SEMIDIFFERENTIAL CALCULUS

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Dedicated to Professor G. Köthe on
his eightieth birthday,
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Let X and Y denote topological vector spaces (TVS's). In this paper we present a theory of differentiation in which the derivative (called by us semiderivative) $f'[x_0]$ of $f: X \to Y$ at x_0 is a sequentially continuous positively homogeneous map from X to Y which is not necessarily additive. Functions $f: X \to Y$ for which all possible one-sided directional derivatives exist are semidifferentiable in the weakest (Gâteaux) sense when these directional derivatives depend sequentially continuously on the "directions". We also discuss Hadamard and Fréchet semiderivatives, in the now customary manner using uniform limits on the sets Σ of a cover of X, due to Sebastião e Silva (cf. [2], [3], [18], [20], [23]).

The first section, §1., discusses the basic concepts. In (1.9) we generalize a theorem of Vainberg asserting the linearity of $f'[x_0]$ when the map $x \to f'[x]$ is continuous at x_0. §2. is concerned with the fundamental theorem of calculus and a slight generalization of an important iterative fixed point result. In §3., we discuss computational rules, especially the composite and derivative-of-the-inverse theorems. Some examples are given in §4. Finally, in §5., we make several comments about generalizations. For deeper versions of some of the results of section 3 in more restricted contexts, see [25] and [26].

§1. Some Basic Denifitions and Results.

Throughout this paper we suppose that X and Y are TVS's (over $\mathbb{R}$ or $\mathbb{C}$). A map $f: X \to Y$ is said to be *sequentially continuous* at x_0 if $x_n \to x_0$ implies $f(x_n) \to f(x_0)$. The map f is sequentially continuous on X if and only if for every open set $U \subset Y$, the set $V = f^{-1}(U)$ is *sequentially open* in X. i.e., every sequence which converges to a point in V is ultimately in V. A map $f: X \to Y$ is said to be *positively homogeneous* if $f(tx) = tf(x)$ holds for all $x \in X$ and all $t > 0$.

Three covers of X are of special interest: Σ_G, the set of sets of the form $\{\ tx: \alpha \leqslant t \leqslant \beta\ \}$ for $x \in X$ and $\alpha, \beta \in \mathbb{R}$; Σ_H, the set of relatively sequentially compact sets (i.e., those in which every sequence has a convergent subsequence); and Σ_F, the set of bounded sets.

Let Σ denote an arbitrary cover of X. The map f: $X \to Y$ is said to be *Σ-semi-differentiable at* $x_0 \in X$ if there is a sequentially continuous positively homogeneous map $f'[x_0]: X \to Y$ such that for every $S \in \Sigma$

$$\lim_{t \downarrow 0} \frac{f(x_0 + th) - f(x_0) - f'[x_0](th)}{t} = 0 \tag{1}$$

holds uniformly on S. Equivalently, we can require that

$$\lim_{n \to \infty} \frac{f(x_0 + t_n h_n) - f(x_0) - f'[x_0](t_n h_n)}{t_n} = 0 \tag{2}$$

for every sequence $0 < t_n \to 0$ and every sequence h_n contained in some $S \in \Sigma$.

If X is normed and $\Sigma = \Sigma_F$, then (1) can be replaced by the more familiar condition

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0) - f'[x_0](h)}{\|h\|} = 0 \tag{3}$$

In the case in which $f'[x_0]$ satisfies $f'[x_0](\alpha x) = \alpha f'[x_0](x)$ for all $x \in X$ and *all* scalars α, the condition (3) defines what is called the uniform differential in [22], which appears to be the first systematic study of non-linear derivatives.

If f is Σ-semidifferentiable at x_0 and $f'[x_0]$ is linear, then f is said to be Σ-differentiable at x_0. The notions of (semi) differentiability defined by the covers Σ_G, Σ_H and Σ_F are associated with the names of Gâteaux, Hadamard and Fréchet, respectively (cf. [2], [3], [18]). We note that $\Sigma_G \subseteq \Sigma_H \subseteq \Sigma_F$ and that $\Sigma_H = \Sigma_F$ when X is an (FM) space ([12], p. 369), in particular, when X is finite dimensional. It is clear that if two covers satisfy $\Sigma \subseteq \Sigma'$, then every Σ'-semidifferentiable map is Σ-semidifferentiable.

There is an alternate characterization of Σ_H-semidifferentiability.

(4) *A map* f: $X \to Y$ *is* Σ_H*-semidifferentiable at* x_0 *if and only if for every path* $x(t): \mathbb{R} \to X$ *for which* $x'(0)$ *exists and* $x(0) = x_0$, *the limit*

$$\text{(5)} \qquad \lim_{t \downarrow 0} \frac{f(x(t)) - f(x_0)}{t}$$

exists. In this case, (5) *is equal to* $f'[x_0](x'(0))$.

Proof: Given any convergent sequence h_n in X and $0 < t_n \to 0$ in $\mathbb{R}$, we see that the path $x(t)$ defined by

$$x(t) = \begin{cases} x_0 + t_n h_n \text{ if } t = t_n \text{ for some } n = 1, 2, \ldots \\ x_0 \text{ if } t \geqslant 0 \text{ and } t \neq t_n \text{ for all } n \\ 2x_0 - x(-t) \text{ if } t < 0 \end{cases}$$

satisfies $x(0) = x_0$ and $x'(0) = h$, where $h = \lim_n h_n$. Thus in place of (5), we can require that

$$\text{(6)} \qquad \lim_{n} \frac{f(x_0 + t_n h_n) - f(x_0)}{t_n}$$

exist for all such sequences h_n and t_n.

If f is Σ_H-semidifferentiable at x_0 and $h_n \to h$ and $0 < t_n \to 0$, then it follows from (2) and the sequential continuity of $f'[x_0]$ that

$$\lim_{n} \frac{f(x_0 + t_n h_n) - f(x_0)}{t_n} = f'[x_0](h).$$

Conversely, let us suppose that the limit in (5) exists for all convergent sequences h_n and $0 < t_n \to 0$. We assert that the value of (6) depends only on the limit h of the sequence h_n: If h_n and h_n' both tend to h and t_n and t_n' are positive null sequences, we define $h_{2m}'' = h_{2m}$, $h_{2m+1}'' = h_{2m+1}'$, $t_{2m}'' = t_{2m}$ and $t_{2m+1}'' = t_{2m+1}'$ for $m = 0, 1, 2, \ldots$. Then since $h_n'' \to h$ and $0 < t_n'' \to 0$ we obtain

$$\lim_{m} \frac{f(x_0 + t_{2m} h_{2m}) - f(x_0)}{t_{2m}} = \lim_{n} \frac{f(x_0 + t_n'' h_n'') - f(x_0)}{t_n''}$$

$$= \lim_{m} \frac{f(x_0 + t_{2m+1}' h_{2m+1}') - f(x_0)}{t_{2m+1}'}$$

from which the assertion follows. Thus we can define a positively homogeneous map $f'[x_0] : X \to Y$ by means of

$$(7) \qquad f'[x_0](h) = \lim_n \frac{f(x_0 + t_n h_n) - f(x_0)}{t_n}$$

where $h_n \to h$ and $0 < t_n \to 0$.

Since

$$\frac{f(x_0 + t_n h_n) - f(x_0) - f'[x_0](t_n h_n)}{t_n} - \frac{f(x_0 + t_n h_n) - f(x_0) - t_n f'[x_0](h)}{t_n}$$

$$= (f'[x_0](h) - f'[x_0](h_n)),$$

the Σ_{II}-semidifferentiability of f at x_0 will follow from (7), once we establish that $f'[x_0]$ is sequentially continuous:

Were this not so, there would exist a neighborhood U of 0 in Y and a convergent sequence $h_n \to h$ in X such that for all n

$$(8) \qquad f'[x_0](h_n) \notin f'[x_0](h) + U.$$

Let V be a symmetric neighborhood of 0 in Y satisfying $V + V \subset U$. It follows from (7) that we can find a sequence $0 < t_n \to 0$ such that

$$\frac{f(x_0 + t_n h_n) - f(x_0)}{t_n} - f'[x_0](h_n) \in V.$$

Then it follows from (7) again that when n is large enough

$$\frac{f(x_0 + t_n h_n) - f(x_0)}{t_n} \in f'[x_0](h) + V.$$

Subtracting the first expression from the second, we obtain

$$f'[x_0](h_n) \in f'[x_0](h) + V + V \subseteq f'[x_0](h) + U$$

which contradicts (8). So $f'[x_0]$ is indeed sequentially continuous and our proof is finished.

This theorem, (4), shows that Σ_H-semidifferentiability is the weakest kind of Σ-semidifferentiability for which the composite theorem (cf. 3. (1) below) holds, and it shows the fundamental nature of our requirement that semiderivatives be sequentially continuous.

It is of interest to know when $f'[x_0]$ is linear. We present a generalization of a well-known theorem of Vainberg [23] (p. 37) which reveals an important limitation of the theory of semiderivatives. Because we do not assume its sequential continuity, we use a different notation for the semiderivative.

A map g: X → Y which is linear when X and Y are considered vector spaces over ℝ is called *real-linear.*

(9) *Let* U *be a set in the* TVS X *whose intersection with every two dimensional subspace* X_2 *of* X *contains a relative neighborhood of* 0. *Let* Y *be a locally convex* TVS *with dual* Y' *and suppose that the map* f: X → Y *is semidifferentiable at every point of the set* $x_0 + U$, *for some* $x_0 \in X$, *in the sense that*

$$RDf[x](h) = \lim_{t \downarrow 0} \frac{f(x + th) - f(x)}{t}$$

exists for all $x \in x_0 + U$ and $h \in X$.

Suppose, too, that the map $x \to RDf[x]$ *has the following continuity property at* x_0:

If $\phi \in Y'$ *then for every subspace* X_2 *of* X *of dimension at most* 2 *and every* $h \in X_2$

$$\lim_{\substack{x \to x_0 \\ x \in x_0 + X_2}} \phi \circ RDf[x](h) = \phi \circ RDf[x_0](h).$$

Then $RDf[x_0]$ *is real-linear.*

Proof: It suffices to show that $\phi \circ RDf[x_0]$ is linear for each real-valued continuous linear functional ϕ, so we can assume that f is real-valued. Since $RDf[x_0]$: : X → ℝ is clearly positively homogeneous we have only show that it is additive. Our proof is a modification of Vainberg's ([23], p. 38-9).

Let $h_1, h_2 \in X$ be given. Define, for $t > 0$,

$$\alpha_1(t) = RDf[x_0](h_1) - (f(x_0 + th_1) - f(x_0))/t$$
$$\alpha_2(t) = RDf[x_0](h_2) - (f(x_0 + th_2) - f(x_0))/t$$
$$\alpha_3(t) = RDf[x_0](h_1 + h_2) - (f(x_0 + th_2 + th_1) - f(x_0))/t$$
$$\alpha_4(t) = (f(x_0 + th_2 + th_1) - f(x_0 + th_2))/t$$
$$\alpha_5(t) = (f(x_0 + th_1) - f(x_0))/t.$$

Then

$$RDf[x_0](h_1) + RDf[x_0](h_2) - RDf[x_0](h_1 + h_2) = \alpha_1(t) + \alpha_2(t) - \alpha_3(t) - \alpha_4(t) + \alpha_5(t). \tag{10}$$

We show that $RDf[x_0](h_1) + RDf[x_0](h_2) = RDf[x_0](h_1 + h_2)$. by showing that the right hand side of (10) approaches 0 as $t \downarrow 0$. We know from the definition of $RDf[x_0]$ that $\alpha_1(t), \alpha_2(t)$ and $\alpha_3(t)$ tend to 0 and that $\alpha_5(t)$ tends to $RDf[x_0](h_1)$ as $t \downarrow 0$. So we have to verify

$$\lim_{t \downarrow 0} \alpha_4(t) = RDf[x_0](h_1). \tag{11}$$

Assume t is small enough that $\{x_0 + tx_2 + \xi x_1 : \xi \in [0, t]\}$ is contained in $x_0 + U$. Then because both one-sided derivatives exist at every point, the function $\xi \to f(x_0 + t_2 x_2 + \xi x_1)$ is continuous on $[0, t]$. Hence we can apply Theorem 3 of [14] to obtain the existence of numbers $\xi_i = \xi_i(t) \in (0, t)$ and $\lambda_i = \lambda_i(t) \in [0, 1]$ with $\lambda_1 + \lambda_2 = 1$ such that

$$\alpha_4(t) = \sum_{i=1}^{2} \lambda_i RDf[x_0 + th_2 + \xi_i h_1](h_1).$$

By the continuity hypothesis, $RDf[x_0 + th_2 + \xi_i h_1](h_1)$ approaches $RDf[x_0](h_1)$ as $t \downarrow 0$ for $i = 1,2$. So (11) follows. This finishes the proof.

Note (as in [23], p. 39) that if $RDf[x_0](h)$ is defined for all $h \in X$, it follows from (10) that it is real-linear if and only if

$$\lim_{t \downarrow 0} \frac{f(x_0 + th_1 + th_2) - f(x_0 + th_1) - f(x_0 + th_2) + f(x_0)}{t} = 0$$

for all $h_1, h_2 \in X$.

It is possible to define *higher order Σ-semiderivatives* after ther manner in which higher order Σ-derivates are defined in [2]. But it follows from (9) that if we make the rather weak assumption that Σ contains all bounded sets whose span has dimension at most 2, then a map f from X to the locally convex space Y, which is n times Σ-semidifferentiable at x_0, is n - 1 times Σ-differentiable at x_0. So we limit our investigation to first order semiderivatives. (For a study of some properties of what might be called higher order Peano semidifferentials cf. [8]).

There are some results which strengthen the continuity requirement of (9) and yield the Σ_L-differentiability of f at x_0 as a consequence, when X and Y are normed. For in this case, we can define a norm on the space $H_\pi (X, Y)$ of continuous positively homogeneous maps from X to Y by defining for each $g \in H_\pi (X, Y)$

(12) $$\|g\| = \sup_{\|h\| \leq 1} \|g(h)\|.$$

Then we apply the fundamental theorem of calculus (established in (2.2) below) exactly as in the proofs of Theorems 1.3 and 1.4 of [18] (pp. 121-2) to generalize these theorems. We state the generalization of Theorem 1.3 of [18] so obtained.

(13) *Let* X *and* Y *be real normed spaces and let* $f: X \to Y$ *be* Σ_G*-semidifferentiable in some neighborhood of* x_0 *in such a way that the map* $x \to f'[x]$ *is continuous at* x_0 *when* $H_\pi (X, Y)$ *is normed as in* (12). *Then* f *is* Σ_L*-differentiable at* x_0.

We close this section with some remarks about sequentially continuous positively homogeneous maps. We start by noting the following.

(14) *Let* Σ *be a cover of* X. *Then any* $f \in H_\pi (X, Y)$ *is* Σ*-semidifferentiable at* 0 *and* $f'[0](h) = f(h)$.

If X and Y are normed spaces, each $f \in H_\pi (X, Y)$ has a decomposition of the form

(15) $$f(x) = \|x\| r(x/\|x\|)\, \sigma(x/\|x\|) \qquad (x \neq 0)$$

where $r(x) = \|f(x)\|$ maps $S_X = \{x : \|x\| = 1\}$ to $[0, \infty]$ and σ maps S_X to S_Y. Conversely, any such pair of maps r, σ define a positively homogeneous map f from X to Y by means of (15) and $f(0) = 0$. Consequently, positively homogeneous maps inherit properties which are familiar for maps between spheres. For example, if $X = Y = \mathbb{R}^n$ $(n \geqslant 0)$, then f is onto if f is one-to-one. Also, the positively homogeneous homeomorphisms of $\mathbb{R}^n$ with the norm topology defined by (12) form a topological group under composition.

§2. The Fundamental Theorem of Calculus

We begin with some notation. If $f: x \to f[x]$ is a map from X to the space Map (X, Y) of all maps from the TVS X to the TVS Y, then we define, for any $x_0, x \in X$

$$\int_{x_0}^{x} f = \int_0^1 f[x_0 + t(x - x_0)](x - x_0)\, dt$$

provided the integral on the right is defined in some sense.

We say that a map f from [0, 1] to the locally convex TVS Y is DP-integrable (Denjoy-Pettis) if there is a $y \in Y$ with the property that for each real-valued continuous linear functional ϕ on Y, the function $\phi \circ f$ is Denjoy integrable (cf. [19] esp. VIII (3.9)) and

$$\phi(y) = (D) \int_0^1 \phi \circ f.$$

In this case we write

$$y = (DP) \int_0^1 f.$$

(1) *Let* X *be a* TVS, *let* Y *be a locally convex* TVS *and let* Σ *be a cover of* X. *If* $f: X \to Y$ *is* Σ*-semidifferentiable at every point of the convex set* C, *then for every* $x, x_0 \in C$

$$f(x) \quad f(x_0) = (DP) \int_{x_0}^{x} f'.$$

Proof: For any real-valued continuous linear functional ϕ on Y, the function $h(t): t \to \phi \circ f(x_0 + t(x - x_0))$ has one-sided derivatives at every point of [0, 1]. So this function is continuous on [0, 1] and satisfies the hypotheses of ([19] VII (10.5)). Therefore ([19] VIII 3.9)

$$h(1) - h(0) = \phi \circ f(x) - \phi \circ f(x_0) = \phi\,(f(x) - f(x_0))$$

is the Denjoy integral of

$$h'(t) = \phi \circ f'[x_0 + t(x - x_0)]\,(x - x_0)$$

over [0, 1], as asserted.

If Y is a normed space, we can prove a strengthened form of (1) in which an integral defined by the norm topology is used, the so-called Denjoy-Bochner integral.

(2) *Let* X *be a* TVS, *let* Y *be a normed space, and let* Σ *be a cover of* X. *If* f: X $\to$ Y *is* Σ*-semidifferentiable at every point of the convex set* C, *then for every* x, $x_0 \in$ C

$$f(x) - f(x_0) = (DB) \int_{x_0}^{x} f'.$$

Proof: We define $g(t) = f(x_0 + t(x - x_0))$ and note that the existence of the right and left derivatives, $g'_+(t)$ and $g'_-(t)$, at each $t \in [0, 1]$ insures that the hypotheses of Lemma 1 and Theorem 3 of [11] hold for g on [0, 1]: Indeed, g is continuous on [0, 1] and for each real-valued continuous linear function ϕ on Y, the real-valued map $t \to \phi \circ g(t)$ is differentiable on the complement of a countable subset of [0, 1] (cf. (17.9) of [10], for example). Finally, since it is the limit of the continuous functions $t \to n(g(t + (1/n)) - g(t))$, the right hand derivative $g'_+(t)$ is Borel measurable and so is essentially separably valued ([6] III.6.9). Thus, by Theorem 3 of [11], g is a.e. differentiable on [0, 1], and obviously $g'(t) = f'[x_0 + t(x - x_0)]\,(x - x_0)$. By lemma 1 of [11], there is a countable cover $\{ C_n : n = 1, 2, \ldots \}$ of [0, 1] such that g is absolutely continuous on each C_n, n = 1, 2, . . . (and cf. also [19] VII 9.1).

Thus, g fulfills the descriptive definition of the indefinite Denjoy integral of $g'(t)$ ([19], p. 241) and we have

$$f(x) - f(x_0) = g(1) - g(0) = (DB) \int_0^1 g' = (DB) \int_{x_0}^{x} f'.$$

To compare this integral with the Lebesgue-Bochner integral and to obtain the DB-integral constructively, as a so-called G-integral, one uses instead the interval function defined by g(t), viz.

$$g(\langle a, b \rangle) = g(b) - g(a)$$

where $< a, b >$ is a relatively open subinterval of $[0, 1]$. The required results are then found in 7. on p. 23 and in 8. on p. 45 of [21].

The fundamental theorem of calculus fails when Y is unrestricted. Consider the case where Y is the locally bounded space $L^p([0, 1])$ ([12] p. 156) with $p \in (0, 1)$. For each $t \in [0, 1]$, let $f(t)$ be the indicator function of the interval $[0, t]$ (cf. Example 1.23 on p. 218 of [2]). Then $f: [0, 1] \to Y$ is a non-constant map whose derivative at every point of $[0, 1]$ is 0: Indeed, if we define

$$||g||_p = \left(\int_0^1 |g(s)|^p ds \right)^{\frac{1}{p}}$$

as usual, we obtain

$$\lim_{t \to t_0} \left|\left| \frac{f(t) - f(t_0)}{t - t_0} \right|\right|_p = \lim_{t \to t_0} |t - t_0|^{\frac{1}{p} - 1} = 0.$$

As an application of (2) we give a generalization of the result which is the starting point for much of the work on Newton's Method in normed spaces (cf. [5], [11], [23]). The proof is as given in [11] (Theorem 1, p. 697) or [23] (Theorem 27.1, p. 260) except that (2) is used when the fundamental theorem of calculus is called for.

(3) *Let* f *be a map from the ball* $\{x : ||x - x_0|| < R\}$ *of the Banach space* X *to* X *which is* Σ_F*-semidifferentiable at every point of the closed ball* $\{x : ||x - x_0|| \leq r\}$ $(0 < r < R)$. *Let* ϕ *be an absolutely continuous real-valued function defined on some interval* $[t_0, t_0 + r]$ *and having the following properties.*

(i) *There is a* $t^* \in [t_0, t_0 + r]$ *such that* $\phi(t^*) = t^*$.

(ii) $||f(x_0) - x_0|| \leq \phi(t_0) - t_0$

(iii) $||f'[x]|| \leq \phi'(t)$ *whenever* $||x - x_0|| \leq t - t_0$ *and* $\phi'(t)$ *exists.*

Then the sequence of successive approximations

$$x_{n+1} = f(x_n) \qquad n = 0, 1, 2, \ldots$$

converges to a fixed point x^* *of* f. *Moreover,* $||x^* - x_0|| \leq t^* - t_0$, *where* t^* *is the smallest number in* $[t_0, t_0 + r]$ *for which* $\phi(t^*) = t^*$.

§3. Computational Results: Composites, Inverses, etc.

There are a number of important theorems in the differential calculus which we have not discussed yet, at least not explicitly. As indicated in the proof of (1.9), there are mean value theorems available for functions which are semidifferentiable on a convex set (cf. [14] and pp. 175-184 of [18]). Also, it is easy to show (cf. (3.8) and (3.15) of [8]) that if Y is, say, a Banach algebra with unit the usual product and quotient rules are valid for Σ-semiderivatives of maps f,g: $X \to Y$. It remains to consider the composite theorem ([24]) and the derivative-of-the-inverse theorem.

We need some additional terminology. Let covers Σ and Σ' of the TVS's X and Y be given. A map f: $X \to Y$ is said to be (Σ, Σ')-*compatible* if for every $S \in \Sigma$ there is an $S' \in \Sigma'$ such that $f(S) \subset S'$. Observe that a sequentially continuous positively homogeneous map from X to Y is (Σ_H, Σ_H)-compatible as well as (Σ_F, Σ_F)-compatible.

The cover Σ' of Y is said to be *sequentially stable* ([24]) if, whenever the sequence r_n tends to 0 in Y and $S' \in \Sigma'$, the set $S' + \{ r_n : n = 1, 2, \ldots \}$ is contained in some $S'' \in \Sigma'$. Σ_H and Σ_F have this property.

A map g' from Y to the TVS Z is said to be *uniformly sequentially continuous on the cover* Σ' of Y if, whenever sequences $y_n + r_n$ and y_n in $S' \in \Sigma'$ with $r_n \to 0$ are given, then $\lim_n (g'(y_n + r_n) - g'(y_n)) = 0$. If g' is sequentially continuous, then g' is uniformly sequentially continuous on Σ_H. A sequentially continuous linear map is uniformly sequentially continuous on every cover.

Now we state the composite theorem.

(1) *Let* X, Y *and* Z *be* TVS's *and let covers* Σ *and* Σ' *of* X *and* Y *respectively be given,* Σ' *being sequentially stable. Suppose that* f: $X \to Y$ *is* Σ*-semidifferentiable at* x_0 *and that* $f'[x_0]$ *is* (Σ, Σ')*-compatible. Let* g: $Y \to Z$ *be* Σ'*-semidifferentiable at* $f(x_0)$ *in such a way that* $g'[f(x_0)]$ *is uniformly sequentially continuous on* Σ'.

Then the composite $g \circ f$ *is* Σ*-semidifferentiable at* x_0 *and* $(g \circ f)'[x_0] = g'[f(x_0)] \circ f'[x_0]$.

Proof: Let sequences $h_n \in S \in \Sigma$ and $0 < t_n \to 0$ be given. Then

$$f(x_0 + t_n h_n) = f(x_0) + t_n f'[x_0](h_n) + t_n r_n$$

where $r_n \to 0$. By the assumptions on $f'[x_0]$ and Σ', the sequence $k_n = f'[x_0](h_n) + r_n$ belongs to a set $S' \in \Sigma'$. Using the Σ'-semidifferentiability of g we obtain

$$g(f(x_0) + t_n k_n) = g(f(x_0)) + t_n\, g'[f(x_0)]\,(k_n) + t_n r_n'$$

and $r_n' \to 0$. Consequently,

$$(g \circ f)\,(x_0 + t_n h_n) = (g \circ f)\,(x_0) + t_n\, g'[f(x_0)] \circ f'[x_0]\,(h_n) + t_n r_n'' \tag{2}$$

where

$$r_n'' = g'[f(x_0)]\,(f'[x_0]\,(h_n) + r_n) \quad g'[f(x_0)]\,(f'[x_0]\,(h_n)) + r_n'$$

Since $g'[f(x_0)]$ is uniformly sequentially continuous on Σ' and $r_n' \to 0$, we have $r_n'' \to 0$, which is what we had to show ((1.2)).

The hypothesis of uniform sequential continuity in (1) is essential, as the next result shows.

(3) *Let* g: $Y \to Z$ *be a* Σ'*-semidifferentiable function at* y_0 *for which* $g'[y_0]$ *is not uniformly sequentially continuous on* Σ' *and let* X *be any infinite dimensional normed space.*

Then there exists a continuous map f: $X \to Y$ *with the following properties.*

(i) $f(0) = y_0$.

(ii) f *is* Σ_F*-semidifferentiable at* 0.

(iii) $f'[0]$ *is* (Σ_F, Σ')*-compatible, provided that for each* $S' \in \Sigma'$ *and* $\delta > 0$ *there exists an* $S'' \in \Sigma'$ *which contains the set* $\{ tS' : t \in [0, \delta] \}$.

(iv) *There exist sequences* $t_n \downarrow 0$ *and* $h_n \in S = \{ x : ||x|| = 1 \}$ *for which*

$$\frac{g \circ f(t_n h_n) \quad g \circ f(0) - g'[y_0] \circ f'[0]\,(t_n h_n)}{t_n} \not\to 0.$$

Proof: By assumption, there exist sequences y_n and $y_n + r_n$ in some $S' \in \Sigma'$ with $r_n \to 0$ but for which

$$g'[y_0]\,(y_n + r_n) \quad g'[y_0]\,(y_n) \not\to 0. \tag{4}$$

Since X is infinite dimensional, the unit sphere $S = \{ x : ||x|| = 1 \}$ is not pre-compact, so we can find a sequence of relatively open sets $O_n \subset S$ such that no sequence x_n satisfying $x_n \in O_n$ has an adherent point. Let a sequence h_n satisfying $h_n \in O_n$ be fixed. Since for each $n = 1, 2, \ldots$ the set $2^{-n+1}S$ is a completely regular topological space ([12] 6, 8. (1)) there exists a real valued continuous function $\phi_n : 2^{-n+1}S \to [0, 1]$ such that $\phi_n(2^{-n+1}h_n) = 1$ and $\phi_n(x) = 0$ if $x \notin 2^{-n+1}O_n$. With the aid of these ϕ_n, we define a sequence f'_n of continuous positively homogeneous maps from X to Y by setting $f_n(0) = 0$ and

$$f'_n(x) = \phi_n(2^{-n+1}x/||x||)\, ||x||\, y_n \qquad (x \neq 0).$$

We define a second sequence f_n by means of

(5i) $f_n(x) = f'_n(x)$ if $||x|| \leq 2^{-n}$;

(5ii) $f_n(x) = \phi_n(2^{-n+1}x/||x||)\, ||x|| (y_n + r_n)$ if $||x|| \geq 2^{-n+1}$;

(5iii) $f_n(x) = (1 \quad t)\, f'_n(x^*) + (2t \quad 1)\, f_n(x^*)$ if $x = tx^*$ for some $x^* \in 2^{-n+1}S$ and some $t \in (1/2, 1)$.

It is readily verified that for any $x \in X$

$$(6) \qquad f_n(x) - f'_n(x) = c(x)\, ||x||\, r_n$$

where $0 \leq c(x) \leq 1$ and $c(2^{-n+1}h_n) = 1$. Also, if $f'_{n_0}(x) \neq 0$ then $f_n(x) = f'_n(x) = 0$ for every $n \neq n_0$, so the maps f and f' defined by $f'(x) = \sum_{n=1}^{\infty} f'_n(x)$ and $f(x) = y_0 + \sum_{n=1}^{\infty} f_n(x)$ are continuous.

Since $f(0) = y_0$ it follows from (6) that

$$\lim_{x \to 0} \frac{f(x) \quad f(0) \quad f'(x)}{||x||} = 0.$$

Hence f is Σ_F-semidifferentiable at 0 and $f'[0] = f'$ ((i), (ii)).

The assertion (iii) follows from the fact that for any $\delta > 0$

$$f'[0](\{ x : ||x|| \leq \delta \}) \subset \{ tS' : t \in [0, \delta] \}.$$

Define the sequence t_n by $t_n = 2^{-n+1}$. Then we obtain from (6) that $f(t_n h_n) = f(0) + t_n f'[0](h_n) + t_n r_n$. Since $f'[0](h_n) = f'(h_n) = f'_n(h_n) = y_n$, we have

$$g'[y_0](f'[0](h_n) + r_n) - g'[y_0](f'[0](h_n))$$
$$= g'[y_0](y_n + r_n) \quad g'[y_0](y_n).$$

The assertion (iv) now follows from (4) and (2).

We will prove the derivative-of-the-inverse theorem for maps on locally bounded spaces. We recall ([12] 15,10.(1)) that in a locally bounded space X the topology can always be obtained from a quasinorm $||\cdot||$, i.e., a map from X to $[0,\infty]$ satisfying

(i) $||x|| = 0$ if and only if $x = 0$,

(ii) $||tx|| = |t|\,||x||$ for all scalars t,

(iii) $||x + x'|| \leqslant K(||x|| + ||x'||)$ for all $x, x' \in X$, where K is independent of x and x'.

(7) *Let* X *and* Y *be locally bounded* TVS's *and let* $f : X \to Y$ *be* Σ_F*-semidifferentiable at* x_0 *in such a way that* $f'[x_0]$ *is a homeomorphism between* X *and* Y *whose inverse* $(f'[x_0])^{-1}$ *is uniformly continuous on the bounded sets,* Σ_F, *of* Y.

Then if f^{-1} *is defined on some neighborhood of* $f(x_0)$ *and continuous at* $f(x_0)$, *it is* Σ_F*-semidifferentiable at* $f(x_0)$ *and* $(f^{-1})'[f(x_0)] = (f'[x_0])^{-1}$.

Proof: $||\cdot||$ will be used ambiguously to denote a quasinorm defining the topology either on X or on Y.

By (1) the map $g(x) = (f'[x_0])^{-1}(f(x) - f(x_0))$ has the identity map on X as its Σ_F-derivative at x_0, i.e.

$$\lim_{x \to x_0} \left|\left| \frac{g(x) - (x - x_0)}{||x - x_0||} \right|\right| = 0.$$

By (iii) above there is a constant $K > 0$ such that

$$\frac{||x - x_0||}{||x - x_0||} = 1 \leqslant K\left(\left|\left|\frac{g(x) - (x - x_0)}{||x - x_0||}\right|\right| + \frac{||g(x)||}{||x - x_0||}\right)$$

for all $x \neq x_0$. It follows from this and from (8) that there exist positive numbers α, δ_1 such that $||x - x_0|| \leqslant \alpha\,||g(x)||$ when $||x - x_0|| \leqslant \delta_1$. Given $\epsilon > 0$ we choose a positive $\delta_2 \leqslant \delta_1$ such that if $||x - x_0|| < \delta_2$, we have

$$(9) \qquad ||g(x) - g(x_0)|| \leqslant (\epsilon/\alpha)\,||x - x_0|| \leqslant \epsilon\,||g(x)||.$$

The inverse of g is defined in some neighborhood of 0 in X and is continuous at 0, where $g^{-1}(0) = x_0$. So we can find a $\delta > 0$ such that if $||z|| < \delta$, then $||g^{-1}(z) - x_0|| < \delta_2$. It now follows from (9) that

$$||z - (g^{-1}(z) - g^{-1}(0))|| \leqslant \epsilon ||z||$$

when $||z|| < \delta$. Since $\epsilon > 0$ was arbitrary, we conclude that g^{-1} has the identity map as its Σ_F-derivative at 0. Using (1) again, we obtain that the map

$$f^{-1}(z) = g^{-1} \circ (f'[x_0])^{-1}(z - f(x_0))$$

is Σ_F-semidifferentiable at $z_0 = f(x_0)$ with semiderivative $(f'[x_0])^{-1}$, as asserted.

§4. Examples

A Riesz space (vector lattice) X is said to be *Dedekind σ-complete* ([13]) if every decreasing sequence which is bounded from below has an infimum. If the Riesz space X has a linear topology with respect to which the map mod (x): $x \to |x| = \sup(x, -x)$ is continuous, it is called a topological Riesz space (TRS). If, moreover, every decreasing sequence x_n with infimum converges to inf x_n in the topology, then X is said to be a *sequentially order continuous* TRS. The L^p-spaces of measurable real functions with $p \in [1, \infty]$ and, more generally, the complete Köthe spaces with absolutely continuous norm ([13], pp. 41-47, and [17]) are examples of Dedekind σ-complete topological Riesz spaces which are sequentially order continuous. For a more general class of examples, cf. [7].

(1) *Let* X *be a* TVS *and let* Y *be a Dedekind σ-complete sequentially order continuous* TRS. *If* $f : X \to Y$ *is a sequentially continuous positively homogeneous map which is subadditive,*

$$f(x + x') \leqslant f(x) + f(x') \text{ for all } x, x' \in X,$$

then f *is* Σ_H*-semidifferentiable at every point* x *in* X *and* $f'[x] : X \to Y$ *is uniformly sequentially continuous on* X *and subadditive.*

Proof: It follows from the proof of ([12] 26,4.(1)) that f is Σ_G-semidifferentiable at any $x \in X$ and $f'[x]$ is subadditive. Let $h_n \to h$ in X and $0 < t_n \to 0$ in $\mathbb{R}$ be given. We have

$$(2)\quad \frac{f(x_0 + t_n h_n) - f(x_0)}{t_n} = \frac{f(x_0 + t_n h) - f(x_0)}{t_n} + \frac{f(x_0 + t_n h_n) - f(x_0 + t_n h)}{t_n}$$

Since f is positively homogeneous and subadditive

$$-f(h - h_n) = -\frac{f(t_n(h - h_n))}{t_n} \leqslant \frac{f(x_0 + t_n h_n) - f(x_0 + t_n h)}{t_n}$$

$$\leqslant \frac{f(t_n(h_n - h))}{t_n} = f(h_n - h).$$

By the sequential continuity of f, $\lim_n f(h - h_n) = \lim_n f(h_n - h) = 0$, so we have

$$\lim_n \frac{f(x_0 + t_n h_n) - f(x_0 + t_n h)}{t_n} = 0.$$

It follows from (2) that

$$\lim_n \frac{f(x_0 + t_n h_n) - f(x_0)}{t_n} = f'[x_0](h),$$

which is equivalent to (1.2) for Σ_H (cf. (1.4), (1.6)).

Finally, a subadditive map g for which $g(0) = 0$ satisfies

$$(3)\qquad \inf(-g(x' - x), -g(x - x')) \leqslant |g(x) - g(x')| \leqslant \sup(g(x' - x), g(x - x'))$$

for all $x, x' \in X$. So if such a map is sequentially continuous at 0, it is uniformly sequentially continuous on X.

We have the following corollary.

(4) *If* C *is a sequentially open convex set in the* TVS X *which contains* 0, *then the Minkowski functional of* C,

$$m_C(x) = \inf\{\rho : x \in \rho C\},$$

is Σ_H*-semidifferentiable at all points of* X, *and for each* $x \in X$, $m_C'[x]$ *is subadditive and uniformly continuous.*

We will apply (4) to get a result which is of particular interest when X is a finite dimensional space, $X = \mathbb{R}^n$. In this case, it follows from (5) below that for any bounded convex open set C in X, there is a $\Sigma_F = \Sigma_H$-semidifferentiable homeomorphism of X which takes the boundary of C to the Euclidean (n 1)-sphere. Consequently, *the boundary of a bounded convex open set in* $\mathbb{R}^n$ *is a compact, "*Σ_F*-semidifferentiable", n 1 dimensional submanifold of* $\mathbb{R}^n$.

(5) *Let X be a normed space and let C be a bounded convex open set in X containing 0. Let* m_C *denote the Minkowski functional of C (cf. (4)). Then the positively homogeneous homeomorphisms f,g : X → X defined by f(0) = g(0) = 0 and*

$$f(x) = \frac{||x||}{m_C(x)}\, x, \text{ resp. } g(x) = f^{-1}(x) = \frac{m_C(x)}{||x||}\, x \quad (x \neq 0)$$

are Σ_H*-semidifferentiable at all points* $x \in X$. *Also, for each* $x \in X$, *the maps* $f'[x]$ *and* $g'[x]$ *are uniformly continuous on X.*

Proof: There exist positive numbers m, M such that $m\,||x|| \leq m_C(x) \leq M\,||x||$; so the maps f and g are bounded, which implies they are Σ_H-semidifferentiable at 0 by (1.15) and, clearly, $f'[0] = f$ and $g'[0] = g$. Since $||\cdot||$ and m_C are everywhere Σ_H-semidifferentiable, by (4), we have for $x \neq 0$

$$f'[x](h) = \frac{||x||}{m_C(x)}\, h + \frac{||\cdot||'[x](h)\cdot m_C(x) - ||x||\, m_C'[x](h)}{(m_C(x))^2}\, x.$$

Since $||\cdot||'[x]$ and $m_C'[x]$ are uniformly continuous on X, it follows that the same is true of $f'[x]$ and, by analogy, also of $g'[x]$.

We turn to another corollary of (1). If X is a Riesz space and $x \in X$, then as usual x^+ and x^- denote the elements sup(x,0) resp. sup(-x,0). If X is Dedekind σ-complete, then for each $x \in X$ the Riesz projection onto the band generated by x exists ([13] 25.1) and we denote it by P_x. For the properties of such projections made use of in the proof of (6) below, cf. 24.6 and 24.7 of [13].

(6) *If X is a Dedekind σ-complete, sequentially order continuous TRS, then the map* mod : $x \to |x|$ *is* Σ_H*-semidifferentiable at each* $x \in X$ *and for any* $h \in X$ *we have*

(7) $$\text{mod}'[x](h) = (I - P_x)\,|h| + P_{x^+}h - P_{x^-}h,$$

where I *denotes the identity map. Thus,* mod′[x] *is linear if and only if* |x| *is a weak unit* (i.e. $P_x = I$).

Proof: The Σ_H-semidifferentiability of mod (x) follows from (1) and it follows as in ([12] 26,4.(1)) that for any x,h ∈ X

$$\mathrm{mod}'[x](h) = \inf_{t>0} \frac{|x+th| - |x|}{t}$$

We obtain (7) by showing that the family y_t defined for all $t > 0$ by

$$y_t = \frac{|x+th| - |x| - t(I - P_x)|h| - tP_{x^+}h + tP_{x^-}h}{t}$$

satisfies $\inf y_t = 0$.

Since $I = (I - P_x) + P_{x^+} + P_{x^-}$, we have

$$|x+th| - |x| = (I - P_x)|th| + (|x^+ + tP_{x^+}h| - x^+) + (|x^- - tP_{x^-}h| - x^-).$$

So we can write

$$\text{(8)}\quad y_t = \frac{|x^+ + tP_{x^+}h| - (x^+ + tP_{x^+}h)}{t} + \frac{|x^- - tP_{x^-}h| - (x^- - tP_{x^-}h)}{t}$$

$$= 2\left(\frac{x^+}{t} + P_{x^+}h\right)^- + 2\left(\frac{x^-}{t} - P_{x^-}h\right)^-$$

$$= 2\left(\frac{P_{h^-}x^+}{t} - P_{x^+}h^-\right)^- + 2\left(\frac{P_{h^+}x^-}{t} - P_{x^-}h^+\right)^-.$$

We now show that if u and v are any two positive elements in X, then the element z defined by

$$z = \inf_{t>0} \left(\frac{P_u v}{t} - P_v u\right)^-$$

is 0: Since

$$z = \inf_{t>0} \left(P_z P_v u - \frac{P_z P_u v}{t}\right)$$

we have

$$0 \leqslant \frac{P_z P_u v}{t} \leqslant P_z P_v u$$

for all $t > 0$. Since a Dedekind σ-complete Riesz space is Archimedean, it follows that $P_z P_u v = 0$. But if $z_1 = P_v u$ and $z_2 = P_u v$, then $P_{z_1} = P_{z_2}$ (from 24.7(i) of [13], for example).
It follows from this and from $P_z z_2 = 0$ that $P_z = P_z P_{z_1} = P_z P_{z_2} = 0$. Hence $z = 0$.

Applying this result to both terms of the final sum in (8), we get $\inf y_t = 0$. This completes the proof.

We note that the Banach lattice ℓ^∞ satisfies all the hypotheses of (6) except that it is not sequentially order continuous. If $x_0 = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$ and $h = (-1, -1, \ldots)$, then in the weak topology

$$\lim_{t \downarrow 0} \frac{|x_0 + th| - |x_0|}{t} = h.$$

However, for all $t > 0$ we have

$$\left\| \frac{|x_0 + th| - |x_0| - th}{t} \right\|_\infty = \sup_{n > t^{-1}} \left| \frac{|\frac{1}{n} - t| - \frac{1}{n} + t}{t} \right| = 2$$

so that mod (x) is not Σ_G-semidifferentiable at x_0.

We close this section by stating, without proof, some results on the norm and modulus functions in some special spaces.

Let (S,M,m) be a measure space. It is known that when $1 < p < \infty$, the norm on L^p (S,M,m) is uniformly Σ_F-differentiable and formulas for the derivative can be found on p. 351 of [12].

(9) *In* L^1 (S,M,m), *the norm has as its* Σ_H*-semiderivative*

$$\|\cdot\|'_1 [x](h) = \int_{\{x=0\}} |h| \, dm + \int_{\{x \neq 0\}} (\operatorname{sgn} x) h \, dm,$$

where $\operatorname{sgn}x(s) = x(s)/|x(s)|$ *when* $x(s) \neq 0$. *The norm is* Σ_F*-semidifferentiable at* $x \neq 0$ *if and only if* x *has the form of a finite sum*

$$(10) \qquad x = \sum_{i=1}^{n} c_i \chi_{a_i} \qquad \text{(a.e.)},$$

where the $c_i \neq 0$ *are scalars and the* a_i *are atoms for* m, *and* χ_{a_i} *denotes the indicator function of* $\{a_i\}$.

(11) *In* $L^\infty(S,M,m)$ *the* Σ_H*-semiderivative of the norm is given by*

$$\|\cdot\|_\infty'[x](h) = \lim_{n\to\infty} \text{ess. sup.}\,((\operatorname{sgn} x)\cdot h \cdot \chi_{E_n}),$$

where χ_{E_n} *denotes the indicator function of the set* $E_n = \{ s \in S : |x(s)| \geqslant \|x\|_\infty \quad 1/n \}$.

The norm $\|\cdot\|_\infty$ *is* Σ_F*-semidifferentiable at* x *if and only if* $x = \|x\|_\infty \chi_E + x'\chi_{S-E}$ *for some* $E \in M$ *and some* $x' \in L^\infty(S,M,m)$ *satisfying* $\|x'\|_\infty < \|x\|_\infty$.

Consequently,

(12) *In any space which is norm-isomorphic to the Banach space* c_0 *of null sequences* (cf. e.g. [4] p. 248) *the norm is* Σ_F*-semidifferentiable.*

Also, in c_0, *the map* mod(x) *is* Σ_F*-semidifferentiable.*

(13) *In* $L^p(S,M,m)$, *where* $p \in [1,\infty]$, *the modulus*, mod(x), *is* Σ_F*-semidifferentiable at* x *if and only if* x *has the form* (10).

§5. Generalizations

We call attention to two variations on the theme of semiderivatives. A map f from the TVS X to The TVS Y is said to be (α)*-homogeneous* for some $\alpha > 0$ if $f(th) = t^\alpha f(h)$ holds for all *positive* scalars t and all $x \in X$. We say that $f: X \to Y$ is (α)*-semidifferentiable at* x_0 *for the cover* Σ of X if there exists a sequentially continuous (α)-homogeneous map $f'[x_0]: X \to Y$ such that

$$(1) \qquad \lim_{t\downarrow 0} \frac{f(x_0 + th) - f(x_0) \quad f'[x_0](th)}{t^\alpha} = 0$$

holds uniformly on the sets of Σ.

Such semiderivatives are a special case of the approximating maps studied in [15], [16] and [8]. We mention them here because for (α)-semidifferentiable maps, analogues of (3.1), (3.3) and (3.7) hold, proofs being similar to the ones given.

However, it is shown in [9] that if f is (α)-semidifferentiable for Σ_G on the convex set C and $\alpha \neq 1$, then $f'[x] = 0$ a.e. on every interval $[x_0, x_1]$ in C. So there are no analogues to (2.1) and to the mean value theorems referred to in 1. and 3..

If in (1) we consider instead the limit as $t \to \infty$, we obtain *asymptotic* (α)*-semiderivatives* (cf. [16]), for which, again, analogues of (3.1), (3.3) and (3.7) are valid. (For the analogue of (3.7) add the hypothesis that f is bounded on bounded sets).

Also, all of these concepts can be defined *relative* to the subspace H of X, i.e. $f'[x_0]$ is defined as a map from H to Y. After obvious modifications, all of the theorems we have presented still hold for relative semiderivatives.

We mention one other result. A map $f : X \to Y$ is said to be Σ*-precompact* if for each $S \in \Sigma$, the image f(S) is precompact. It follows as in [8] (§3) and [16] that if Y is locally convex and $f : X \to Y$ is Σ-precompact, then every (asymptotic) (α)-semiderivative of f is Σ-precompact.

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