SOME CLASS OF INTEGRAL FUNCTIONS REPRESENTED BY DIRICHLET SERIES OF SEVERAL COMPLEX VARIABLES HAVING FINITE ORDER

by

S. DAOUD

ABSTRACT:

In this paper, we consider two classes A and B of the family of integral functions defined by Dirichlet series of finite order. We establish some results concerning the integral functions represented by Dirichlet series of finite order ρ_1 and ρ_2 with respect to variables s_1 and s_2.

1. Consider the double* integral function defined by a every-where absolutely convergent Dirichlet Series [1]

$$f(s_1, s_2) = \sum_{m,n=0}^{\infty} a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2) \tag{1.1}$$

of complex variables s_1 and s_2, where the coefficients $a_{m,n}$ are complex numbers, $\lambda_o = \mu_o = 0$, $(\lambda_m)_{m \geq 1}$, $(\mu_n)_{n \geq 1}$ are two sequences of real increasing numbers whose limits are infinity and further

$$\limsup_{m+n \to \infty} \frac{\log(m+n)}{\lambda_m + \mu_n} = D < \infty \tag{1.2}$$

$$\limsup_{m+n \to \infty} \frac{\log|a_{m,n}|}{\lambda_m + \mu_n} = -\infty \tag{1.3}$$

* For the sake of simplicity, we have considered only two variables. For the corresponding in the case of several variables, the analytical work is similar to the corresponding analysis for functions of two complex variables.

Let

$$M(\sigma_1,\sigma_2) = \sup_{-\infty<t_1,t_2<\infty} |f(\sigma_1+it_1,\sigma_2+it_2)|$$

be the maximum modulus of the integral function $f(s_1,s_2)$, for Re $s_1 \leqslant \sigma_1$, Re $s_2 \leqslant \sigma_2$. From the maximum modulus principles of analytic functions, it follows that, if $f(s_1,s_2)$ is not constant with respect to any one of the variables s_1 and s_2 then

$$M(\sigma_1',\sigma_2) > M(\sigma_1,\sigma_2) \quad , \text{for } \sigma_1' > \sigma_1$$

(1.4)
$$M(\sigma_1,\sigma_2') > M(\sigma_1,\sigma_2) \quad , \text{for } \sigma_2' > \sigma_2$$

$$M(\sigma_1',\sigma_2') > M(\sigma_1,\sigma_2) \quad , \text{for } \sigma_1' > \sigma_1 \text{ and } \sigma_2' > \sigma_2$$

We shall consider the class A of the family of integral functions defined by Dirichlet series of finite order as a special subclass of the class of integral functions represented by Dirichlet series [1].

DEFINITION 1. We shall say that an integral function $f(s_1,s_2)$ given by Dirichlet series of finite order belongs to the class A, if there exists $K_1 > 0$, $\beta_1 > 0$; $K_2 > 0$, $\beta_2 > 0$ such that

(i) For any fixed value of $\sigma_2 > 0$, there exists a number $\sigma^{(1)} = \sigma^{(1)}(K_1, \beta_1, \sigma_2)$ such that

$$M(\sigma_1,\sigma_2) < \exp\{K_1 \exp(\sigma_1\beta_1)\} \quad , \text{for } \sigma_1 \geqslant \sigma^{(1)}$$

(ii) For any fixed value of $\sigma_1 > 0$, there exists a number $\sigma^{(2)} = \sigma^{(2)}(K_2, \beta_2, \sigma_1)$ such that

$$M(\sigma_1,\sigma_2) < \exp\{K_2 \exp(\sigma_2\beta_2)\} \quad , \text{for } \sigma_2 \geqslant \sigma^{(2)}$$

and so there exists a number $\sigma = \sigma(K_1, K_2, \beta_1, \beta_2)$ such that

$$M(\sigma_1,\sigma_2) < \exp\{K_1 \exp(\sigma_1\beta_1) + K_2 \exp(\sigma_2\beta_2)\} \quad , \text{for } \sigma_1,\sigma_2 \geqslant \sigma$$

DEFINITION 2. An integral function $f(s_1,s_2)$ defined by Dirichlet series has a finite order ρ_1 and ρ_2 with respect to variables s_1 and s_2 if

(i) For any arbitrary small $\epsilon > 0$, and any $\sigma_2 > 0$, there exists a number $\sigma^{(1)} = \sigma^{(1)}(\epsilon,\sigma_2)$ such that

$$M(\sigma_1,\sigma_2) < \exp\exp\{\sigma_1(\rho_1+\epsilon)\} \quad , \text{for } \sigma_1 \geqslant \sigma^{(1)} \tag{1.5}$$

In addition, there exists at least one value of σ_2, say $\sigma_2^o(\epsilon)$ and corresponding an arbitrary large values of $\sigma_1 : \{\sigma_{1i}\}$ such that

$$M(\sigma_{1i},\sigma^o{}_2(\epsilon)) > \exp\exp\{\sigma_{1i}(\rho_1-\epsilon)\} \tag{1.6}$$

The assertion (i) is equivalent to

$$\limsup_{\sigma_2\to\infty}\{\limsup_{\sigma_1\to\infty}\frac{\log\log M(\sigma_1,\sigma_2)}{\sigma_1}\} = \rho_1 \tag{1.7}$$

(ii) For any arbitrary small $\epsilon > 0$, and any $\sigma_1 > 0$, there exists a number $\sigma^{(2)} = \sigma^{(2)}(\epsilon,\sigma_1)$ such that

$$M(\sigma_1,\sigma_2) < \exp\exp\{\sigma_2(\rho_2+\epsilon)\} \quad , \text{for } \sigma_2 \geqslant \sigma^{(2)} \tag{1.8}$$

In addition, there exists at least one value of σ_1, say $\sigma_1^o(\epsilon)$ and corresponding an arbitrary large values of $\sigma_2 :\{\sigma_{2j}\}$ such that

$$M(\sigma_1^o(\epsilon),\sigma_{2j}) > \exp\exp\{\sigma_{2j}(\rho_2-\epsilon)\} \tag{1.9}$$

The assertion (ii) is equivalent to

$$\limsup_{\sigma_1\to\infty}\{\limsup_{\sigma_2\to\infty}\frac{\log\log M(\sigma_1,\sigma_2)}{\sigma_2}\} = \rho_2 \tag{1.10}$$

DEFINITION 3. An integral function $f(s_1,s_2)$ defined by Dirichlet series has a finite order (ρ_1,ρ_2) if

(i) $f(s_1,s_2) \in A$

(ii) $f(s_1,s_2)$ has a finite order ρ_1 and ρ_2 with respect to variables s_1 and s_2 as definition 2.

(iii) In addition, for $\epsilon > 0$, there exists a number $\sigma = \sigma(\epsilon)$, such that

$$(1.11) \qquad M(\sigma_1,\sigma_2) < \exp\{\exp(\rho_1+\epsilon)\sigma_1 + \exp(\rho_2+\epsilon)\sigma_2\} \quad , \text{for } \sigma_1,\sigma_2 \geqslant \sigma$$

THEOREM 1. If

$$f(s_1,s_2) = \sum_{m,n=0}^{\infty} a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2)$$

is an integral function of order (ρ_1,ρ_2), $(0 < \rho_1 < \infty, 0 < \rho_2 < \infty)$, then

$$\limsup_{\sigma_1,\sigma_2\to\infty} \left\{ \frac{\log\log M(\sigma_1,\sigma_2)}{\sigma_1\rho_1 + \sigma_2\rho_2} \right\} = 1$$

PROOF. In view of eq. (1.11) and for given $\mu_1 > \rho_1$ and $\mu_2 > \rho_2$, one has

$$\limsup_{\sigma_1,\sigma_2\to\infty} \left\{ \frac{\log\log M(\sigma_1,\sigma_2)}{\sigma_1\mu_1+\sigma_2\mu_2} \right\} \leqslant 1$$

Further, if $\sigma_1 > 0$, $\sigma_2 > 0$, $\mu = \max(\mu_1,\mu_2)$, $\rho = \min(\rho_1,\rho_2)$, then we have

$$\frac{\log\log M(\sigma_1,\sigma_2)}{\sigma_1\rho_1+\sigma_2\rho_2} \leqslant \frac{\log\log M(\sigma_1,\sigma_2)}{(\sigma_1+\sigma_2)\rho} =$$

$$\frac{\log\log M(\sigma_1,\sigma_2)}{\sigma_1\mu_1+\sigma_2\mu_2} \cdot \frac{\sigma_1\mu_1+\sigma_2\mu_2}{(\sigma_1+\sigma_2)\rho}$$

$$\leqslant \frac{\mu}{\rho} \quad \frac{\log \log M(\sigma_1, \sigma_2)}{\sigma_1 \mu_1 + \sigma_2 \mu_2}$$

then

$$\limsup_{\sigma_1, \sigma_2 \to \infty} \frac{\log \log M(\sigma_1, \sigma_2)}{\sigma_1 \rho_1 + \sigma_2 \rho_2} = a \leqslant \frac{\mu}{\rho} < +\infty \tag{1.12}$$

thus, it is necessary to prove that $a = 1$.

Suppose that $a > 1$, then there exist a', a" such that $1 <$a" $<$a' $<$a, and in view of eq. (1.12), there exist two sequences $\{\sigma_{1i}\}$, $\{\sigma_{2j}\}$ such that

$$\frac{\log \log M(\sigma_{1i}, \sigma_{2j})}{\sigma_{1i} \rho_1 + \sigma_{2j} \rho_2} > a' \qquad i,j = 1, 2, \ldots$$

i.e.

$$M(\sigma_{1i}, \sigma_{2j}) > \exp \exp \{ a'(\sigma_{1i} \rho_1 + \sigma_{2j} \rho_2) \} \qquad i,j = 1, 2, \ldots$$

and for $j = j_o$, one has

$$M(\sigma_{1i}, \sigma_{2j}) > \exp \exp \{ a'' (\sigma_{1i} \rho_1) \} \qquad \text{for } i \geqslant i_o$$

This contradicts the hypothesis that the integral function $f(s_1, s_2)$ has order ρ_1 with respect to variable s_1, because for sufficiently small $\epsilon > 0$, $a'' \rho_1 > \rho_1 + \epsilon$.

Now, take $a < 1$, then there exist a', a", such that a $<$a' $<$a" <1, and in view of (1.12), one has

$$\frac{\log \log M(\sigma_1, \sigma_2)}{\sigma_1 \rho_1 + \sigma_2 \rho_2} < a' \qquad \text{for } \sigma_1, \sigma_2 \geqslant \sigma \text{ (say)}$$

i.e.

(1.13) $$M(\sigma_1,\sigma_2) < \exp\exp\{a'(\sigma_1\rho_1+\sigma_2\rho_2)\} \qquad \text{for } \sigma_1,\sigma_2 \geqslant \sigma$$

From (1.13) and in virtue of (1.4), we have

$$M(\sigma_1,\sigma_2) < \exp\exp\{a'(\sigma_1\rho_1+\sigma_o\rho_2)\} \qquad \text{for } \sigma_1 \geqslant \sigma_o, 0 < \sigma_2 \leqslant \sigma_o$$

now, choose $\sigma^* \geqslant \sigma_o$ such that

$$a'(\sigma_1\rho_1+\sigma_o\rho_2) < a''\sigma_1\rho_1 \qquad \text{for } \sigma_1 \geqslant \sigma^*$$

then, we have for any $0 < \sigma_2 \leqslant \sigma_o$

(1.14) $$M(\sigma_1,\sigma_2) < \exp\exp\{a''(\sigma_1\rho_1)\}$$

If $\sigma_2 > \sigma_o$, one can choose $\sigma^* \geqslant \sigma_O$ such that the following inequality holds

$$a'(\sigma_1\rho_1+\sigma_2\rho_2) < a''\sigma_1\rho_1 \qquad \text{for } \sigma_1 \geqslant \sigma^*$$

i.e.

(1.15) $$M(\sigma_1,\sigma_2) < \exp\exp\{a''(\sigma_1\rho_1)\} \qquad \text{for } \sigma_1 \geqslant \sigma^*(\sigma_2)$$

This contradics the hypthesis that the integral function $f(s_1,s_2)$ has order ρ_1 with respect to the variable s_1, because for sufficiently small $\epsilon > 0$, $\rho_1 a'' < \rho_1 - \epsilon$.

2. Let, for j=1,2

$$f_{s_j}(s_1,s_2) = \partial/\partial s_j f(s_1,s_2)$$

and

$$M^{(j)}(\sigma_1,\sigma_2) = \sup_{-\infty<t_1,t_2<\infty} |f_{s_j}(\sigma_1+it_1, \sigma_2+it_2)|$$

LEMMA 2.

(a) For any fixed value of $\sigma_2 > 0$, there exists a number $\sigma^{(1)} = \sigma^{(1)}(f, \sigma_2)$ such that

$$M^{(1)}(\sigma_1,\sigma_2) \geqslant \frac{M(\sigma_1,\sigma_2)\log M(\sigma_1,\sigma_2)}{\sigma_1} \qquad \text{for } \sigma_1 \geqslant \sigma^{(1)} \tag{2.1}$$

(b) For any fixed value of $\sigma_1 > 0$, there exists a number $\sigma^{(2)} = \sigma^{(2)}(f,\sigma_1)$ such that

$$M^{(2)}(\sigma_1,\sigma_2) \geqslant \frac{M(\sigma_1,\sigma_2)\log M(\sigma_1,\sigma_2)}{\sigma_2} \qquad \text{for } \sigma_2 \geqslant \sigma^{(2)} \tag{2.2}$$

PROOF. It can easily be shown that for a fixed value of $\sigma_2 > 0$, the function

$$g(\sigma_1,\sigma_2) = \frac{\log M(\sigma_1,\sigma_2)}{\sigma_1}$$

is monotonic increasing, for $\sigma_1 \geqslant \sigma^{(1)}(f,\sigma_2)$.

Let ξ_1, such that Re $\xi_1=\sigma_1$ and $|f(\xi_1,s_2)| = M(\sigma_1,\sigma_2)$, then we have

$$M^{(1)}(\sigma_1,\sigma_2) \geqslant |f_{\xi_1}(\xi_1,s_2)|$$

$$= \left| \lim_{h\to 0} \frac{f(\xi_1,s_2) - f(\xi_1-h,s_2)}{h} \right|$$

$$\geqslant \lim_{h\to 0} \frac{M(\sigma_1,\sigma_2) - M(\sigma_1-h,\sigma_2)}{h}$$

$$= \lim_{h\to 0} \frac{\exp\{\sigma_1 g(\sigma_1,\sigma_2)\} - \exp\{(\sigma_1-h)g(\sigma_1-h,\sigma_2)\}}{h}$$

$$\geqslant \lim_{h\to 0} \frac{\exp\{\sigma_1 g(\sigma_1,\sigma_2)\} - \exp\{(\sigma_1-h)g(\sigma_1,\sigma_2)\}}{h}$$

$$= g(\sigma_1,\sigma_2)\exp\{\sigma_1 g(\sigma_1,\sigma_2)\}$$

then

$$M^{(1)}(\sigma_1,\sigma_2) \geqslant \frac{\log M(\sigma_1,\sigma_2)}{\sigma_1} M(\sigma_1,\sigma_2)$$

and the proof of the 2nd. part is similar to that of the 1st. part.

THEOREM 3. If

$$f(s_1,s_2) = \sum_{m,n=0}^{\infty} a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2)$$

is an integral function of finite order (ρ_1,ρ_2), ρ_1,ρ_2 with respect to the variables s_1 and s_2, then

$$\limsup_{\sigma_2\to\infty} \left\{ \limsup_{\sigma_1\to\infty} \frac{\log\{M^{(1)}(\sigma_1,\sigma_2)/M(\sigma_1,\sigma_2)\}}{\sigma_1} \right\} \geqslant \rho_1 \tag{2.3}$$

$$\text{(2.4)} \qquad \limsup_{\sigma_1\to\infty} \left\{ \limsup_{\sigma_2\to\infty} \frac{\log\{M^{(2)}(\sigma_1,\sigma_2)/M(\sigma_1,\sigma_2)\}}{\sigma_2} \right\} \geqslant \rho_2$$

Further, if $a_{m,n} \geqslant 0$, then

$$\text{(2.5)} \qquad \limsup_{\sigma_2\to\infty} \left\{ \limsup_{\sigma_1\to\infty} \frac{\log\{M^{(1)}(\sigma_1,\sigma_2)/M(\sigma_1,\sigma_2)\}}{\sigma_1} \right\} \leqslant 2\,\rho_1$$

$$\text{(2.6)} \qquad \limsup_{\sigma_1\to\infty} \left\{ \limsup_{\sigma_2\to\infty} \frac{\log\{M^{(2)}(\sigma_1,\sigma_2)/M(\sigma_1,\sigma_2)\}}{\sigma_2} \right\} \leqslant 2\,\rho_2$$

PROOF. From eq. (2.1), we have

$$\log \sigma_1 + \log\left\{\frac{M^{(1)}(\sigma_1,\sigma_2)}{M(\sigma_1,\sigma_2)}\right\} \geqslant \log\log M(\sigma_1,\sigma_2)$$

or, in view of eq. (1.7), we have

$$\text{(2.7)} \qquad \limsup_{\sigma_2\to\infty} \left\{ \limsup_{\sigma_1\to\infty} \frac{\log\{M^{(1)}(\sigma_1,\sigma_2)/M(\sigma_1,\sigma_2)\}}{\sigma_1} \right\} \geqslant$$

$$\limsup_{\sigma_2\to\infty} \left\{ \limsup_{\sigma_1\to\infty} \frac{\log\log M(\sigma_1,\sigma_2)}{\sigma_1} \right\} = \rho_1$$

Now, we suppose that $a_{m,n} \geqslant 0$, then for any fixed value of $\sigma_2 > 0$, we have

$$M(\sigma_1,\sigma_2) = f(\sigma_1,\sigma_2) \quad , \quad M^{(1)}(\sigma_1,\sigma_2) = \partial/\partial\sigma_1 f(\sigma_1,\sigma_2)$$

Further, for any fixed value of $\sigma_2 > 0$, the function $\log M(\sigma_1,\sigma_2)$ is increasing convex function of σ_1, then one can write $\log M(\sigma_1,\sigma_2)$ as

$$\log M(2\sigma_1,\sigma_2) = \log M(\sigma_1,\sigma_2) + \int_{\sigma_1}^{2\sigma_1} \frac{\partial/\partial t_1\, M(t_1,\sigma_2)}{M(t_1,\sigma_2)}\, dt_1 \tag{2.8}$$

$$\geqslant 2\,\sigma_1 \frac{M^{(1)}(\sigma_1,\sigma_2)}{M(\sigma_1,\sigma_2)}$$

therefore

$$\limsup_{\sigma_2\to\infty}\left\{\limsup_{\sigma_1\to\infty} \frac{\log\{M^{(1)}(\sigma_1,\sigma_2)/M(\sigma_1,\sigma_2)\}}{\sigma_1}\right\} \leqslant 2\,\rho_1$$

Similarly, by using eq. (2.2) of lemma 2, one can prove that eqs. (2.4) and (2.6) holds.

THEOREM 4. Let

$$f(s_1,s_2) = \sum_{m,n=0}^{\infty} a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2)$$

be an integral function of finite order ρ_1,ρ_2 with respect to variables s_1 and s_2, then

$$\limsup_{\sigma_2\to\infty}\left\{\limsup_{\sigma_1\to\infty} \frac{\log\{M^{(1)}(\sigma_1,\sigma_2)/M(\sigma_1,\sigma_2)\}}{\sigma_1}\right\} = \rho_1 \tag{2.9}$$

$$\limsup_{\sigma_1\to\infty}\left\{\limsup_{\sigma_2\to\infty} \frac{\log\{M^{(2)}(\sigma_1,\sigma_2)/M(\sigma_1,\sigma_2)\}}{\sigma_2}\right\} = \rho_2 \tag{2.10}$$

PROOF. For some fixed $\eta > 0$, we have

$$\log M(\sigma_1+\eta,\sigma_2) = \log M(\sigma_1,\sigma_2) + \int_{\sigma_1}^{\sigma_1+\eta} \frac{\partial/\partial t_1\, M(t_1,\sigma_2)}{M(t_1,\sigma_2)}\, dt_1$$

$$> \eta\, \frac{M^{(1)}(\sigma_1,\sigma_2)}{M(\sigma_1,\sigma_2)}$$

Hence

$$\limsup_{\sigma_2\to\infty}\left\{\limsup_{\sigma_1\to\infty}\frac{\log\log M(\sigma_1,\sigma_2)}{\sigma_1}\right\}\geqslant$$

$$\limsup_{\sigma_2\to\infty}\left\{\limsup_{\sigma_1\to\infty}\frac{\log\{M^{(1)}(\sigma_1,\sigma_2)/M(\sigma_1,\sigma_2)\}}{\sigma_1}\right\}$$

or

$$\rho_1\geqslant\limsup_{\sigma_2\to\infty}\left\{\limsup_{\sigma_1\to\infty}\frac{\log\{M^{(1)}(\sigma_1,\sigma_2)/M(\sigma_1,\sigma_2)\}}{\sigma_1}\right\}\tag{2.11}$$

From (2.7) and (2.11) follows (2.9).

Similarly, we can prove that (2.10) hold.

DEFINITION 4. We shall say that an integral function $f(s_1,s_2)$ represented by Dirichlet series of finite order belonges to the class B, if the following conditions are satisfied

(i) For any fixed value of $\sigma_2>0$, there exists $K_1>0$, $\beta_1>0$ and $\sigma^{(1)}=\sigma^{(1)}(K_1,\beta_1,\sigma_2)$ such that

$$\frac{M^{(1)}(\sigma_1,\sigma_2)}{M(\sigma_1,\sigma_2)}<\exp(K_1\sigma_1\beta_1)\qquad\text{for } \sigma_1\geqslant\sigma^{(1)}$$

(ii) For any fixed value of $\sigma_1>0$, there exists $K_2>0$, $\beta_2>0$ and $\sigma^{(2)}=\sigma^{(2)}(K_2,\beta_2,\sigma_1)$ such that

$$\frac{M^{(2)}(\sigma_1,\sigma_2)}{M(\sigma_1,\sigma_2)}<\exp(K_2\sigma_2\beta_2)\qquad\text{for } \sigma_2\geqslant\sigma^{(2)}$$

and so, there exists a number $\sigma = \sigma(K_1, K_2, \beta_1, \beta_2)$ such that

$$\frac{M^{(1)}(\sigma_1,\sigma_2)}{M(\sigma_1,\sigma_2)} + \frac{M^{(2)}(\sigma_1,\sigma_2)}{M(\sigma_1,\sigma_2)} < \exp(K_1\sigma_1\beta_1 + K_2\sigma_2\beta_2),\ \sigma_1,\sigma_2 \geqslant \sigma$$

THEOREM 5. If

$$f(s_1,s_2) = \sum_{m,n=0}^{\infty} a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2)$$

be an integral function of order (ρ_1,ρ_2) $(0 < \rho_1 < \infty, 0 < \rho_2 < \infty)$, then

$$\limsup_{\sigma_1,\sigma_2 \to \infty} \left[\frac{\log\{M^{(1)}(\sigma_1,\sigma_2)/M(\sigma_1,\sigma_2) + M^{(2)}(\sigma_1,\sigma_2)/M(\sigma_1,\sigma_2)\}}{\sigma_1\rho_1 + \sigma_2\rho_2}\right] = 1$$

PROOF. From eqs. (2.9) and (2.10), we have

(i) For any $\epsilon > 0$, and any $\sigma_2 > 0$, there exists a number $\sigma^{(1)} = \sigma^{(1)}(\epsilon,\sigma_2)$ such that

$$\frac{M^{(1)}(\sigma_1,\sigma_2)}{M(\sigma_1,\sigma_2)} < \exp\{\sigma_1(\rho_1+\epsilon)\} \qquad \text{for } \sigma_1 \geqslant \sigma^{(1)}$$

(ii) For any $\epsilon > 0$, and any $\sigma_1 > 0$, there exists a number $\sigma^{(2)} = \sigma^{(2)}(\epsilon,\sigma_1)$ such that

$$\frac{M^{(2)}(\sigma_1,\sigma_2)}{M(\sigma_1,\sigma_2)} < \exp\{\sigma_2(\rho_2+\epsilon)\} \qquad \text{for } \sigma_2 \geqslant \sigma^{(2)}$$

and so, there exists a number $\sigma = \sigma(\epsilon)$, such that

$$\frac{M^{(1)}(\sigma_1,\sigma_2)}{M(\sigma_1,\sigma_2)} + \frac{M^{(2)}(\sigma_1,\sigma_2)}{M(\sigma_1,\sigma_2)} < \exp\{\sigma_1(\rho_1+\epsilon) + \sigma_2(\rho_2+\epsilon)$$

for $\sigma_1, \sigma_2 \geqslant \sigma$. Then

$$\limsup_{\sigma_1,\sigma_2\to\infty} \left[\frac{\log \{M^{(1)}(\sigma_1,\sigma_2)/M(\sigma_1,\sigma_2)+M^{(2)}(\sigma_1,\sigma_2)/M(\sigma_1,\sigma_2)\}}{\sigma_1\rho_1+\sigma_2\rho_2} \right] = A \leqslant 1$$

Now, it is necessary to prove that $A = 1$,

Let $A < 1$ and $A < A' < A'' < 1$. Then

(2.12)

$$\frac{M^{(1)}(\sigma_1,\sigma_2)}{M(\sigma_1,\sigma_2)} + \frac{M^{(2)}(\sigma_1,\sigma_2)}{M(\sigma_1,\sigma_2)} < \exp\{A'(\sigma_1\rho_1+\sigma_2\rho_2)\}, \quad \sigma_1,\sigma_2 \geqslant \sigma$$

From (2.12), we obtain that for any $\sigma_2 > 0$, there exists a number $\sigma^{(1)} = \sigma^{(1)}(\sigma_2)$, such that

$$\frac{M^{(1)}(\sigma_1,\sigma_2)}{M(\sigma_1,\sigma_2)} < \exp\{A''(\sigma_1\rho_1)\} \qquad \text{for } \sigma_1 \geqslant \sigma^{(1)}$$

this contradicts the hypothesis that the integral function $f(s_1,s_2)$ has order ρ_1 with respect to variable s_1, because for sufficiently small $\epsilon > 0$, $\rho_1 A'' < \rho_1 - \epsilon$, hence $A = 1$.

REFERENCE

1.S. DAOUD. On the space of integral Dirichlet functions of several complex variables (to appear).

U.E.R. Sciences-Mathématiques.
40 Avenue du Recteur Pineau,
86002 Poitiers (FRANCE)