ON IDEALS OF OPERATORS AND OF LOCALLY CONVEX SPACES

by

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ABSTRACT

This paper investigates some special operator ideals on Banach spaces and the Grothendieck spaces ideals they generate, from the point of view of their structural and stability properties. The results are given in terms of the sequence ideals associated to the operator ideals and an analysis of some special sequence ideals is also made.

INTRODUCTION

The purpose of this paper is to investigate some special operator ideals on Banach spaces and the Grothendieck space ideals they generate, from the point of view of their structural and stability properties. The general results are given in Section 1, while Section 2 is devoted to examples and applications. Since our concern is with suitable extensions of proper operator ideals on Hilbert spaces, the results will naturally be in terms of the associated proper sequence ideals λ (cf. [9], 15.2 and 15.3).

We refer to [9] and also to [7] and [12] for the notation and for the notions, by now classical, of operator ideals on Banach and Hilbert spaces, of ideals of locally convex spaces (space ideals), of Grothendieck space ideals and of sequence ideals.

Finally, as is [6] and [7], to every sequence ideal λ we associate its (unique) *ideal kernel* $\bar{\lambda}$ defined by

$$\bar{\lambda} = \left\{ \xi \in \lambda \ : \ (\sup_{k \geq n} | \xi_k |)_n \ \in \lambda \right\} .$$

The ideal kernel was first introduced in [6] and was extensively used in [6] and [7]. Its fundamental importance will be reaffirmed in Section 2.

1. OPERATOR AND SPACE IDEALS

From now on λ will always stand for a proper sequence ideal (i.e. $\lambda \subset c_0$) and, according to [7], §2, $\mathcal{S}_\lambda$ will be the corresponding ideal on Hilbert spaces (or on ℓ^2). If $\mathcal{A}_0$ is any operator ideal on Hilbert spaces, we denote by $\mathcal{A}_0^h$ *an extension of* $\mathcal{A}_0$ *to the class of Banach spaces with the property that* $(\mathcal{A}_0^h)^m \subset \mathcal{H}$ *for some* m, where $\mathcal{H}$ is the ideal of all operators that factor through a Hilbert space.

1.1 Remark. From the definitions of the inferior extension $\mathcal{A}_0^{inf}$ and of the superior extension $\mathcal{A}_0^{sup}$ ([9], 15.6.4 and 15.6.2), we see that $\mathcal{A}_0^{inf}$ itself and $(\mathcal{A}_0^{sup})^m \cap \mathcal{H}$, for any m, provide examples of extensions $\mathcal{A}_0^h$. Also clearly $\mathcal{A}_0^{inf}$ is this smallest such extension. When $\mathcal{A}_0 = \mathcal{S}_\lambda$ further extensions $\mathcal{S}_\lambda^h$ are exhibited by the ideals $\mathcal{A}_\lambda$ and $\mathcal{N}_\lambda$ $(\lambda \subset \ell^1)$ introduced in [7].

Recall that two operator ideals $\mathcal{A}$ and $\mathcal{B}$ are said to be *equivalent,* written $\mathcal{A} \sim \mathcal{B}$, if there exist m and n such that $\mathcal{A}^m \subset \mathcal{B}$ and $\mathcal{B}^n \subset \mathcal{A}$ (cf. [5], 7.1.5). Denoting by Groth ($\mathcal{A}$), resp. Groth ($\mathcal{B}$), the Grothendieck space ideal generated by $\mathcal{A}$, resp. $\mathcal{B}$, it is then clear that Groth ($\mathcal{A}$) = Groth ($\mathcal{B}$) if $\mathcal{A} \sim \mathcal{B}$. From this, Remark 1.1 above and Lemma 4 of [12], it follows the immediate but useful

1.2 Lemma. $\mathcal{A}_0^h \sim \mathcal{A}_0^{inf}$ *and hence* Groth ($\mathcal{A}_0^h$) = Groth ($\mathcal{A}_0^{inf}$).

1.3 Corollary. Groth ($\mathcal{S}_\lambda^{inf}$) = Groth ($\mathcal{A}_\lambda$) (= Groth ($\mathcal{N}_\lambda$) *if* $\lambda \subset \ell^1$).

In view of the above lemma, there is no ambiguity in denoting by Groth ($\mathcal{A}_0$) the Grothendieck space ideal generated by any extension $\mathcal{A}_0^h$ and, when $\mathcal{A}_0 = \mathcal{S}_\lambda$, we shall set $\mathbb{N}_\lambda$ = Groth ($\mathcal{S}_\lambda$). The space ideal $\mathbb{N}_\lambda$ may rightly be called the *ideal of* λ*–nuclear spaces.* We note the fact that, due to the definition of $\mathcal{S}_\lambda^h$ and $\mathbb{N}_\lambda$, *a locally convex space E belongs to* $\mathbb{N}_\lambda$, *if and only if it has a basis* $\mathcal{U}$ *of hilbertain neighbourhoods of* 0 *such that every* U $\in \mathcal{U}$ *contains a* V $\in \mathcal{U}$ *for which the sequence of diameters* $(d_n(V,U)) \in \lambda$.

1.4 Remark. Extensions of $\mathcal{S}_\lambda$ which are not of type $\mathcal{S}_\lambda^h$ may, a priori, generate Grothendieck space ideals different from $\mathbb{N}_\lambda$. However, by Theorem 9 of [7] this is not possible if $\lambda = \ell^1 . \lambda$, since then $\mathcal{S}_\lambda$ has exactly one extension, say $\mathcal{S}_\lambda^e$. In this case, it is worth noting that, by [9], 15.6.15-17, we have.

1.5 Proposition. $\mathcal{S}^{e}_{\lambda}$ *is injective, surjective and completely symmetric.*

A further stability result may be obtained as follows. For $\xi \in \lambda$ define $\hat{\xi}$ by

$$\hat{\xi}_k = \xi_{n_k}, \quad \text{where } n_k = \max\{n : n^2 \leq k\}. \tag{1}$$

Then the results of [13], § 3 give

1.6 Proposition. *If* $\hat{\xi} \in \lambda$ *whenever* $\xi \in \lambda$, *then the tensor product of two maps in* $\mathcal{S}_{\lambda}$, *resp.* $\mathcal{S}^{\inf}_{\lambda}$, *belongs to* $\mathcal{S}_{\lambda}$, *resp.* $\mathcal{S}^{\inf}_{\lambda}$.

We now come to stability properties of Grothendieck space ideals. Recall that a *variety* is a class of locally convex spaces which is stable under the formation of isomorphic images, subspaces, quotients and arbitrary products (cf. [2]). Then combining Propositions 7.1.6, 7.2.2, 7.2.3 and 7.2.6 of [5] we have.

1.7 Theorem. Groth ($\mathcal{A}^{h}_{o}$) *is a variety.*

1.8 Corollary. $\mathbb{N}_{\lambda}$ *is a variety.*

Regarding stability with respect to the formation of countable direct sums, we can generalize Proposition 7.2.7 of [5] as follows. Recall that a sequence (x_n) in a linear topological space E is said to be *very weakly convergent* if there exists a sequence (ξ_n) of non-zero scalars such that $\xi_n x_n \to 0$ in E. It is clear that this is a notion which depends only on the bornology of E, since we might as well require $(\xi_n x_n)$ to be bounded in E. Denote by $\mathcal{C}$ the class of all spaces in which every sequence is very weakly convergent (a characterization of such a class is given in [11]; in particular, $\mathcal{C}$ contains all metrizable spaces). We have

1.9 Theorem. *Suppose that on* $\mathcal{A}^{h}_{o}$ *there is a quasi-complete linear topology* τ *such that* $(\mathcal{A}^{h}_{o}, \tau) \in \mathcal{C}$. *Then* Groth ($\mathcal{A}^{h}_{o}$) *is stable under the formation of countable direct sums.*

Proof. The proof is the same as that of Proposition 7.2.7 of [5], except that the use of Proposition 6.2.4 there is motivated by the following argument, which takes the place of Corollary 1.4.7:

Since every sequence in $(\mathcal{A}^{h}_{o}, \tau)$ is very weakly convergent, if maps $T_n \in \mathcal{A}^{h}_{o}$ are given (with the same domain and range) then there are scalars $\xi_n \neq 0$ such that the sequence $(\xi_n T_n)$ is bounded in $(\mathcal{A}^{h}_{o}, \tau)$. Thus, by quasi-completeness there are scalars $\eta_n \neq 0$ such that the series $\sum_n \eta_n \xi_n T_n$ converges to a map $T \in \mathcal{A}^{h}_{o}$.

For the varieties $\mathbb{N}_\lambda$ a further criterion is exhibited by Theorem I.5 of [6] which we recall here for completeness.

Putting $\lambda^+ = \{\xi \in \lambda : \xi_1 \geq \xi_2 \geq \ldots > 0\}$, we have

1.10 Theorem. Suppose that λ satisfies the following hypothesis:

(H) *If $(\xi^{(k)})$ is a sequence of elements of λ^+, then there is a $\xi \in \lambda^+$ such that* $\sup_n \xi_n^{-1} \xi_n^{(k)} < \infty$ *for all* k.

Then $\mathbb{N}_\lambda$ is stable under the formation of countable direct sums.

To understand the relationship between Theorems 1.9 and 1.10 we need the following lemma, where $\bar{\lambda}_\nu$ denotes the ideal kernel $\bar{\lambda}$ endowed with its normal topology $\nu(\bar{\lambda}, \bar{\lambda}^x)$.

1.11 Lemma. If $\bar{\lambda}$ is perfect, then the following assertions are equivalent:

(i) λ (equivalently, $\bar{\lambda}$) *satisfies (*H*).*

(ii) $\bar{\lambda}_\nu \in \mathcal{C}$.

Proof: (i) $\Longrightarrow$ (ii): Let $(\xi^{(k)})$ be a sequence of elements of $\bar{\lambda}$. There are elements $\eta^{(k)} \in \lambda^+$ such that $|\xi_n^{(k)}| \leq \eta_n^{(k)}$ for all k,n. By assumption there is $\xi \in \lambda^+$ for which $\sup_n \xi_n^{-1}\eta_n^{(k)} < \infty$ for all k, whence also $\sup_n \xi_n^{-1} |\xi_n^{(k)}| = c_k < \infty$.

A defining system of semi-norms for $\bar{\lambda}_\nu$ is given by the family $(p_\zeta : \zeta \in \bar{\lambda}^x)$, where $p_\zeta(\eta) = \sum_n |\eta_n \zeta_n|$. Given $\zeta \in \bar{\lambda}^x$ and putting $\sum_n \xi_n |\zeta_n| = c_\zeta$, we obtain

$$p_\zeta((kc_k)^{-1} \xi^{(k)}) = (kc_k)^{-1} \sum_n |\xi_n^{(k)} \zeta_n|$$

$$\leq k^{-1} \sum_n \xi_n |\zeta_n| = k^{-1} c_\zeta \to 0.$$

Since ζ was arbitrary, it follows that $(\xi^{(k)})$ is very weakly convergent and hence that $\bar{\lambda}_\nu \in \mathcal{C}$.

(ii) $\Longrightarrow$ (i): Let $(\xi^{(k)})$ be a sequence of elements of λ^+. Since $\lambda^+ \subset \bar{\lambda}$ and $\bar{\lambda}_\nu \in \mathcal{C}$, there is a sequence of numbers $\gamma_k > 0$ such that $\gamma_k \xi^{(k)} \to 0$ in $\bar{\lambda}_\nu$. Because $\bar{\lambda}$ is perfect, $\bar{\lambda}_\nu$ is complete and hence, if $\rho_k = 2^{-k} \gamma_k$, then there exists $\xi \in \bar{\lambda}$ such that $\sum_k \rho_k \xi^{(k)} = \xi$. This implies $\xi \in \lambda^+$ and

$$\rho_k \xi_n^{(k)} \leq \sum_j \rho_j \xi_n^{(j)} = \xi_n ,$$

which shows that (H) is satisfied.

Recalling that the ideal kernel $\bar{\lambda}$ is *additive* ([7], Lemma 1), i.e. that $\xi, \eta \in \bar{\lambda}$ implies that the sequence $(|\xi_1|, |\eta_1|, |\xi_2|, |\eta_2| \ldots)$, rearranged in decreasing order of magnitude, also belongs to $\bar{\lambda}$, we can now give the

1.12 Theorem. On $\mathcal{S}_\lambda$ *and* $\mathcal{S}_\lambda^{\inf}$ *there are natural linear topologies* τ *and* τ' *arising from* $\nu(\bar\lambda, \bar\lambda^x)$. *If* $\bar\lambda$ *is perfect, then (i) and (ii) of Lemma 1.11 are equivalent to* $(\mathcal{S}_\lambda, \tau) \in \mathcal{C}$ *and to* $(\mathcal{S}_\lambda^{\inf}, \tau') \in \mathcal{C}$.

Proof. Consider the defining family $(p_\zeta : \zeta \in \bar\lambda^x)$ of semi-norms on $\bar\lambda_\nu$. Denoting by $a_n(T)$ the approximation numbers of $T \in \mathcal{S}_\lambda$, we put $q_\zeta(T) = p_\zeta(a_n(T))$ for all $\zeta \in \bar\lambda^x$ and all $T \in \mathcal{S}_\lambda$. q_ζ is well-defined, since $T \in \mathcal{S}_\lambda$ if and only if $(a_n(T)) \in \bar\lambda$; also, $q_\zeta(\gamma T) = |\gamma| q_\zeta(T)$ for all scalars γ. Consider the sets

$$U(\zeta, \varepsilon) = \{ T \in \mathcal{S}_\lambda : q_\zeta(T) < \varepsilon \} .$$

For each $\zeta \in \bar\lambda^+$ let $\zeta'_n = \zeta_{2n}$ for all n. Since $\bar\lambda$ is additive, the sequence $(a_1(T), a_1(T), a_2(T), a_2(T), \ldots)$ belongs to $\bar\lambda$ and hence $q_{\zeta'}(T) < \infty$. Now let $U(\zeta, \varepsilon)$ be given. Without loss of generality we may assume ζ positive and non-decreasing. Then if $S,T \in U(\zeta', \varepsilon/4)$ we have

$$q_\zeta(S+T) = \sum_n \zeta_n a_n(S+T) \leq 2 \sum_n \zeta_{2n} a_{2n-1}(S+T) \leq$$

$$\leq 2 \sum_n \zeta_{2n}(a_n(S) + a_n(T)) = 2(q_{\zeta'}(S) + q_{\zeta'}(T)) < \varepsilon$$

so that $S + T \in U(\zeta, \varepsilon)$. This shows that $\{U(\zeta, \varepsilon) : \zeta \in \bar\lambda^x, \varepsilon > 0\}$ is a basis of neighbourhoods of 0 for a linear topology τ on $\mathcal{S}_\lambda$. A linear topology τ' may then be defined on $\mathcal{S}_\lambda^{\inf}$ by setting, for $T \in \mathcal{S}_\lambda^{\inf}$ and $T = RT_oS$ with $R, S \in \mathcal{L}$ and $T_o \in \mathcal{S}_\lambda$, $r_\zeta(T) = \inf \|R\| q_\zeta(T_o) \|S\|$ (the infimum being taken over all possible factorizations) and proceeding as above. Finally, since $q_\zeta(\gamma T) = |\gamma| q_\zeta(T)$ for all scalars γ, it is clear that the assertions $\bar\lambda_\nu \in \mathcal{C}$, $(\mathcal{S}_\lambda, \tau) \in \mathcal{C}$ and $(\mathcal{S}_\lambda^{\inf}, \tau') \in \mathcal{C}$ are all equivalent.

To conclude our discussion of the stability properties of the space ideals $\mathbb{N}_\lambda$, we determine when such ideals are *stability classes*, i.e. varieties that are stable under the formation of countable direct sums, completions and projective tensor products (cf. [4]). Recalling (1) we find

1.13 Theorem. Suppose that λ *satisfies (H) and that there is a* $p > 0$ *such that* $\xi \in \lambda^p \equiv \{ \xi : (\xi_n^p) \in \lambda \}$ *whenever* $\xi \in \lambda$. *Then* $\mathbb{N}_\lambda$ *is a stability class.*

Proof. Since a Grothendieck space ideal is always stable under completions, by Corollary 1.8 and Theorem 1.10 it remains to check that $\mathbb{N}_\lambda$ is stable under the formation of tensor products. But this follows from [13], § 2 and the characterization of $\mathbb{N}_\lambda$ noted after its definition. The full force of Proposition 1.6 is not needed and the hypothesis of the theorem suffices.

2. Examples and applications

To begin with, we recall some classical ideals of type $\mathbb{N}_\lambda$ (these are all varieties, by Corollary 1.8).

$\mathbb{N}_\phi$ = the ideal $\mathbb{W}$ of weak spaces,
$\mathbb{N}_{\ell o}$ = the ideal $\mathbb{N}_o$ of strongly nuclear spaces,
$\mathbb{N}_{\ell 1}$ = the ideal $\mathbb{N}$ of nuclear spaces,
$\mathbb{N}_{c_o}$ = the Schwartz-Hilbert ideal $\mathbb{S}\,\mathbb{H}$ ($= \mathbb{S} \cap \mathbb{H}$) (cf. [1]).

The sequence ideals that we shall consider will be those that are generated by the power series spaces $\Lambda_r(\alpha)$ (cf. [8], 6.2) and their duals $\Gamma_s(\alpha)$. Precisely, let $0 < r \leq \infty$ and let $\alpha = (\alpha_n)$ be a sequence such that $0 < \alpha_n \nearrow \infty$. Then

$$\Lambda_r(\alpha) = \{\xi : \sum_n t^{\alpha_n} |\xi_n| < \infty \text{ for all } t < r\}$$

is called a *power series space of finite or infinite type* according to whether $r < \infty$ or $r = \infty$. Dually, for $0 < s < \infty$ we define

$$\Gamma_s(\alpha) = \{\xi : \sup_n t^{\alpha_n} |\xi_n| < \infty \text{ for some } t > s\} = \Lambda'_{1/s}(\alpha).$$

In order to be able to obtain sequence ideals (i.e., contained in c_o) from the above spaces, we must have $1 < r \leq \infty$ and $1 \leq s < \infty$. We shall assume $1 < r < \infty$ throughout for the finite type case and we shall treat $\Lambda_\infty(\alpha)$ (and $\Gamma_1(\alpha)$, when necessary) separately. Also, we will only consider the case when $\Lambda_r(\alpha)$ and $\Gamma_s(\alpha)$ (resp. $\Lambda_\infty(\alpha)$) are *nuclear* i.e., as well-known, when $\sum_n R^{-\alpha_n} < \infty$ for each (resp. for some) $R > 1$ (cf. [8], 6.1.5). Now observe that in general, outside the classical case, it is very difficult to define a sequence ideal directly. The most comon procedure amounts to taking an ideal kernel $\overline{\lambda}$ (which is much easier to define) and then to set

$$\lambda \equiv \{\xi \in c_o : \overline{\xi} \in \overline{\lambda}\},$$

where $\bar{\xi}$ is the sequence $(|\xi_n|)$ rearranged in decreasing order of magnitude. It is then clear that λ is a sequence ideal. Thus we have to ascertain when $\Lambda_r(\alpha)$, $\Gamma_s(\alpha)$ and $\Lambda_\infty(\alpha)$ are ideal kernels. We have

2.1 Lemma. (a) $\Lambda_\infty(\alpha)$ *and* $\Gamma_1(\alpha)$ *are ideal kernels if and only if*

$$\sup_n \frac{\alpha_{2n}}{\alpha_n} < \infty \quad \text{(i.e., } \alpha \textit{ is stable}). \tag{2}$$

(b) $\Lambda_r(\alpha)$ *and* $\Gamma_r(\alpha)$ $(r > 1)$ *are ideal kernels if and only if*

$$\lim_n \frac{\alpha_{2n}}{\alpha_n} = 1. \tag{3}$$

Proof. To begin with, note that conditions ($\bar{S}1$) – ($\bar{S}$ 3) of Lemma 1 of [7] (cf. also Lemma 2), characterize an ideal kernel. Now $\Lambda_r(\alpha)$, $\Gamma_s(\alpha)$ and $\Lambda_\infty(\alpha)$ $\subset c_0$. Also, $(\sup_{k\geqslant n} |\xi_k|) \in \Lambda_r(\alpha)$ or $\Lambda_\infty(\alpha)$ whenever $\xi \in \Lambda_r(\alpha)$ or $\Lambda_\infty(\alpha)$ by Lemma (2.8) of [3], which applies also in the finite type case, while it is immediate to see that this holds for $\Gamma_s(\alpha)$ as well. Thus the property of being ideal kernels is equivalent to that of being additive (definition just before Theorem 1.12).

(a) It is easily seen (and it is also shown in [3], Theorem (2.10) for $\Lambda_\infty(\alpha)$ that additivity is equivalent to (2).

(b) Assume (3) and for each $1 < t < r$ let q,ϵ be such that $t^{1+\epsilon} \leqslant q < r$. Then choose m so that $\alpha_{2n} \leq (1+\epsilon)\alpha_n$ for all $n \geq m$. If $\xi, \eta \in \Lambda_r(\alpha)$, considering the sequence $(|\xi_1|, |\eta_1|, |\xi_2|, |\eta_2| \dots)$ we have

$$\sum_{n\geq m} t^{\alpha_{2n}} |\eta_n| \leq \sum_{n\geq m} q^{\alpha_n} |\eta_n| < \infty$$

and similarly for $(|\xi_n| : n \geq m)$, since $\alpha_{2n-1} \leq \alpha_{2n}$. Thus $\Lambda_r(\alpha)$ is additive. Conversely, suppose that $\Lambda_r(\alpha)$ is additive and that (3) does not hold. Then there are a $\delta > 0$ and a strictly increasing sequence (n_k) of integers such that $\alpha_{2n_k} \geq (1+\delta)\alpha_{n_k}$ for all k. Choose $r_k \nearrow r$ and define the sequence ξ by

$$\xi_{n_k} = r_k^{-\alpha_{n_k}} \text{ and } \xi_n = 0 \text{ for } n \notin (n_k).$$

For each j we have, by nuclearity,

$$\sum_{n \geq n_{j+1}} r_j^{\alpha_n} \xi_n = \sum_{k \geq j+1} \left(\frac{r_j}{r_k}\right)^{\alpha_{n_k}} \leq \sum_{k \geq j+1} \left(\frac{r_j}{r_{j+1}}\right)^{\alpha_k} < \infty \tag{4}$$

and hence $\xi \in \Lambda_r(\alpha)$. Now let $\eta = (\xi_1, \xi_1, \xi_2, \xi_2, \ldots)$ and choose j so that $r_j^{1+\delta} \geq r$. Then

$$\sum_{n \geq n_j} r_j^{\alpha_n} \eta_n \geq \sum_{n \geq n_j} r_j^{\alpha_{2n}} \eta_{2n} =$$

$$\sum_{k \geq j} r_j^{\alpha_{2n_k}} r_k^{-\alpha_{n_k}} \geq \sum_{k \geq j} \left(\frac{r}{r_k}\right)^{\alpha_{n_k}} = +\infty$$

and $\Lambda_r(\alpha)$ is not additive. Similarly for $\Gamma_r(\alpha)$.

The above lemma shows that, contrary to the infinite type case (cf. Theorem (2.10) of [3]), stability (i.e. condition (2)) is not sufficient for additivity in the finite type case. This leads us to introduce the classes of *nuclear exponent sequences* α which satisfy (2) and (3) respectively. Precisely, we put

$$\sigma_\infty = \left\{\alpha : \sup_n \frac{\alpha_{2n}}{\alpha_n} < \infty \text{ and } \sup_n \frac{\log n}{\alpha_n} < \infty\right\},$$

$$\sigma'_\infty = \left\{\alpha : \sup_n \frac{\alpha_{2n}}{\alpha_n} < \infty \quad \text{and} \quad \lim_n \frac{\log n}{\alpha_n} = 0\right\},$$

$$\sigma_1 = \left\{\alpha : \lim_n \frac{\alpha_{2n}}{\alpha_n} = 1 \quad \text{and} \quad \lim_n \frac{\log n}{\alpha_n} = 0\right\},$$

so that $\Lambda_\infty(\alpha)$ (resp. $\Gamma_1(\alpha)$, resp. $\Lambda_r(\alpha)$, $\Gamma_r(\alpha)$ with $r > 1$) is a nuclear ideal kernel if and only if $\alpha \in \sigma_\infty$ (resp. $\alpha \in \sigma'_\infty$; resp. $\alpha \in \sigma_1$). Observe that the definition of σ'_∞ shows that $\Gamma_1(\alpha)$ *is a mixed case*, in so far as it is of infinite type with respect to additivity and of finite type with respect to nuclearity.

Now noting that $(n).\Gamma_s(\alpha) = \Gamma_s(\alpha)$ by nuclearity, we may supplement Corollary 2 to Theorem 13 of [7] by

2.2 Proposition. *If $\alpha \in \sigma_1$ (resp. $\alpha \in \sigma_\infty$; resp. $\alpha \in \sigma_\infty$) then $\Lambda_r(\alpha)$ and $\Gamma_r(\alpha)$ (resp. $\Lambda_\infty(\alpha)$; resp. $\Gamma_1(\alpha)$) are examples of ideal kernels $\overline{\lambda}$ for which S_λ has a unique* *extension (which then enjoys the properties of Proposition 1.5). $\Lambda_r(\alpha)$ and $\Gamma_r(\alpha)$ never are idempotent for $r > 1$ (while $\Lambda_\infty(\alpha)$ and $\Gamma_1(\alpha)$ are).*

2.3 ***Remark.*** We note that the proof given in [6], p. 19, to show that (2) implies

$$\sup_n n^{-p}\alpha_n < \infty \text{ for some } p > 0,$$

can be adapted to yield that (3) implies

$$\lim_n n^{-p}\alpha_n = 0 \quad \text{for all } p > 0. \tag{5}$$

The converse is not true, as the following example shows.

2.4 ***Example.*** For every k let $n_k = 2^{2^k}$ and put $\alpha_n = (\log_2 n)^k$ for $n_{k-1} \leq n \leq n_k$ ($n_o = 2$). We have, for any given $p > 0$,

$$\frac{\alpha_n}{n^p} \leq \frac{\alpha_{n_k}}{n_{k-1}^p} = 2^{k^2 - p2^{k-1}} \longrightarrow 0 \text{ as } k \to \infty$$

and (5) holds. On the other hand

$$\frac{\alpha_{2n_k}}{\alpha_{n_k}} = \frac{(2^k+1)^{k+1}}{2^{k^2}} = (1 + \frac{1}{2^k})^k (2^k+1) \longrightarrow \infty \text{ as } k \to \infty$$

and α does not even satisfy (2). Note that $\Lambda_r(\alpha)$ is nuclear.

Now we wish to obtain upper and lover bounds for the families $\{\Lambda_r(\alpha): r > 1, \alpha \in \sigma_1\}$ and $\{\Gamma_r(\alpha) : r > 1, \alpha \in \sigma_1\}$.

For this we need a

2.5 ***Lemma.*** *The following assertions are equivalent:*

$$\limsup_n \frac{\alpha_n}{\beta_n} \leq c. \tag{i}$$

$$\Lambda_{r^c}(\beta) \subset \Lambda_r(\alpha). \tag{ii}$$

Proof. (i) $\Longrightarrow$ (ii): For a given $t < r$ choose $\varepsilon > 0$ such that $t^{c+\varepsilon} < r^c$ and then q such that $t < q < r$ and $t^{c+\varepsilon} \leq q^c$. Finally, pick an m for which

$$\sup_{n \geq m} \frac{\alpha_n}{\beta_n} \leq c + \varepsilon .$$

If $\xi \in \Lambda_{r^c}(\beta)$ we have

$$\sum_{n \geq m} t^{\alpha_n} |\xi_n| \leq \sum_{n \geq m} t^{(c+\varepsilon)\beta_n} |\xi_n| \leq \sum_{n \geq m} q^{c\beta_n} |\xi_n| < \infty$$

and hence $\xi \in \Lambda_r(\alpha)$.

(ii) $\Longrightarrow$ (i): Suppose that there is a strictly increasing sequence (n_k) of integers such that $\frac{\alpha_{n_k}}{\beta_{n_k}} \geq d > c$. Choosing a sequence $r_k \uparrow r$, put

$$\xi_{n_k} = r_k^{-c\beta_{n_k}} \quad \text{and} \quad \xi_n = 0 \quad \text{for } n \notin (n_k).$$

As in (4) we see that $\xi \in \Lambda_{r^c}(\beta)$. However, for j such that $r_j^{d/c} > r$,

$$\sum_n r_j^{\alpha_n} \xi_n \geq \sum_k \left(\frac{r_j^{\alpha_{n_k}/\beta_{n_k}}}{r_k^c}\right)^{\beta_{n_k}} \geq \sum_k \left(\frac{r_j^{d/c}}{r_k}\right)^{c\beta_{n_k}} = \infty ,$$

showing that $\xi \notin \Lambda_r(\alpha)$.

2.6 ***Corollary***. $\Lambda_t(\beta) \subset \Lambda_r(\alpha)$ *(whatever* $r, t > 1$*) if and only if* $\lim_n \frac{\alpha_n}{\beta_n} = 0$

Hence the latter condition is also equivalent to

$$\bigcup_{t > 1} \Lambda_t(\beta) \subset \bigcap_{r < \infty} \Lambda_r(\alpha) = \Lambda_\infty(\alpha).$$

Statements similar to those of Lemma 2.5 and Corollary 2.6 hold for the spaces $\Gamma_s(\alpha)$ and we leave their formulation to the reader. Now we can give the

2.7 Proposition. (i) $\bigcup_{p>0} \Lambda_\infty(n^p) = \bigcup_{p>0} \Lambda_r(n^p) =$

$= \cup \{\Lambda_r(n^p) : r > 1, p > 0\} \subset \bigcap_{\alpha \in \sigma_1} \Lambda_\infty(\alpha) =$

$= \cap \{\Lambda_r(\alpha) : r > 1, \alpha \in \sigma_1\}$

(ii) $\bigcup_{\alpha\epsilon\sigma_1} \Lambda_\infty(\alpha) = \cup \{\Lambda_r(\alpha) : r > 1, \alpha \in \sigma_1\} = s\,(= \Lambda_\infty(\log n))$.

Proof. Let $\xi \in \Lambda_r(n^p)$ with $r > 1$ and $p > 0$ fixed. If $1 < t < r$, we must have

$$\sum_n t^{n^p} \mid \xi_n \mid < \infty .$$

Choose any $q < p$. Then for any given k we have

$$\sum_n e^{kn^q} \mid \xi_n \mid = \sum_n e^{(kn^q - n^p \log t)} t^{n^p} \mid \xi_n \mid < \infty$$

and hence $\xi \in \Lambda_\infty(n^q)$. This shows that the first three unions in the statement of the proposition are equal. Also, by (5) and Corollary 2.6 the third union is contained in the first intersection, which equals the second. Next, by the first intersection, which equals the second. Next, by the definition of σ_1, $\Lambda_r(\alpha) \subset s$ for $\alpha \in \sigma_1$. Finally, let $\xi \in s$. Choose integers n_k such that $n_{k+1} \geq 2n_k$ and $\sum_{n \geq n_k} n^{k^2} \mid \xi_n \mid \leq k^{-2}$ and then define the sequence α by

$\alpha_n = 1$ for $1 \leq n < n_1$ and $\alpha_n = k \log n$ for $n_k \leqslant n < n_{k+1}$.

Since $\frac{\log n}{\alpha_n} = \frac{1}{k} \to 0$ and $\frac{\alpha_{2n}}{\alpha_n} \leqslant \frac{(k+2)\log 2n}{k \log n} \to 1$, $\alpha\epsilon\sigma_1$.

Also given any j we have

$$\sum_{n \geq n_j} e^{j\alpha_n} \mid \xi_n \mid \leq \sum_{k \geq j} \sum_{n \geq n_k} n k^2 \mid \xi_n \mid \leq \sum_{k \geq j} k^{-2} < \infty$$

and hence $\xi \in \Lambda_\infty(\alpha)$.

From the above, by duality arguments and noting that $\bigcap_{s < \infty} \Gamma_s(\alpha) = \Lambda_\infty(\alpha)$, we immediately obtain

2.8 ***Proposition.*** *Proposition* 2.7 *continues to hold when* Λ_∞ *is replaced by* Γ_1 *and/or* Λ_r *by* Γ_r.

We now come to checking when the condition of Proposition 1.6 is satisfied. We find

2.9 ***Proposition.*** *For* $\alpha \in \sigma_\infty$ *(resp.* $\alpha \in \sigma'_\infty$*) the following assertions are equivalent:*

(i) $\sup_n \dfrac{\alpha_{n^2}}{\alpha_n} < \infty$

(ii) $\xi \in \Lambda_\infty(\alpha)$ *(resp.* $\Gamma_1(\alpha)$*) implies* $\hat{\xi} \in \Lambda_\infty(\alpha)$ *(resp.* $\Gamma_1(\alpha)$*).*

Proof. (i) $\Longrightarrow$ (ii): Assume $\sup_n \dfrac{\alpha_{n^2}}{\alpha_n} = c < \infty$, so that $\sup_n \dfrac{\alpha_{2n}}{\alpha_n} \leq c$, and let $\xi \in \Lambda_\infty(\alpha)$. By (1) we have $\hat{\xi}_k = \xi_n$ for $n^2 \leq k < (n+1)^2$ and hence, for any j,

$$\sum_{k \geq 4} e^{j\alpha_k} \mid \hat{\xi}_k \mid = \sum_{n \geq 2} \sum_{k = n^2}^{n^2 + 2n} e^{j\alpha_k} \mid \xi_n \mid \leq$$

$$\leq \sum_{n \geq 2} \mid \xi_n \mid (2n+1)\, e^{j\alpha_{n^2+2n}} \leq$$

$$\leq 3 \sum_{n \geq 2} \mid \xi_n \mid n\, e^{j\alpha_{2n^2}} \leq 3 \sum_{n \geq 2} \mid \xi_n \mid n\, e^{jc\alpha_{n^2}} \leq$$

$$\leq 3 \sum_{n \geq 2} \mid \xi_n \mid n\, e^{jc^2\alpha_n} \leq 3 \sum_{n \geq 2} \mid \xi_n \mid e^{m\,\alpha_n} < \infty\,,$$

where $\sup_n (jc^2 + \dfrac{\log n}{\alpha_n}) \leq m < \infty$ by the fact that $\alpha \in \sigma_\infty$. Thus $\hat{\xi} \in \Lambda_\infty(\alpha)$.

(ii) $\Longrightarrow$ (i): If (i) does not hold, then for every m there is an integer n_m such that $\alpha_{n_m^2} \geq m\,\alpha_{n_m}$. If we put $\xi_{n_m} = e^{-m\,\alpha_{n_m}}$ and $\xi_n = 0$ for $n \notin (n_m)$, then $\xi \in \Lambda_\infty(\alpha)$, since for every j,

$$\sum_n e^{j\,\alpha_n}\,\xi_n = \sum_m e^{(j-m)\,\alpha_{n_m}} < \infty .$$

However,

$$\sum_k e^{j\alpha_k}\,\hat{\xi}_k = \sum_m e^{-m\alpha_{n_m}} \sum_{k\,=\,n_m^2}^{n_m^2+2n_m} e^{j\alpha_k} \geq$$

$$\geq \sum_m e^{j\alpha_{n_m^2}-m\alpha_{n_m}} \geq \sum_m e^{(j-1)\,m\alpha_{n_m}} = \infty$$

and hence $\hat{\xi} \notin \Lambda_\infty(\alpha)$.

A similar proof holds for $\Gamma_1(\alpha)$.

2.10 ***Remark.*** Similarly to what observed in Remark 2.3, condition (i) of Proposition 2.9 implies

$$\sup_n\ (\log n)^{-p}\alpha_n < \infty \qquad \text{for some } p > 0 \tag{6}$$

and hence for some $p \geq 1$ by nuclearity of α (put $\sup_n \alpha_{n^2}/\alpha_n = 2^p$ and argue on n in the range $2^{2^{k-1}} < n \leq 2^{2^k}$).

2.11 ***Proposition.*** *For* $\bar{\lambda} = \Lambda_r(\alpha)$ *or* $\Gamma_r(\alpha)$ $(r > 1,\ \alpha \in \sigma_1)$ *it is never the case that* $\xi\epsilon\bar{\lambda}$ *implies* $\hat{\xi}\epsilon\bar{\lambda}$.

Proof. Inspection of the proof of Proposition 2.9 shows that in the finite type case condition (i) of the above proposition should be replaced by the condition

$$\lim_n \frac{\alpha_{n^2}}{\alpha_n} = 1.$$

But this (similarly to (5)) would yield, in place of (6),

$$\lim_n (\log n)^{-p} \alpha_n = 0 \qquad \text{for all } p > 0$$

and it is immediate to see that such condition contradicts the fact that $\alpha \in \sigma_1$.

To conclude, we discuss the varieties $\mathbb{N}_\lambda$ generated by the sequence spaces considered. To begin with, we recall the following definition from [10]. As we agreed after Corollary 1.3, $\mathbb{N}_{\Lambda_r(\alpha)}$ is the class of $\Lambda_r(\alpha)$ - nuclear spaces for $\alpha \in \sigma_1$. Then following [10] we say that a locally convex space is $\Lambda_N(\alpha)$ - nuclear if it is $\Lambda_r(\alpha)$ - nuclear for all $r > 1$.

2.12 Proposition. *If $\alpha \in \sigma_1$, then $\mathbb{N}_{\Gamma_r(\alpha)} = \mathbb{N}_{\Lambda_r(\alpha)} = \mathbb{N}_{\Lambda_N(\alpha)}$ for all $r > 1$.*

Proof. Recalling that $\Gamma_r(\alpha) \subset \Lambda_r(\alpha) \subset \Gamma_{r^{1/2}}(\alpha)$, the assertion follows from (3) and Proposition 2.14 of [10].

We then have the following theorem which includes part of Theorem (2.10) of [3] and Proposition 3.2 of [10].

2.13 Theorem. *$\mathbb{N}_{\Lambda_\infty(\alpha)}$ $(\alpha \in \sigma_\infty)$ and $\mathbb{N}_{\Lambda_N(\alpha)}$ $(\alpha \in \sigma_1)$ are stable under the fomation of countable direct sums.*

Proof. The assertion follows from Proposition 2.12 and Theorem 1.10, since $\Lambda_\infty(\alpha)$ and $\Lambda_r(\alpha)$ satisfy (H) by Lemma II.1 of [6].

2.14 Remark. The above theorem does not hold for the space $\Gamma_1(\alpha)$. In fact, let $\alpha \in \sigma'_\infty$; then $\Lambda_1(\alpha)$ belongs to $\mathbb{N}_{\Gamma_1(\alpha)}$ while the direct sum of countably many copies of $\Lambda_1(\alpha)$ does not. Since $\Gamma_1(\alpha)$ does not satisfy condition (H), the above example shows that no countable direct sum stability can be expected in the absence of such condition.

2.15 Theorem. *Suppose that α satisfies condition (i) of Proposition 2.9. Then the varieties $\mathbb{N}_{\Lambda_\infty(\alpha)}$ $(\alpha \in \sigma_\infty)$, $\mathbb{N}_{\Gamma_1(\alpha)}$ $(\alpha \in \sigma'_\infty)$ and $\mathbb{N}_{\Lambda_N(\alpha)}$ $(\alpha \in \sigma_1)$ are stable under the formation of projective tensor products.*

Proof. Apply Theorem 1.13. For $\mathbb{N}_{\Lambda_\infty(\alpha)}$ and $\mathbb{N}_{\Gamma_1(\alpha)}$ the result follows directly from Proposition 2.9, while for $\mathbb{N}_{\Lambda_N(\alpha)}$ the assertion is a consequence of the easily checked fact (see the first part of the proof of Proposition 2.9) that, if $\sup_n \alpha_{n^2}/\alpha_n = c < \infty$, then $\xi \in \Lambda_r(\alpha)$ implies $\hat{\xi} \in \Lambda_{r^{c^2}}(\alpha)$.

Examples of sequences $\alpha \in \sigma_\infty$, σ'_∞ or σ_1 and satisfying Proposition 2.9 (i) are afforded by the sequences α^p ($p > 1$) defined by $\alpha_n^p = (\log n)^p$.

Now a combination of Theorems 2.13 and 2.15 yields

2.16 Theorem. *$\mathbb{N}_{\Lambda_\infty(\alpha)}$ ($\alpha \in \sigma_\infty$) and $\mathbb{N}_{\Lambda_N(\alpha)}$ ($\alpha \in \sigma_1$) are stability classes.*

2.17 Remark. The above classes provide additional examples to those of [4] of stability classes. Moreover, of the four typical varieties mentioned in the introduction, $\mathbb{N}_o$, $\mathbb{N}$ and $\mathbb{S}\,\mathbb{H}$ are stability classes as well-known (this also follows from Theorems 1.10 and 1.13), but $\mathbb{W}$ is not. Indeed, we can assert.

2.18 Proposition: *The largest space ideal of the form $\mathbb{N}_\lambda$, namely $\mathbb{S}\,\mathbb{H}$, is a stability class and hence it is the largest stability class of this type. There is no smallest stability class of the form $\mathbb{N}_\lambda$.*

Proof. For the second assertion, out of each sequence $\beta = (\beta_n)$ such that $0 < \beta_n \nearrow \infty$, form the sequence space

$$\lambda(\beta) = \{ \xi : \sum_n n^k \beta_{n^k} | \xi_n | < \infty \qquad \text{for all } k \}$$

It is easily seen that $\lambda(\beta)$ is an ideal kernel satisfying $\hat{\xi} \in \lambda(\beta)$ if $\xi \in \lambda(\beta)$. Moreover $\lambda(\beta)$, being a Köthe space, satisfies (H) by Lemma II.1 of [6] . Thus $\mathbb{N}_{\lambda(\beta)}$ is a stability class by Theorem 1.13. Now it is evident that, for each sequence ideal λ, there exists a sequence β as above such that $\lambda(\beta) \underset{\neq}{\subset} \bigcap_{p>o} \bar{\lambda}^p$, so that $\mathbb{N}_{\lambda(\beta)} \underset{\neq}{\subset} \mathbb{N}_\lambda$. It also follows from this that $\bigcap_\beta \mathbb{N}_{\lambda(\beta)} = \Omega =$ the stability class, introduced in [4], of all locally convex spaces of maximal diametral dimension.

Finally, observing that $\bigcap_{p>o} \Lambda_r^p(\alpha) = \bigcap_{r<\infty} \Lambda_r(\alpha) = \Lambda_\infty(\alpha)$ is Fréchet, idempotent and clearly $\Lambda_N(\alpha)$-nuclear but not $\Lambda_\infty(\alpha)$-nuclear, and applying Theorem I-3 and Corollary I-2 of [6] we conclude with the

2.19 Theorem. (a) *Let $\alpha \in \sigma_1$. Then $\Lambda_\infty(\alpha)$ is a universal generator for $\mathbb{N}_{\Lambda_N(\alpha)}$ and hence the sub-ideal $\mathbb{F}\mathbb{N}_{\Lambda_N(\alpha)}$ of Fréchet spaces has $\Lambda_\infty(\alpha)^{\mathbb{N}}$ as a universal space.*

(b) $\mathbb{F}\mathbb{N}_{\Lambda_\infty(\alpha)}$ *has no universal space.*

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