

# THE CENTRAL LIMIT THEOREM FOR STOCHASTIC INTEGRALS

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## ABSTRACT.

The results of Giné & Marcus (6) on the C.L.T. for stochastic integrals w.r.t. Lévy processes are extended to the multidimensional time-parameter situation. To reach this extension some useful convergence criteria for  $[0, 1]^q$ -indexed processes are proven, extending wellknown results of Billingsley, Chentsov, Bickel and Wichura (3), (2).

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## 0. INTRODUCTION

In (6) Giné & Marcus proved that processes defined by stochastic integration, i.e.

$$F_t = \int_0^t f.dM, \quad t \in [0, 1],$$

where  $M$  is a random measure with independent increments uniformly in the domain of normal attraction of an stable law of index  $p \in (0, 2]$ , satisfy the C.L.T. in  $D[0, 1]$  if  $p \neq 2$  and in  $C[0, 1]$  if  $p = 2$ . They prove that if  $(F_t^k)_{k=1}^\infty$  are independent copies of the process  $F_t$ , then the normalized sums

$$n^{-1/p} (F_t^1 + \dots + F_t^n)$$

converge, in the usual weak sense, to an stable process in the space  $D[0, 1]$  (or  $C[0, 1]$ ).

It is wellknown that in this kind of problems what is relevant is to have good thightness criteria in the corresponding spaces of values. The first part of the present work is devoted to find the appropriate tightness conditions in our situation, i.e. in the  $D[0, 1]^q$  (or  $C[0, 1]^q$ ) case. With this results in hand we show the main lines of the construction of the stochastic integral in a second part, and in a third part we prove the C.L.T. and give some applications: we prove that Itô's and diffusion processes with multidimensional time parameter satisfy the central limit theorem. It may be worth mentioning that in the proof of the fundamental maximal inequality, lemma (3.1) of II, an extension of Ledoux's 2-dimensional Doob's inequality to  $q$  dimensions is needed (Thm. (2.1), II).

### ***I. Convergence and regularity of stochastic processes indexed by $[0, 1]^q$ .***

In (2) P. Bickel & M. Wichura obtain fluctuation inequalities for processes indexed by  $[0, 1]^q$ , extending results of Chentsov and Billingsley, (3), (4). Here we extend their theorem 3 to the case that marginals of  $m$ , the control measure, are not necessarily continuous. Bickel & Wichura (op. cit. pag. 1665, final) announce a possible extension to the case that  $m$  depends on  $n$ , and the measures  $m_n$  converge weakly to a measure with continuous marginals. Our extension has a different character:  $m$  will be fixed (independent of  $n$ ), we will suppose instead that our processes have independent increments, and the constants that appear in their theorem 1 will depend on  $m$ ,  $q$ ,  $\gamma$ , and  $\beta$  in our case. This is the content of point 2. Point 3 is devoted to give applications of the fluctuation inequalities to the convergence of processes indexed by  $[0, 1]^q$ . An application to the regularity of processes with independent increments over  $[0, 1]^q$  is given in point 4. On this later result it is worthy to say that R. Morkvenas (10), using Dynkin-Kinney's type conditions, proves that all stochastically continuous processes with independent increments have versions in  $D[0, 1]^q$ . Our Thm. (4.1) is not enclosed in his result because we only impose right stochastic continuity.

As most of the proofs offer no difficulties we only give short sketches or just skip them completely for the sake of saving space, thus keeping the paper reasonably sized.

### **1. DEFINITIONS AND PREVIOUS RESULTS.**

Notation is much like in (2). Let  $q$  be a positive integer and  $T_1, T_2, \dots, T_q$  subsets of  $[0, 1]$ , each of which contains 0 and 1, and is finite or  $[0, 1]$ . Let

$(X_t)_{t \in T}$  be a stochastic process indexed by  $T = T_1 \times T_2 \times \dots \times T_q$  and with values in a normed space  $(E, \|\cdot\|)$ . We suppose that  $X$  is separable and vanish on the lower boundary of  $T$ , i.e. the set of points of  $T$  with some coordinate equal to 0.

For each  $p$ ,  $1 \leq p \leq q$ , and each  $t \in T_p$  define

$$X_t^{(p)} : T_1 \times \dots \times T_p \times \dots \times T_q \longrightarrow E \quad \text{by}$$

$$X_t^{(p)}(t_1, \dots, t_{p-1}, t_{p+1}, \dots, t_q) = X(t_1, \dots, t_{p-1}, t, t_{p+1}, \dots, t_q)$$

and if  $s \leq t \leq u$  in  $T_p$  define

$$m_p(s, t, u)(X) = \min \{ \|X_t^{(p)} - X_s^{(p)}\|, \|X_u^{(p)} - X_t^{(p)}\| \} \quad \text{where}$$

$\|\cdot\|$  is the supremum norm.

**Definition (1.1):** With the notations just introduced

$$M_p''(X) = \sup \{ m_p(s, t, u)(X) : s \leq t \leq u, s, t, u \in T_p \}$$

$$M''(X) = \max_{1 \leq p \leq q} M_p''(X)$$

$$M(X) = \sup \{ |X(t)| : t \in T \} \quad ::$$

The following proposition is very useful and quite elementary.

**Proposition (1.2):** If  $1_q = (1, \dots, 1)$ , then

$$M(X) \leq \sum_{p=1}^q M_p''(X) + |X(1_q)| \leq qM''(X) + |X(1_q)| \quad ::$$

We say that  $B \subset T$  is a block if  $B = \prod_{p=1}^q (s_p, t_p]$ , in this case we also write  $B = (s, t]$  where  $s = (s_1, \dots, s_q)$  and  $t = (t_1, \dots, t_q)$ .

Denote by  $X(B)$  the rectangular increment of  $X$  over the block  $B$ , i.e.:

$$X(B) = \sum_{\epsilon_1=0}^1 \sum_{\epsilon_2=0}^1 \dots \sum_{\epsilon_q=0}^1 (-1)^{q-\sum_{p=1}^q \epsilon_p} \epsilon_p X(s_1 + \epsilon_1(t_1 - s_1), \dots, s_q + \epsilon_q(t_q - s_q))$$

and say that  $X$  has independent increments if  $X(B_i)$ ,  $i = 1, \dots, n$ , are independent random variables whenever  $B_i$ ,  $i = 1, \dots, n$ , are disjoint blocks.

**Definition (1.3):** We write  $X \in C_1^m(\beta, \gamma)$  if  $X$  has independent increments and for all positive  $\lambda$

$$P \{ |X(B)| \geq \lambda \} \leq \lambda^{-\gamma} \cdot (m(B))^\beta$$

holds for all block  $B \subset T$ , for some positive finite measure,  $m$ , on  $T$ , vanishing over  $\partial_{\text{inf}} T$  and some fixed  $\gamma > 0$  and  $\beta > 0$ .

Here we recall Thm.1 of Bickel & Wichura (2), for later use.

**Theorem (1.4):** If  $X \in C^m(\beta, \gamma)$ , i.e. if for all pair of disjoint blocks  $B$  and  $C$  in  $T$ , and all positive  $\lambda$

$$P \{ |X(B)| \geq \lambda, |X(C)| \geq \lambda \} \leq \lambda^{-\gamma} (m(B \cup C))^\beta$$

for some fixed  $\gamma > 0$  and  $\beta > 1$ , then for all positive  $\lambda$  and all  $p$ ,  $1 \leq p \leq q$ , the following inequalities hold

$$P \{ M_p''(X) \geq \lambda \} \leq K_q(\beta, \gamma) \lambda^{-\gamma} \cdot (m(T))^\beta$$

$$P \{ M''(X) \geq \lambda \} \leq L_q(\beta, \gamma) \lambda^{-\gamma} \cdot (m(T))^\beta \quad ::$$

The identification  $D_q = D([0, 1]^q; \mathbb{R}) = D([0, 1]; D_{q-1})$  suggests the introduction of the following moduli.

**Definition (1.5):** For  $x \in D_q$  and  $\delta > 0$  we define

$$w_x^{(p)}(\delta) = \sup \{ \min(\|x_t^{(p)} - x_s^{(p)}\|, \|x_u^{(p)} - x_t^{(p)}\|) : s, t, u \in T_p,$$

$$s \leq t \leq u, u - s \leq \delta \}$$

$$w''(\delta) = \max \{ w^{(p)}(\delta) : 1 \leq p \leq q \} \quad ::$$

In what follows we shall also need Thm. (3.1) of Neuhaus (11).

## 2. FLUCTUATION INEQUALITIES.

The following theorem is the main technical tool in the proof of the remaining results of this part. We sketch its proof in the following. From now on  $\beta > 1/2$ .

**Theorem (2.1):** Let  $X$  be a stochastic process on  $T \subset [0, 1]^q$ , as above. Suppose that for some  $\beta > 1/2$  and  $\gamma > 0$  the process  $X$  satisfies condition  $C_i^m(\beta, \gamma)$  for  $0 < \lambda < 1$  (see Definition (1.3)). Then there exists a constant  $\underline{K} = K(q, \beta, \gamma, m(T))$  such that

$$P \{ M_p''(X) \geq \lambda \} \leq \underline{K} \lambda^{-4\gamma} (m(T))^{2\beta} \left(1 - \frac{J_p[0, 1]}{m_p[0, 1]}\right)^\beta$$

for all  $\lambda$ ,  $0 < \lambda < 1$ , and all  $p$ ,  $1 \leq p \leq q$ . Where  $J_p[0, 1]$  is the maximum jump of the distribution function,  $F_{m_p}$ , of the  $p$ -th. marginal,  $m_p$ , of the measure  $m$  over  $T$ .

**Proof.** (Sketch). It goes through three steps.

**Step 1.** Suppose, in a first step, that  $T = \{t_0, t_1, \dots, t_m\}$ , where  $t_0 = 0 < t_1 < \dots < t_m = 1$ . In this case

$$M''(X) = \sup_{0 \leq t_k \leq t_i \leq t_j \leq 1} \min \{ \|X_{t_j} - X_{t_i}\|, \|X_{t_i} - X_{t_k}\| \}.$$

If we let  $\xi_k = X_{t_k} - X_{t_{k-1}}$ ,  $\xi_0 = 0$  and  $S_j = \sum_{k=0}^j \xi_k$ , then

$$M''(X) = \sup_{0 \leq k \leq i \leq j \leq m} \min \{ \|S_j - S_i\|, \|S_i - S_k\| \}$$

$P \{ \|S_j - S_i\| \geq \lambda \} \leq \lambda^{-\gamma} \left( \sum_{i < k \leq j} m(t_k) \right)^\beta$ , and by independence of the increments

$$P \{ \|S_j - S_i\| \geq \lambda, \|S_i - S_k\| \geq \lambda \} \leq \lambda^{-2\gamma} \left( \sum_{i < e \leq j} m(t_e) \right)^\beta$$

$$\left( \sum_{k < e \leq i} m(t_e) \right)^\beta.$$

Hence, we are in position to apply thm. (12.6) of Billingsley (3), to get the result we want with  $\lambda^{-2\gamma}$  instead of  $\lambda^{-4\gamma}$ . (We will use this fact in step 3).

**Step 2.**  $q = 1$ ,  $T = [0, 1]$ ,  $m$  arbitrary.

Use discretization, step 1 and separability to show that

$$P \{ M_1''(X) \geq \lambda \} \leq K \lambda^{-2\gamma} (m(T))^{2\beta} \left(1 - \frac{J_m[0, 1]}{m(T)}\right)^\beta.$$

**Step 3.**  $q \geq 2$ ,  $T$  and  $m$  arbitrary.

Assume  $p = 1$ , for other  $p$  the argument is the same. Like in step 5 of Bickel & Wichura's proof of theorem 1, the key point is that the version for  $q = 1$  of our theorem works for the function valued process  $(X_t^{(1)})_{t \in T_1}$ . To show this it is enough to find bounds for its increments.

For fixed  $s$  and  $t$  let  $Y = X_t^{(1)} - X_s^{(1)}$  over  $T^* = T_2 \times \dots \times T_q$ . It is easy to see that if  $m^*(B') := m((s, t] \times B')$ , then

$$Y \in C_1^{m^*}(\beta, \gamma)$$

for all  $\lambda$ ,  $0 < \lambda < 1$ , hence

$$\begin{aligned} P \{ M_1''(Y) \geq \lambda r_1 (q-1)^{-1} \} &\leq \lambda^{-2\gamma} K_q(\gamma, \beta) r_1^{-2\gamma} (m_1(s, t))^{2\beta} \leq \\ &\leq \lambda^{-2\gamma} K_q(m(T))^\beta r_1^{-2\gamma} (m_1(s, t))^\beta \end{aligned}$$

for all positive  $\lambda < 1$  and  $r_1$ , by thm. (1.4). This, together with the trivial bound (see prop. (1.2))

$$\begin{aligned} P \{ \|X_t^{(1)} - X_s^{(1)}\| \geq \lambda \} &\leq P \{ M_1''(Y) \geq \lambda r_1 (q-1)^{-1} \} + \\ &P \{ |Y(1_{q-1})| \geq \lambda r_2 \} \end{aligned}$$

valid for all  $0 < \lambda < 1$  and all non negative  $r_1, r_2$  such that  $r_1 + r_2 = 1$ , leads to

$$P \{ \|X_t^{(1)} - X_s^{(1)}\| \geq \lambda \} \leq (r_1^{-2\gamma} (m(T))^\beta K_q + r_2^{-2\gamma}) \lambda^{-2\gamma} (m_1(s, t))^\beta$$

Now, by the theorem in dimension 1:

$$P \{ M_1''(X) \geq \lambda \} \leq (r_{10}^{-2\gamma} (m(T))^\beta K_q + r_{20}^{-\gamma} (\lambda^{-4\gamma}) (m(T))^{2\beta} (1 - \frac{J_1(T_1)}{m_1(T_1)})^\beta)$$

where  $r_{10}, r_{20}$  are chosen to minimize the constant, i.e.  $r_{10}$  is the solution of the equation  $2K_q (m(T))^\beta (1 - r_1)^{\gamma+1} = r_1^{2\gamma+1}$  over  $(0, 1)$ , and  $r_{20} = 1 - r_{10}$ . ::

### 3. COVERGENCE OF PROCESSES INDEXED BY $[0, 1]^d$ .

Using arguments very near to those in Billingsley (3), pg. 133-134, we can now proof the following result.

**Theorem (3.1):** Let  $(X_n)_{n=1}^\infty$  be processes over  $T = [0, 1]^d$  vanishing on  $\partial_{\text{inf}} T$ . Suppose that  $X_n \in C_1^m(\beta, \gamma)$  for some  $\beta > 1/2$  and for  $n = 1, 2, \dots$ . Then:

$$\lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} P \{ w_{X_n}^{(p)}(\delta) \geq \epsilon \} = 0$$

for all  $\epsilon > 0$  ::

**Remark.** The previous theorems also hold if condition  $X \in C_1^m(\beta, \gamma)$  is replaced by:  $X$  defined over  $(\Omega_1 \times \Omega_2, P_1 \times P_2)$  and for all  $u_1 \in \Omega_1, X_t(u_1, \cdot) \in C_1^m(\beta, \gamma)$ . We then say that  $X \in \tilde{C}_1^m(\beta, \gamma)$ .

**Theorem (3.2):** A sequence  $(P_n)_{n=1}^\infty$  of probability measures over  $(D_q, \mathcal{D}_q)$  is tight if:

i) For all  $\eta > 0$ , there exists a  $\epsilon \in \mathbb{R}$  such that

$$P_n \{ x : \sup_t |x(t)| > \epsilon \} \leq \eta, \quad n = 1, 2, \dots$$

ii) For all positive  $\epsilon, \eta$  there exists  $\delta, 0 < \delta < 1$ , and  $n_0$ , such that for all  $n \geq n_0$ :

$$a) \quad P_n \{ x : w_x''(\delta) \geq \epsilon \} \leq \eta.$$

$$b) P_n \{ x : w_x^{(p)} [0, \delta] \geq \epsilon, \text{ for some } p, 1 \leq p \leq q \leq \eta \}.$$

$$c) P_n \{ x : w_x^{(p)} [1-\delta, 1] \geq \epsilon, \text{ for some } p, 1 \leq p \leq q \leq \eta \} \leq \eta.$$

**Proof:** Apply the arguments of Billingsley (3), thm. (14.4), to the functions  $t \longrightarrow \|x_t^{(p)}\|$ . ::

As an application of the previous results we get the following theorem, that generalizes theorem (2.3) of Giné & Marcus (6), to processes indexed by  $[0, 1]^q$ .

**Theorem (3.3):** Let  $(X_n)_{n=1}^\infty$ ,  $X$  be  $D_q$ -valued random variables, vanishing on  $\partial_{\inf} T$ , and such that:

i) The finite dimensional distributions of the  $X_n$  converge weakly to the corresponding distributions of  $X$ .

ii)  $X_n \in \widetilde{C}_1^m(\beta, \gamma)$ ,  $n = 1, 2, \dots$ , for some  $\beta > 1/2$ , and  $\gamma > 0$ .

iii) For all  $\epsilon > 0$

$$\lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} P_n \{ x : w_x^{(p)} [1-\delta, 1] > \epsilon, \text{ for some } p, 1 \leq p \leq q \} = 0$$

Then  $\{P_n = L(X_n)\}_{n=1}^\infty$  converge weakly to  $L(X)$ , as a sequence of probability measures on  $(D_q, \mathcal{D}_q)$ .

**Proof:** Follows by induction on  $q$  ::

In applications quite frequently we don't know that  $X \in D_q$ . It is then useful to have the following variant of the previous theorem, whose proof requires no new arguments.

**Theorem (3.4):** Let  $(X_n)_{n=1}^\infty$  be as in thm. (3.3). Suppose that:

i) The finite dimensional distributions of  $X_n$  are weakly convergent and

$$\lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} P_n \{ x : \|x_\delta^{(p)}\| \geq \epsilon \} = 0$$



for all  $\epsilon > 0$  and all  $p, 1 \leq p \leq q$ .

ii)  $X_n \in C_i^m(\beta, \gamma)$ ,  $n = 1, 2, \dots$ , for some  $\beta > 1/2$  and  $\gamma > 0$ .

iii) For all positive  $\epsilon$

$$\lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} P_n \{ x : w_x^{(p)} [1-\delta, 1] \geq \epsilon \text{ for some } p, 1 \leq p \leq q \} = 0.$$

Then  $\{ P_n = L(X_n) \}_{n=1}^{\infty}$  is weakly convergent ::

#### 4. REGULARITY OF PROCESSES WITH INDEPENDENT INCREMENTS.

A straightforward extension of the proof of theorem (15.7) of (3), leads to

**Theorem (4.1):** If  $X \in C^m(\beta, \gamma)$ , where  $\beta > 1/2$ ,  $\gamma > 0$ , then  $X$  has a version with sample paths in  $D[0, 1]^q$  ::

### II. Construction of the stochastic integral.

Here we define integrals of the form

$$\int_0^t f(s, \omega) \cdot M(ds, \omega)$$

where  $f(s, \omega)$  is a process over  $T = [0, 1]^q$  and  $M$  is a random measure satisfying certain conditions to be specified below. The main technical complication is solved by extending to dimension  $q > 2$ , whatever, a Doob's kind inequality for strong martingales, that allows as to derive the maximal inequality, lemma (3.1), in much the same way as in Giné & Marcus (6).

#### 1. DEFINITIONS AND NOTATIONS.

**Definition (1.1):** Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $L^0(\Omega, \mathcal{F}, P)$  the set of measurable functions over  $(\Omega, \mathcal{F}, P)$ . Let  $\mathcal{B}$  denote the Borel  $\sigma$ -field of  $[0, 1]^q$ . Say that

$$M : \mathcal{B} \longrightarrow L^0(\Omega, \mathcal{F}, P)$$

is a random measure if it satisfies the following two conditions:

$$(i) \quad M\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n M(A_i) \quad \text{a.s.}$$

$$(ii) \quad \text{Pr.-}\lim_{n \rightarrow \infty} M\left(\bigcup_{i=1}^n A_i\right) = M(A) \quad \text{a.s.}$$

whenever  $A_1, A_2, \dots$  are disjoint Borel sets in  $[0, 1]^q$ , (where Pr.-lim means “limit in probability” and  $A = \bigcup_{i=1}^{\infty} A_i$ ).

**Definition (1.2):** We say that a random measure,  $M$ , is an independently scattered random measure with increments uniformly in the domain of normal attraction (D.N.A.) of a symmetric stable law of index  $p \in (0, 2]$  and control measure  $m$ , if  $m$  is a finite positive measure over  $([0, 1]^q, \mathcal{B})$  and the following conditions hold:

i)  $M(A) \in \text{D.N.A. } (m(A)^{1/p}\theta)$ , for all  $A \in \mathcal{B}$ , where  $\theta$  is a symmetric stable r.v. of index  $p$ .

$$ii) \quad \sup_{A \in \mathcal{B}} \Lambda_p(m(A)^{-1/p} M(A)) = c < \infty$$

where the sup is taken over those Borel sets with  $m(A) \neq 0$ , and  $\Lambda_p$  is the weak  $L_p$ -norm (sometimes Lorentz norm).

iii)  $M(A_1), \dots, M(A_n)$  are independent symmetric r.v.'s whenever  $A_1, \dots, A_n$  are disjoint Borel sets.

If  $p = 2$  we use the same notations but instead of i) and ii) we shall require that

$$E M^2(A) = m(A), \text{ for all } A \in \mathcal{B} ::$$

**Definition (1.3):** Let  $f(t, \omega)$ ,  $t \in [0, 1]^q$ , be a stochastic process and  $(F_t)_{t \in [0, 1]^q}$  the natural filtration associated with the process  $M(0, t]$ , i.e.  $F_t$  is the  $\sigma$ -field generated by negligible sets and the random variables  $M(A)$ , where  $A \in \mathcal{B} \cap [0, t]$ . Say that  $f$  is *simple and non-anticipative w.r.t.  $(F_t)$*  if there exist partitions of  $[0, 1]$ ,  $0 = s_1^{(1)} < s_2^{(1)} < \dots < s_{n_1+1}^{(1)} = 1, \dots, 0 = s_1^{(q)} < s_2^{(q)} < \dots < s_{n_q+1}^{(q)} = 1$ ,

say, and random variables  $\{\alpha_0, \alpha_{(i_1 \dots i_q)}\}_{(i_1 \dots i_q)}$ , where  $1 \leq i_1 \leq n'_1, \dots, 1 \leq i_q \leq n'_q$ , such that  $\alpha_0$  is  $F_0$ -measurable and if  $z'_{i_1 \dots i_q} = (s_{i_1}^{(1)}, s_{i_2}^{(2)}, \dots, s_{i_q}^{(q)}) \in [0, 1]^q$ ,  $\alpha_{(i_1 \dots i_q)}$  is  $F_{z'_{(i_1 \dots i_q)}}$ -measurable, and our process may be represented as

$$f(t, \omega) = \alpha_0(\omega) I_{\{0\}}(t) + \sum_{i_1 \dots i_q=1}^{n'_1 \dots n'_q} \alpha_{(i_1 \dots i_q)}(\omega) I_{\Delta'_{(i_1 \dots i_q)}}(t)$$

$$\text{where } \Delta'_{(i_1 \dots i_q)} = (z'_{(i_1 \dots i_q)}, z'_{(i_1+1, \dots, i_q+1)}] = \prod_{p=1}^q (s_{i_p}^{(p)}, s_{i_p+1}^{(p)}]$$

and  $\{0\} = \partial_{\inf}[0, 1]^q$ , the set of points in  $[0, 1]^q$  with some coordinate equal to 0 ::

**Definition (1.4):** Let  $f$  be a simple non-anticipative process as in the previous definition. Define its stochastic integral w.r.t.  $dM$  as the process

$$\begin{aligned} \int_0^t f.dM &= \int_0^t f(s, \omega) M(ds, \omega) = \alpha_0(\omega) \cdot M\{0\} + \\ &+ \sum_{i_1 \dots i_q=1}^{n'_1 \dots n'_q} \alpha_{(i_1 \dots i_q)}(\omega) \cdot M(\Delta'_{(i_1 \dots i_q)})_t \end{aligned}$$

$$\text{where } M(\Delta'_{(i_1 \dots i_q)})_t = M(\Delta'_{(i_1 \dots i_q)} \cap [0, t]) \quad ::$$

We shall always choose a separable version of  $M$  such that the process  $X(t) = M(0, t]$  has sample paths in  $D[0, 1]^q$ , (use Thm. (4.1), part I, here), so that the previously defined process

$$F(t) = \int_0^t f.dM$$

is a non-anticipative process with trajectories in the space  $D[0, 1]^q$ .

**Definition (1.5):** Say that a process  $(X_t)_{t \in \mathbb{N}^q}$  is a strong martingale w.r.t. the

$\sigma$ -fields  $(F_t)_{t \in \mathbb{N}^q}$ , if it vanishes over  $\partial_{\inf}(\mathbb{N}^q)$ , the random variable  $X_t$  is integrable and  $F_t$ -measurable for all  $t \in \mathbb{N}^q$  and

$$E(\Delta X(R_t) \mid \bigvee_{p=1, \dots, q} F_{t_p}^{p, \infty}) = 0$$

for all rectangle  $R_t \geq t$  with vertex at  $t$ , where  $\Delta X(R_t)$  is the usual rectangular increment of the process  $X$  over  $R_t$ , and

$$F_{t_p}^{p, \infty} = \bigvee_{z \in Q(t_p)} F_z$$

$Q(t_p)$  being the set of points in  $\mathbb{N}^q$  with  $t_p$  as  $p$ -th. coordinate ( $\bigvee$  means “ $\sigma$ -field generated by”)  $\therefore$

## 2. DOOB'S INEQUALITY FOR STRONG MARTINGALES.

Using induction on  $q$  and arguments very similar to those in (8) the following theorem is easily proved.

**Theorem (2.1):** Let  $X$  be a strong martingale over  $\mathbb{N}^q$ ,  $k$  a positive integer and  $r$  a positive real number. Then, the following inequality holds:

$$\begin{aligned} & r.P \left\{ \sup_{n_1 \leq k, \dots, n_q \leq k} |X_{(n_1 \dots n_q)}| > 4^{q-1} r \right\} \leq \\ & \leq (5/2)^{q-1} E \left( |X_{(k \dots k)}| I_{\left\{ \sup_{n_1 \leq k, \dots, n_q \leq k} |X_{(n_1 \dots n_q)}| \geq r \right\}} \right) \therefore \end{aligned}$$

A standard argument then leads to

**Corollary (2.2):** Let  $X$  be a square integrable strong martingale over  $\mathbb{N}^q$ , like in the theorem. Then

$$E \left( \sup_{n_1 \leq k, \dots, n_q \leq k} (X_{(n_1, \dots, n_q)})^2 \right) \leq 2^{q-1} 10^{q/2} 5^{-1/2} E (X_{(k, \dots, k)})^2 \therefore$$

## 3. THE MAXIMAL INEQUALITY AND THE EXTENSION OF THE INTEGRAL.

Let  $f$  be a simple non-anticipative process, as previously defined, and let

$$F(t) = \int_0^t f \cdot dM$$

be its stochastic integral. Define

$$\|F\| = \sup_{t \in [0, 1]^q} \left| \int_0^t f \cdot dM \right|.$$

**Lemma (3.1):** (Maximal inequality). There exists a constant,  $c_p$ , such that

$$\Lambda_p(\|F\|) \leq c_p \cdot \|f\|_{L^p(P \times m)}$$

for all  $f \in L^p(P \times m)$ .

**Proof.** With Corollary (2.2) in hand the proof goes through the same lines as the corresponding 1-dimensional result in (6) ::

**Definition (3.2):** We say that  $f$  belongs to the class  $M^p(M, m)$ , where  $(M, m)$  is a random measure like the one in DEF. (1.2), II, if it belongs to  $L^p(P \times m)$  and there exists a sequence,  $\{f_n\}$ , of simple non-anticipative processes, such that  $f_n \rightarrow f$  in  $L^p(P \times m)$ . Suppose that

$$\left\{ \int_0^t f_n \cdot dM \right\}_{n=1}^{\infty}$$

converge uniformly a.s. Then, define the stochastic integral of  $f$  w.r.t.  $dM$  as the limit

$$F(t) = \int_0^t f \cdot dM = \lim_{n \rightarrow \infty} \int_0^t f_n \cdot dM$$

for all  $t \in [0, 1]^q$  ::

**Remark.** The definition above makes sense because of the maximal inequality.

**Lemma (3.3):** Let  $\{ (M^v, f^v) : v = 1, 2, \dots \}$  be independent copies of  $(M, f)$ , where  $f \in M^p(M, m)$ . Then

$$\begin{aligned} \Lambda_p \left( \sup_t \left| n^{-1/p} \sum_{v=1}^n \int_0^t f^v \cdot dM^v \right| \right) &\leq \\ &\leq c_p \cdot \left( \int_{[0,1]^q} E |f|^p \cdot dm \right)^{1/p}. \end{aligned}$$

**Proof.** Prove it first for simple non-anticipative  $f$ , and then use an approximation argument, like in (6) ::

### III. The Central Limit Theorem.

#### 1. THE CASE $p \neq 2$

Now we can state and prove the C.L.T. we were looking for.

**Theorem (1.1):** Let  $(M, m)$  be a random measure like in DEF. (1.2), part II, and  $f$  a process in  $M^p(M, m)$ . Then there exists a stable process,  $S_t$ ,  $t \in [0, 1]^q$ , of index  $p$  and sample paths in  $D[0, 1]^q$ , such that the process

$$F(t) = \int_0^t f \cdot dM$$

is in its domain of normal attraction (D.N.A.), i.e. if  $(F^k)_{k=1}^\infty$  are independent copies of  $F$ , then

$$w^* - \lim_{n \rightarrow \infty} n^{-1/p} \left( \sum_{k=1}^n \int_0^t f^k \cdot dM^k \right) = S_t$$

in the usual sense of weak convergence of  $D[0, 1]^q$ -valued random variables.

We need several lemmas to completely prove this theorem. We state them all and prove the most relevant one (to this work).

**Lemma (1.2):** Let  $(X_n)_{n=1}^\infty, (Y_n^m)_{n=1}^\infty$ ,  $m = 1, 2, \dots$ , be sequences of  $D[0, 1]^q$ -valued random variables. Assume that the following conditions hold:

i) For all  $m = 1, 2, \dots$  the sequence of measures

$$\{L(Y_n^m)\}_{n=1}^{\infty}$$

is weakly convergent.

$$\text{ii) } \lim_{m \rightarrow \infty} \sup_n \Lambda_p(\|X_n - Y_n^m\|) = 0$$

Then  $\{L(X_n)\}_{n=1}^{\infty}$  is weakly convergent and

$$w^* \text{-}\lim_{n \rightarrow \infty} L(X_n) = w^* \text{-}\lim_{m \rightarrow \infty} (w^* \text{-}\lim_{n \rightarrow \infty} L(Y_n^m)) \quad ::$$

(M,m) will be, all the time, a random measure like that in theorem (1.1).

**Lemma (1.3):** Let  $f$  be a simple non-anticipative process, then

$$\int_{[0,1]^q} f.dM$$

belongs to the domain of normal attraction of

$$\|f\|_{L^p(P \times m)} \cdot \theta$$

where  $\theta$  is a random variable like in DEF. (1.2), II.

**Proof:** Looking at definition (1.4), II, we see that the argument is the same as that in the proof of lemma (4.1), in (6) ::

**Lemma (1.4):** Let  $\{\Delta_{(i_1, \dots, i_q)}\}_{(i_1, \dots, i_q)}$  be as in DEF. (1.3), II. For each multiindex  $(i_1, \dots, i_q)$  let  $\{Z_n^{(i_1, \dots, i_q)}\}_{n=1}^{\infty}$  be a sequence of  $D[0,1]^q$ -valued random variables such that:

$$\text{i) } Z_n^{(i_1, \dots, i_q)}(z) = 0 \text{ if } z \in [0,1]^q \setminus \{z: z > z_{(i_1, \dots, i_q)}\}.$$

$$\text{ii) } Z_n^{(i_1, \dots, i_q)}(z) = Z_n^{(i_1, \dots, i_q)}(z_{(i_1+1, \dots, i_q+1)}) \text{ if}$$

$$z \geq z_{(i_1+1, \dots, i_q+1)}.$$

$$\text{iii) } Z_n^{(i_1, \dots, i_q)}(z) = Z_n^{(i_1, \dots, i_q)}(\hat{z}) \text{ if } z \in \hat{\Delta}_{(i_1, \dots, i_q)} \setminus \Delta_{(i_1, \dots, i_q)}$$

$$\text{where } \hat{\Delta}_{(i_1, \dots, i_q)} = \{ z : z > z_{(i_1, \dots, i_q)} \} \setminus \{ z : z_{(i_1+1, \dots, i_q+1)} < z \}$$

and  $\hat{z}$  is the orthogonal projection of  $z$  on the frontier of  $\Delta_{(i_1, \dots, i_q)}$  if  $z \in \hat{\Delta}_{(i_1, \dots, i_q)} \setminus \Delta_{(i_1, \dots, i_q)}$ , and  $\hat{z} = z$  otherwise.

Suppose that for all  $(i_1, \dots, i_q)$  the sequence of probability measures

$$\{ L(Z_n^{(i_1, \dots, i_q)} | \bar{\Delta}_{(i_1, \dots, i_q)}) \}_{n=1}^{\infty}$$

(where “|” means “restriction to”), is tight as a sequence of probability measures over  $D[\bar{\Delta}_{(i_1, \dots, i_q)}]$ .

Then the sequence of probability distributions

$$\{ L(\Sigma_{(i_1, \dots, i_q)} Z_n^{(i_1, \dots, i_q)}) \}_{n=1}^{\infty}$$

is tight in the same sense.

**Proof.** Use appropriate moduli and THM. (3.1) of Neuhaus (11), the analog of THM. (15.2) in (3), in the present situation ::

**Lemma (1.5):** Let  $M$  and  $f$  be as in lemma (1.3). Then the finite dimensional distributions of the process

$$\int_0^t f.dM$$

are in the D.N.A. of the corresponding distributions of a stable process,  $S_t$ , over  $[0, 1]^q$  ::

To state the following lemma we need some more notation: Let us define

$$G_{\{0\}}(t) = M_{\{0\}} I_{[0, 1]^q}(t) \quad \text{and}$$



$$G_{(i_1, \dots, i_q)}(t) = M(z_{(i_1, \dots, i_q)}, z) I_{\hat{\Delta}_{(i_1, \dots, i_q)}}(t) + \\ + M(\Delta_{(i_1, \dots, i_q)}) I_{(z_{(i_1+1, \dots, i_q+1)}, 1_q)}(t)$$

where the notations are the same as in lemma (1.4). Then it is easy to see that for all  $t \in [0, 1]^q$  we have the following equality

$$\int_0^t f.dM = \alpha_0 G_{\{0\}}(t) + \sum_{i_1, \dots, i_q} \alpha_{(i_1, \dots, i_q)} \cdot G_{(i_1, \dots, i_q)}(t).$$

Next lemma is the C.L.T. statement for a process like

$$\alpha_{(i_1, \dots, i_q)} \cdot G_{(i_1, \dots, i_q)}(t)$$

when  $E |\alpha_{(i_1, \dots, i_q)}|^p < \infty$ .

**Lemma (1.6):** Let us suppose that  $\alpha_{(i_1, \dots, i_q)}$  is independent of  $G_{(i_1, \dots, i_q)}$  and that  $E |\alpha_{(i_1, \dots, i_q)}|^p < \infty$ . Consider a sequence

$$\left\{ \alpha_{(i_1, \dots, i_q)}^k, G_{(i_1, \dots, i_q)}^k \right\}_{k=1}^{\infty}$$

of independent copies of the pair  $(\alpha_{(i_1, \dots, i_q)}, G_{(i_1, \dots, i_q)})$ .

Then the sequence of probability measures

$$\left\{ L(n^{-1/p} \sum_{k=1}^n \alpha_{(i_1, \dots, i_q)}^k \cdot G_{(i_1, \dots, i_q)}^k | \bar{\Delta}_{(i_1, \dots, i_q)}) \right\}_{n=1}^{\infty}$$

over the space  $(D[\bar{\Delta}_{(i_1, \dots, i_q)}], \mathcal{D}[\bar{\Delta}_{(i_1, \dots, i_q)}])$  is tight, in fact it is weakly convergent to the law of a stable process indexed by  $\bar{\Delta}_{(i_1, \dots, i_q)}$ .

**Proof:** In what follows we will not write the indexes  $(i_1, \dots, i_q)$ .

Let us assume, in a first step, that  $\alpha$  is a.s. bounded. We want to use THM. (3.4), I, with  $\bar{\Delta}$  (the closure of  $\Delta_{(i_1, \dots, i_q)}$ ) in the place of  $[0, 1]^q$ .

For  $n = 1, 2, \dots$  let  $X_n$  be the random variable whose distributions is the  $n$ -th. term of the sequence (of distributions) in the lemma; we already know

that the finite dimensional distributions of  $X_n$  converge weakly to the corresponding distributions of a stable process. Note that  $X_n$  vanishes on the inferior frontier of  $\bar{\Delta}$ , and also

$$\begin{aligned} P \{ |X_n(B)| \geq \lambda \} &= P \{ |n^{-1/p} \sum_{k=1}^n \alpha^k M^k(B)| \geq \lambda \} = \\ &= E_\alpha (P_M \{ |n^{-1/p} \sum_{k=1}^n \alpha^k M^k(B)| \geq \lambda \}) \leq E_\alpha (\lambda^{-p} K_p n^{-1} \sum_{k=1}^n |\alpha^k|^p \cdot m(B)) \\ &= K'_p \cdot \lambda^{-p} \cdot m(B) \text{ for all positive } \lambda, \text{ where } K'_p \text{ depends on } \alpha \text{ and } p. \end{aligned}$$

This proves that  $X_n \in \tilde{C}_1^m(\gamma, \beta)$  with  $\gamma = p$  and  $\beta = 1$ . Now, an argument like that in step 3, proof of THM. (2.1), I, shows that for all positive  $\epsilon$

$$\lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} P_n \{ x : \|x_\delta^{(p)}\| \geq \epsilon \} = 0.$$

Hence it remains only to verify condition iii) of the above mentioned theorem (3.4). Fix  $p$ ,  $1 \leq p \leq q$ , and define  $\tilde{M}$  over

$$T_\delta^p = T_1 \times \dots \times T_{p-1} \times [t_{i+1} - \delta, t_{i+1}] \times T_{p+1} \times \dots \times T_q$$

by

$$\tilde{M}(A) = M(A) - M(A \cap \partial_{\sup} T_\delta^p), \quad M(\emptyset) = 0.$$

Then  $\tilde{M}$  is a measure with independent increments and control measure

$$\tilde{m}(A) = m(A) - m(A \cap \partial_{\sup} T_\delta^p).$$

We have

$$\begin{aligned} w_{X_n}^{(p)}[t_{i+1} - \delta, t_{i+1}] &\leq 2 \cdot \sup_{s \in [t_{i+1} - \delta, t_{i+1}]} \| (X_n)_s^{(p)} \| \leq \\ &\leq 2 \sup_{(z)_p \in [t_{i+1} - \delta, t_{i+1}]} |X_n(z)| \end{aligned}$$

hence

$$P \{ w_{X_n}^{(p)}[t_{i+1} - \delta, t_{i+1}] > \epsilon \} \leq P \left\{ \sup_{(z)_p \in [t_{i+1} - \delta, t_{i+1}]} |n^{-1/p} \sum_{k=1}^n \alpha^k M^k(z_{i_1} \dots i_q, z)| > \epsilon/2 \right\}$$

$$\leq 2^p \cdot \epsilon^{-p} \cdot (c'_p)^p E |\alpha|^p \cdot \tilde{m}(T_\delta^p) \leq 2^p \cdot \epsilon^{-p} \cdot (c'_p)^p E |\alpha|^p \cdot m_p [t_{i+1} - \delta, t_{i+1})$$

goes to zero, uniformly in  $n$ , when  $\delta \downarrow 0$ .

This proves the lemma in the a.s. bounded case. Now the general case follows like in (6), pg. 71, by a truncation-approximation argument where lemma (1.2) plays its role.  $\therefore$

Now the proof of theorem (1.1) follows:

If  $f^k$  is a simple non-anticipative process

$$\int_0^t f^k \cdot dM^k = \alpha_0^k \cdot M^k(\partial_{\inf} T) + \sum_{i_1 \dots i_q} \alpha_{(i_1 \dots i_q)}^k \cdot G_{(i_1 \dots i_q)}^k$$

with notation specified before lemma (1.6).

By lemma (1.6)

$$\left\{ L(n^{-1/p} \sum_{k=1}^n \alpha_{(i_1 \dots i_q)}^k \cdot G_{(i_1 \dots i_q)}^k \mid \Delta_{(i_1 \dots i_q)}) \right\}_{n=1}^\infty$$

weakly converges to the law of a stable process,  $S^{(i_1 \dots i_q)}$ , over  $\Delta_{(i_1 \dots i_q)}$ . This, together with lemma (1.4), gives the tightness of the sequence

$$\left\{ L(n^{-1/p} \sum_{k=1}^n \int_0^t f^k \cdot dM^k) \right\}_{n=1}^\infty$$

and, in fact, by lemma (1.5), this sequence is convergent in law to  $L(S_t)$ , where  $S_t$  is a stable process over  $[0, 1]^q$ .

Now, for an arbitrary  $f$ , choose a sequence,  $\{f_i\}_{i=1}^\infty$ , of simple non-anticipative processes such that  $f_i \rightarrow f$  in  $L^p$ .

Then, if for each  $i = 1, 2, \dots$ ,  $\{(f_i^k, M^k)\}_{k=1}^\infty$  is a sequence of independent copies of  $(f_i, M)$ , we have

$$w^* \cdot \lim_{n \rightarrow \infty} L(n^{-1/p} \sum_{k=1}^n \int_0^t f_i^k \cdot dM^k) = L(S_t).$$

By the extension of the maximal inequality (lemma (3.3)):

$$\begin{aligned} \lim_{i \rightarrow \infty} \sup_n \Lambda_p (n^{-1/p} \sum_{k=1}^n \int_0^t (f_i^k - f^k) dM^k) &\leq \\ &\leq c_p \lim_{i \rightarrow \infty} \|f_i - f\|_{L^p} = 0. \end{aligned}$$

Hence, by the approximation lemma (1.2):

$$\begin{aligned} w^* \text{-} \lim_{n \rightarrow \infty} L (n^{-1/p} \sum_{k=1}^n \int_0^t f^k \cdot dM^k) &= \\ = w^* \text{-} \lim_{i \rightarrow \infty} (w^* \text{-} \lim_{n \rightarrow \infty} L (n^{-1/p} \sum_{k=1}^n \int_0^t f_i^k \cdot dM^k)) &= \\ = w^* \text{-} \lim_{i \rightarrow \infty} (L (S_t^i)) &= L (S_t) \quad \because \end{aligned}$$

**Remark.** Previous results are easily extended to cover the  $[0, \infty)^q$  time domain case, using wellknown techniques (7).

## 2. THE CASE $p = 2$ . APPLICATION TO ITÔ'S AND DIFFUSION PROCESSES.

If the control measure,  $m$ , is a continuous one, and the process  $M [0, t]$  has a  $C [0, 1]^q$ -valued version, then the C.L.T. holds in  $C [0, 1]^q$ . In this case,  $M$  is necessarily a Gaussian measure (a wellknown fact, at least in dimension  $q = 1$ . See (5) and also (7) where the proof of Thm. (19.1) of (3) is generalized to an arbitrary dimension).

In the present case define the class  $M^2 (M, m)$  exactly like in DEF. (1.2), II, and remember that  $M$  satisfies

$$E M^2 (A) = m (A)$$

for all  $A \in \mathcal{B}$ .

Then if  $f$  is a simple non-anticipative process

$$E \left( \int_0^t f \cdot dM \right)^2 = \int_0^t E \|f\|^2 \cdot dm$$

for all  $t \in [0, 1]^q$ , and as we may work with right continuous versions, the maximal inequality

$$E (\| \int_0^t f.dM \|^2)^{1/2} \leq K. ( \int_{[0, 1]^q} E |f|^2 . dm )^{1/2}$$

is satisfied, and this allows us to define the stochastic integral for  $f \in M^2(M, m)$  in the same way as in the case  $p \neq 2$ . The following theorem then holds:

**Theorem (2.1):** Let  $(M, m)$  be a random measure like before, and  $m$  continuous. Suppose that  $M [0, t]$  has continuous trajectories. Then  $M [0, t]$  is a Gaussian process with continuous paths and if  $f$  belongs to the class  $M^2(M, m)$ , there exists a process,  $S_t$ , Gaussian with continuous trajectories, such that the process

$$\int_0^t f.dM$$

belongs to its D.N.A. in the sense of weak convergence of probability measures in  $C [0, 1]^q$  ::

We skip the proof of this theorem and go directly into some applications.  
It is wellknown that Itô's processes are processes of the form

$$I_t = \int_0^t a(s, \omega) ds + \int_0^t b(s, \omega) dW_s$$

where we suppose that  $t \in [0, 1]^q$ . Under some conditions on  $a$  and  $b$  we can apply the previous theorem to show that  $I_t$  satisfy the C.L.T. in  $C [0, 1]^q$ .

**Theorem (2.2):** Let  $a$  and  $b$  be non-anticipative processes w.r.t. the  $\sigma$ -fields

$$F_t = \sigma \{ W_s : s \leq t \}$$

where  $W$  is Wiener process. Assume that

$$\int_0^T E (b^2(t, .)) dt < \infty \text{ and } \sup_{s \leq T} E |a^{q+1}(s, .)| < \infty$$

for all  $T \in \mathbb{R}_+^q$ . Then the process  $I_t$  satisfies the C.L.T. in  $C [0, \infty)^q$  ::

We write the result for diffusion processes only in the case  $q = 2$ .

**Theorem (2.3):** Let  $a(z,x)$  and  $b(z,x)$  be two functions over  $[0, \infty)^2 \times (-\infty, \infty)$ , and suppose that for all  $x$  and  $z$

$$a^2(z,x) + b^2(z,x) \leq c \cdot (1 + x^2).$$

Let  $(X_z)_{z \in \mathbb{R}^2}$  be the diffusion process defined by the stochastic differential equation (some Lipschitz condition on  $a$  and  $b$  is assumed)

$$X_{st} = \int_{R_{st}} a(z, X_z) dz + \int_{R_{st}} b(z, X_z) dW_z$$

where  $R_{st} = [0, s] \times [0, t]$ . Then the process  $X_{st}$  satisfies the C.L.T. in  $C[0, \infty)^2$ .

**Proof.** In view of the preceding result it is enough to show that

$$\sup_{s,t \leq T} E(X_{st})^{2m} < \infty$$

for all  $T < \infty$ . But it is an easy matter to show that in fact

$$E(X_{st})^{2m} \leq K_m \cdot e^{st}$$

for all  $s, t$  (standard arguments apply)  $\therefore$

### **Final remarks and comments**

a) It will be very interesting to get a result like THM. (2.1), part (I), for processes with non necessarily independent increments. I don't know at present how to do this.

b) Extension of the previous results to the Banach valued case offer no difficulties, at least when the space of values is  $(p + \epsilon)$ -uniformly smooth, for some positive  $\epsilon$ . Complete characterisation of those spaces where the results hold are not yet known.

c) I want to express my indebtedness and gratitude to Professor E. Giné, who suggested this problems to me and has given efficient help whenever needed.

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