frac{q}{p}}$$

Hence $x_i \leqslant C_1 \left(\frac{q}{p}\right)^{\frac{1}{q}} \cdot \psi(i)^{-\frac{1}{p}} \cdot \|x\|_{\phi,\psi,p,q}$ if $1 \leqslant p < q < \infty$.

In case of $1 \leqslant q < p < \infty$, $\left\{\psi(K)^{\frac{q}{p}-1}\right\}$ is nonincreasing and therefore $\|x\|^q_{\phi,\psi,p,q} \geqslant x_i^q \, \psi(i)^{\frac{q}{p}-1} \cdot \phi(i) \geqslant x_i^q \, \psi(i)^{\frac{q}{p}}$.

Hence $x_i \leqslant \psi(i)^{-\frac{1}{p}} \|x\|_{\phi,\psi,p,q}$

We can prove now the inclusion relation

Proposition 4. 1) *Let* $1 \leqslant p < \infty$, $1 \leqslant q < q_1 \leqslant \infty$. *Then* $\ell_{\psi,\psi,p,q} \subset \ell_{\phi,\psi,p,q_1}$ *for every* $\phi \in \mathcal{N}$, $\psi \in \mathcal{N}^*$ *and for every* $x \in \ell_{\phi,\psi,p,q}$ $\|x\|_{\phi,\psi,p,q_1} \leqslant K \left(\frac{q}{p}\right)^{\frac{1}{q}-\frac{1}{q_1}}$ $\|x\|_{\phi,\psi,p,q}$ *if* $p < q$ *and* $\|x\|_{\phi,\psi p,q_1} \leqslant \|x\|_{\phi,\psi,p,q}$ *if* $p \geqslant q$, $\phi \sim \phi_1$ *if* $p < q$.

2) *Let either* $1 \leqslant p < p_1 \leqslant \infty$, $1 \leqslant q < \infty$ *or* $1 \leqslant p < p_1 < \infty$, $q = \infty$. *Then* $\ell_{\phi,\psi,p,q} \subset \ell_{\phi,\psi,p_1,q}$ *and for every* $x \in \ell_{\phi,\psi,p,q}$ $\|x\|_{\phi,\psi,p_1,q} \leqslant \|x\|_{\phi,\psi,p,q}$.*)

Proof. Let $x \in \ell_{\phi,\psi,p,q}$ and $1 \leqslant p < q < q_1 < \infty$. Then by using Lemma 3 we have

$$\|x\|^{q_1}_{\phi,\psi,p,q_1} = \phi\left(\left\{\psi(i)^{\frac{q_1}{p}-1} x_i^{q_1}\right\}\right) = \phi\left(\left\{\psi(i)^{\frac{q_1}{p}-1} x_i^{q_1-q} x_i^q\right\}\right) \leqslant$$

$$\phi\left(\left\{\psi(i)^{\frac{q_1}{p}-1} c\left(\frac{q}{p}\right)^{\frac{q_1-q}{q}} \psi(i)^{-\frac{q_1-q}{p}} \|x\|^{q_1-q}_{\phi,\psi,p,q} \cdot x_i^q\right\}\right) =$$

(*) In (1) if $q < p$ it is necessary that $\phi(\eta) \geqslant \psi(\eta)$, $\eta = 1, 2, \ldots$

$$= c\left(\frac{q}{p}\right)^{\frac{q_1-q}{q}} \|x\|^{q_1-q}_{\varphi,\psi,p,q} \cdot \phi\left(\left\{\psi(i)^{\frac{q}{p}-1} x_i^q\right\}\right) = c\left(\frac{q}{p}\right)^{\frac{q_1-q}{q}} \|x\|^{q_1}_{\varphi,\psi,p,q}$$

Whence we obtain the first part of (1) $(K = c^{\frac{1}{q_1}})$.

In case of $p \geqslant q$ we deduce

$$\|x\|^{q_1}_{\phi,\psi,p,q_1} = \phi\left(\psi(i)^{\frac{q_1}{p}-1} x_i^{q_1-q} x_i^q \right) \leqslant$$

$$\phi\left(\left\{\psi(i)^{\frac{q_1}{p}-1-\frac{q_1-q}{p}} \|x\|^{q_1-q}_{\phi,\psi,p,q} x_i^q\right\}\right) =$$

$$= \|x\|^{q_1-q}_{\phi,\psi,p,q}\, \phi\left(\left\{\psi(i)^{\frac{q}{p}-1} x_i^q\right\}\right) = \|x\|^{q_1}_{\varphi,\psi,p,q}.$$

For $1 \leqslant p < q < q_1 = \infty$ and $1 \leqslant q < p < q_1 = \infty$ the relations result directly from Lemma 3. The proof of (2) is easy using the monotonicity of the functions $\phi \in \mathcal{N}$.

Remark. *These relations are valid for the spaces* $\ell_{\phi,p,q}$ *for all* $\phi \in \mathcal{N}^*$ *if* $p \geqslant q$ *since* $\psi(\eta) = \varphi(\eta)$ *and for* $\phi \sim \phi_\pi$ *if* $p < q$. *For* ϕ_π *see* [2].

Problem. *If* $\psi(\eta) > \phi(\eta)$, $\eta = 1, 2, \ldots$ *is the relation* $\ell_{\phi,\psi,p,q} \subset \ell_{\phi,\psi,p,q_1}$ *true for* $q < p$, $q < q_1$?

2. Inequalities of Lewis type

By means of the spaces $\ell_{\phi,p,q}$ we define some operator ideals. Let E,F be normed spaces and $\mathcal{L}$ (E,F) the set of all linear and bounded operators $T: E \to F$. The n^{th} approximation number of T is defined to be $a_\eta(T) = \inf_K \|T - K\|$, $K \in \mathcal{L}$ (E,F), rank $K < n$, $n = 1, 2, \ldots$ For the definition of others s-numbers of T we refere to [5], [6], [7], [9]. It is well known that the approximation numbers are additive and multiplicative i.e.

$$a_{m+n-1}(T_1 + T_2) \leqslant a_m(T_1) + a_n(T_2) \quad ; \quad a_{m+n-1}(T_1 T_2) \leqslant a_m(T_1)\, a_n(T_2).$$

**) From the definition of $\ell_{\phi,p,q}$ it results that $\{ a_\eta(T) \} \in C_0$ hence T is an approximable operator.

We denote by $\mathcal{L}^{(a)}_{\phi,p,q}$ (E,F) $= \{ T \in \mathcal{L}$ (E,F) $: \{ a_\eta (T) \} \in \ell_{\phi,p,q} \}$ **). This class $\mathcal{L}^{(a)}_{\phi,p,q}$ (E,F) is an operator ideal (cf. [5], [7], [9], [11]) and $L^{(a)}_{\phi,p,q}(T) =$ $= \{ \phi (\{ \phi (\eta)^{\frac{q}{p}-1} a_\eta (T)^q \}) \}^{\frac{1}{q}}$ is a quasinorm on $\mathcal{L}^{(a)}_{\phi,p,q}$ (E,F).

Let $[\mathfrak{U}_1, A_1]$ and $[\mathfrak{U}_2, A_2]$ be quasinormed operator ideals. If there are the constants $\lambda \geqslant o$, $c \geqslant 1$ such that the quasinorms verifie $A_1 (S) \leqslant c.n^\lambda A_2 (S)$ for $S \in \mathcal{L}$ (E,F), rank $S \leqslant n$, $n = 1, 2, \ldots$, then this inequality is called inequality of Lewis type [1], [5].

In the sequel we present a generalized form of this inequality for the operator ideals $\mathcal{L}^{(a)}_{\phi,p,q}$ (E,F), using the results from Proposition 4.

Thus we obtain

Proposition 5. *Let* $1 \leqslant p < q \leqslant \infty$, $1 \leqslant$ u, $v < \infty$, *then*

$L^{(a)}_{\phi,p,\mu}(S) \leqslant c \cdot \phi (\eta)^{\frac{1}{p} - \frac{1}{q}} L^{(a)}_{\phi,q,\nu} (S)$ *for* $S \in \mathcal{L}$ (E,F), rank $S \leqslant n$, $n = 1, 2, \ldots$ *and* $\phi \in \mathcal{N}^*$, $\phi \sim \phi_\pi$.

Proof. Put $\frac{1}{\tau} = \frac{1}{p} - \frac{1}{q}$. In case $1 \leqslant v < \infty$ we have

$$L^{(a)}_{\phi,p,\mu}(S) = \{ \phi (\{ \phi (i)^{\frac{\mu}{p}-1} a_i (S)^\mu \}_{i=1}) \}^{\frac{1}{\mu}} =$$

$$= \{ \phi (\{ \phi (i)^{\frac{\mu}{p} - \frac{\mu}{q} - 1 + \frac{\mu}{q}} a_i (S)^\mu \}^n_{i=1}) \}^{\frac{1}{\mu}}$$

$$\leqslant \{ \phi (\{ \phi (i)^{\frac{\mu}{\tau}-1} \}^{\eta}_{i=1}) \}^{\frac{1}{\mu}} \cdot L^{(a)}_{\phi,q,\infty}(S) \leqslant c \cdot \phi (\eta)^{\frac{1}{\tau}} L^{(a)}_{\phi,q,\nu}(S).$$

Remarks. 1) This inequality is alfo valid for any other additive s-number (for example the Gelfand and Kolmogorov numbers [5], [6], [9]).

2) Since $\phi (\eta) \leqslant \phi_1 (\eta) = \eta$ it results that $L^{(a)}_{\phi,p,\mu}(S) \leqslant c \cdot \eta^{\frac{1}{p} - \frac{1}{q}} L^{(a)}_{\phi,q,\nu}(S)$.

REFERENCES

[1] B. CARL, Entropy numbers, s-numbers and eigenvalue problems. J. functional anal. 41 (1981), 290-306.

[2] I. GOHBERG, M. KREIN, Introduction into theory of nonsel fadjoints operators . . ., Providence, 1969 (engl. transl.)

[3] G. KOTHE, Topologische lineare Räume, Berlin-Heidelberg-New York (1966).

[4] M. KATO, On Lorentz spaces $\ell_{p,q}$ { E } , Hiroshima Math.· J6, 1, 1976, 73-93.

[5] A. PIETSCH, Operator ideals, Berlin, 1978.

[6] –"– S-numbers of operators in Banach spaces, Studia Math. 51 (1974), 201-223.

[7] –"– Weyl numbers and eigenvalues of operators in Banach spaces. Math. Ann. 247 (1980), 149-178.

[8] R. SCHATTEN, Norm ideals of completely continuous operators, Berlin, 1960.

[9] N. TIŢA, $\ell_{\phi,\varphi}$ operators and (ϕ, φ)-spaces, Collect. Math. 30,1. 1979, 3-10.

[10] –"– On a class of $\ell_{\phi,\varphi}$ operators, Collect. Math. 32, 3, 1981, 275-279.

[11] –"– Complemente de ideale de operatori, Braşov University press. 1983.

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