ON THE GRADUATED DERIVATIVES

By

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0. INTRODUCTION

In the present paper we start out from the concepts of linear connection, connection of second order developed in [2] by means of a pair of operators $(\mathfrak{C}, \mathfrak{D})$ and connection of order $n \geqslant 2$ -- see [3] - all of them intrinsically defined on a differentiable manifold.

Some differences between the second order connections and those of the order n, when $n = 2$, have led us to the problem of characterizing algebraically these concepts which are defined in the module of vectors on a ring, and of extending them to the case of derivations, although they are not covariant, according to all that is made in [4]. The instrument we construct to unify these ideas is the graduated extension of an additive map, inspired by [5], and the consequent notion of derivative of degree n. Our study is made following two equivalent ways, the one by derivations and the other by differentiations.

In the paragraphs 1, 2 and 3 we translate classical definitions of connections and derivatives on a manifold to the A-module M of vectors of a ring A and we deduct their main properties. Primary derivatives are the algebraic translation of the derivatives with respect to a linear connection which, in the case of being with respect to the vector zero, we call banal derivatives. Bompiani derivatives and connections belong to the transfer to the A-module M of connections of the second order defined by a pair of operators $(\mathfrak{C}, \mathfrak{D})$ introduced locally in [1] and developed in [2] afterwards. The differentiations and connections of order $n \geqslant 2$ that we define on M are the same which are treated in [3] on manifolds, from which we have excluded the axiom (D_3) inherent to the differentiable structure, as we are only interested in their algebraic behaviour.

In the paragraphs 4 and 5 the concept of graduated extension of an additive map and one of additive map of degree n are introduced. These are going to make up essential algebraic instruments for the definition in the paragraph 6 of derivative of degree n.

In the paragraph 7 we compare the derivative definition of degree n with all that we said in the above mentioned paragraphs 1, 2 and 3, and we find its definition to be equivalent to the banal derivative if $n = 0$, equivalent to the non-banal primary derivative if $n = 1$, a generalization of the Bompiani derivative if $X_1 \neq 0 \neq X_2$, and equivalent to the definition of derivative of order $n \geq 2$ if $X_i \neq 0$, $1 \leq i \leq n$. In virtue of which, of course, we deduced that the derivatives of degree n form the underlying algebraic structure of the different derivatives on a manifold from which we have started out except in the case of the Bompiani derivatives , and that the degree is the algebraic translation of the order.

1. Primary derivations

Let A be a commutative integral domain of characteristic zero.

A *vector* of A is a map $X : A \longrightarrow A$ verifying:

$$X(a + b) = X(a) + X(b) \ ; \ X(ab) = aX(b) + bX(a) \ ; \ \text{for all } a, b \in A.$$

For the usual definitions of addition, multiplication by elements of A and bracket, the vectors of A form an A-module and a Lie algebra, which we shall denote by $\mathfrak{V}(A)$. Henceforth, let M be the A-module $\mathfrak{V}(A)$.

If P and Q are two A-modules, we shall denote the A-modules of maps, additive homomorphisms and A-linear maps of P into Q by $\mathcal{A}(P, Q)$, $\mathcal{H}(P, Q)$ and $\mathcal{L}(P, Q)$, respectively.

Definition 1.1

If $X \in M$, it is said that D_X is a *primary derivative* with respect to X, when D_X is a map $M \longrightarrow M$, verifying the conditions:

$$\text{(c) } D_X(Y + Z) = D_X Y + D_X Z \ ; \ \text{(d) } D_X(aY) = aD_X Y + X(a)Y \ ; \ Y, Z \in M, a \in A.$$

Let $\mathfrak{D}(M)$ be the set of primary derivatives on M. An easy verification shows that $\mathfrak{D}(M)$ forms an A-module and a Lie algebra for the usual operations, and that $D_X + D'_Y$, aD_X and $[D_X, D'_Y]$ are, respectively, primary derivatives with respect to $X + Y$, aX and $[X, Y]$.

In particular, the difference of two derivatives with respect to a same vector X, D_X D'_X is a derivative D''_O with respect to vector zero.

Definition 1.2

A primary derivative is called *banal* when it is a derivative with respect to vector zero.

It is easy to verify that every banal derivative is an A-linear map and that the set $\mathfrak{D}_O(M)$ of banal derivatives is a submodule and a Lie subalgebra of $\mathfrak{D}(M)$.

Definition 1.3

The map $D : M \longrightarrow \mathfrak{D}(M)$, with $D : X \longrightarrow D_X$, is called *primary derivation.* If $D_{X+Y} = D_X + D_Y$, the primary derivation is said *additive.* If D is A-linear, that is, additive and $D_{aX} = aD_X$, the primary derivation is said *covariant.* A covariant primary derivation is also called a *primary connection.*

Therefore, a primary connection is defined by the following axioms: (a) $D_{X+Y}Z = D_XZ + D_YZ$; (b) $D_{aX}Y = aD_XY$; (c) $D_X(Y + Z) = D_XY + D_XZ$; (d) $D_X(aY) = aD_XY + X(a)Y$; for all $X, Y, Z \in M$, $a \in A$.

Definition 1.4

If $X \in M$, the map $\bar{D}X : M \longrightarrow M$ is called *primary differential* of X when, if $Y \in M$, then $(\bar{D}X)Y = D_YX$ is a primary derivative of X with respect to Y. The map $\bar{D} : M \longrightarrow \mathcal{L}(M, M)$, such that $X \longrightarrow \bar{D}X$, is called *primary differentiation.*

Definition 1.5

We shall call *exterior differentiation* on M the map d, that augments the degree of each exterior form by one unit, and which is defined – since $\mathfrak{T}_s^r \approx \mathcal{L}(M^r, \mathfrak{T}^s)$ – by:

(1) $da(X) = X(a)$, $a \in A$, $X \in M$.

(2) $r!d\omega_r(X_0, X_1, ..., X_r) = \sum_{i=0}^{r} (-1)^i X_i(\omega_r(X_0, ..., \hat{X}_i, ..., X_r)) +$
$+ \sum_{i<j} (-1)^{i+j} \omega_r([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_r)$, $\omega_r \in \mathfrak{T}_r$, $(X_0, ..., X_r) \in M^{r+1}$

Proposition 1.1

If D is a primary derivation on M, its primary differentiation satisfies: (C) $\bar{D}(X + Y) = \bar{D}X + \bar{D}Y$; (D) $\bar{D}(aX) = a\bar{D}X + X \otimes da$; $X, Y \in M$, $a \in A$.

If D is additive, the differentiation verifies:

(A) $(\bar{D}X)(Y + Z) = (\bar{D}X)Y + (\bar{D}X)Z$. Then we shall say $\bar{D}$ is *additive.*

If D is covariant, the differentiation verifies also:

(B) $(\bar{D}X)(aY) = a(\bar{D}X)Y$. Then we shall say $\bar{D}$ is *covariant.*

Proof

(D) is a consequence from (d): $(a\bar{D}X + X \otimes da)Y = aD_YX + da(Y)X =$ $= aD_YX + Y(a)X = D_Y(aX) = \bar{D}(aX)Y$. Analogously, (A), (B) and (C) are consequences, respectively, from (a), (b) and (c). □

Proposition 1.2

If $\bar{D}$ is a map $M \longrightarrow \mathcal{A}(M,M)$ satisfying (C) and (D), and we define the following map for all $X \in M$:

$$D_X : M \longrightarrow M$$
$$Y \longrightarrow (\bar{D}Y)X ,$$

then one deduces (c) and (d).

If $\bar{D}$ verifies (A), (respectively (B)), D_X verifies (a), (respectively (b)).

Proof

Trivial. --

Therefore, it is indifferent to define, in the first place, the primary derivative and to deduce the primary differential from it, or to define first the primary differential and to deduce the primary derivative from it.

The next proposition can be easily verified.

Proposition 1.3

We can define a primary connection, indistinctly, by means of a covariant derivation $D : M \longrightarrow \mathfrak{D}(M)$, which verifies (a), (b), (c) and (d), or by means of a covariant differentiation $\bar{D}$: $M \longrightarrow \mathcal{C}_1^1 \approx \mathcal{L}(M,M)$, which satisfies (A), (B), (C) and (D). Both are related by: $D_Y X = (\bar{D}X)Y$.

2. Bompiani derivations

Bompiani defined the notion of connection of second order in a differentiable manifold V by its components in each local chart $(U,x^1 \ldots x^n)$. Such components consist of two system of functions C^{kq}_{ijp}, D^k_{ijp} such that, for each global vector field X, the expression

$$\partial_{ij}X^k + C^{kq}_{ijp}\,\partial_q X^p + D^k_{ijp}X^p$$

defines a global tensor field of type (1,2).

Di Comite defined the notion of connection of order n in an analogous way. The conditions which must fulfill the components C,D in the intersection of two local charts are, of course, complicated. We are going to give an algebraic intrinsic version of these concepts.

Definition 2.1

Let $\mathcal{C}$ be a map between the A-module MxM and the A-module of additive homomorphisms $\mathcal{H}(M,\mathcal{T}^2)$

$$\mathcal{C}: M\times M \longrightarrow \mathcal{H}(M,\mathcal{T}^2)$$
$$(X_1,X_2) \longrightarrow \mathcal{C}_{(X_1,X_2)}$$

which verifies the axiom:

(C_3) $\mathcal{C}_{(X_1,X_2)}(aY) = a\mathcal{C}_{(X_1,X_2)}Y + X_1(a)X_2\otimes Y + X_2(a)X_1\otimes Y$; $X_1,X_2,Y\epsilon M, a\epsilon A$.

Let $\mathcal{D}$ be a map between the A-module MxM and the A-module of additive homomorphisms $\mathcal{H}(M,M)$:

$$\mathcal{D}: M\times M \longrightarrow \mathcal{H}(M,M)$$
$$(X_1,X_2) \longrightarrow \mathcal{D}_{(X_1,X_2)}$$

satisfying the axiom:

(C_4) $\mathcal{D}_{(X_1,X_2)}(aY) = a\mathcal{D}_{(X_1,X_2)}Y + c_1^1(da\otimes\mathcal{C}_{(X_1,X_2)}Y) + X_2(X_1(a))Y$; $a\epsilon A$, $X_1, X_2, Y\epsilon M$; where c_1^1 is the contraction of the first contravariant index and the first covariant index.

Then, the map $D = \mathcal{D}$ is called *Bompiani derivation* defined by the pair of operators $(\mathcal{C},\mathcal{D})$, and $D_{(X_1,X_2)}$ is said *Bompiani derivative* with respect to (X_1,X_2).

Definition 2.2

A Bompiani derivation D defined by the pair $(\mathcal{C},\mathcal{D})$ is called *biadditive* if $\mathcal{C}$ and $\mathcal{D}$ are biadditive homomorphisms. The Bompiani derivation is said *covariant* or *Bompiani connection* if it is biadditive, $\mathcal{D}$ is A-bilinear and $\mathcal{C}$ verifies the axioms:

(C_1) $\mathcal{C}_{(aX_1,X_2)}Y = a\mathcal{C}_{(X_1,X_2)}Y \quad X_2(a)X_1\otimes Y$; $a\epsilon A$, $X_1, X_2, Y\epsilon M$.

(C_2) $\mathcal{C}_{(X_1,aX_2)}Y = a\mathcal{C}_{(X_1,X_2)}Y$; $a\epsilon A$, $X_1, X_2, Y\epsilon M$.

Definition 2.3.

If $Y\epsilon M$, we shall call *Bompiani differential* of Y the following map:

$$\bar{D}Y : M\times M \longrightarrow M$$
$$(X_1,X_2) \longrightarrow \mathcal{D}_{(X_1,X_2)}Y\ ,$$

and *Bompiani differentiation* the next map:

$$\bar{D} : M \longrightarrow \mathcal{A}(M^2, M)$$
$$Y \longrightarrow \bar{D}Y.$$

If the derivation D is biadditive or covariant, the differentiation is said *biadditive* or *covariant*, respectively.

Proposition 2.1

Let ∇ be a primary derivation on the A-module M, and we define for all $X_1, X_2, Y \in M$: $\mathcal{C}_{(X_1, X_2)}Y = X_1 \otimes \nabla_{X_2} Y + X_2 \otimes \nabla_{X_1} Y - (\nabla_{X_2} X_1) \otimes Y$, $\mathcal{D}_{(X_1, X_2)}Y = \nabla_{X_2} \nabla_{X_1} Y - \nabla_{\nabla_{X_2} X_1} Y$. Then, the pair $(\mathcal{C}, \mathcal{D})$ induces a Bompiani derivation on M.

Moreover, if ∇ is additive, the Bompiani derivation is biadditive, and if ∇ is covariant, also the Bompiani derivation is covariant.

Proof

If ∇ is a primary derivation, then $\mathcal{C}_{(X_1, X_2)} \in \mathcal{H}(M, \mathcal{T}^2)$ and $\mathcal{D}_{(X_1, X_2)} \in \mathcal{H}(M, M)$, as consequence of Definition 1.1., axiom (c). The axioms (c) and (d) imply:

$$(C_3)\ \mathcal{C}_{(X_1, X_2)}(aY) = X_1 \otimes \nabla_{X_2}(aY) + X_2 \otimes \nabla_{X_1}(aY) - (\nabla_{X_2} X_1) \otimes (aY) =$$
$$= X_1 \otimes (a\nabla_{X_2} Y + X_2(a)Y) + X_2 \otimes (a\nabla_{X_1} Y + X_1(a)Y) - a(\nabla_{X_2} X_1) \otimes Y =$$
$$= a\mathcal{C}_{(X_1, X_2)} Y + X_1(a) X_2 \otimes Y + X_2(a) X_1 \otimes Y. \text{ and}$$
$$(C_4)\ \mathcal{D}_{(X_1, X_2)}(aY) = \nabla_{X_2} \nabla_{X_1}(aY) - \nabla_{\nabla_{X_2} X_1}(aY) = \nabla_{X_2}(a\nabla_{X_1} Y + X_1(a)Y)$$
$$- a\nabla_{\nabla_{X_2} X_1} Y - (\nabla_{X_2} X_1)(a)Y = a\nabla_{X_2} \nabla_{X_1} Y + X_2(a)\nabla_{X_1} Y + X_1(a)\nabla_{X_2} Y +$$
$$+ X_2(X_1(a))Y - a\nabla_{\nabla_{X_2} X_1} Y - (\nabla_{X_2} X_1)(a)Y = a\mathcal{D}_{(X_1, X_2)} Y +$$
$$+ c_1^1(da \otimes \mathcal{C}_{(X_1, X_2)} Y) + X_2(X_1(a))Y.$$

If ∇ is additive, it may be easily proved that $\mathcal{C}$ and $\mathcal{D}$ are biadditive maps because of (a) and (c).

If ∇ is covariant, $\mathcal{D}$ is A-bilinear and (C_1), (C_2) are verified because:

$$\mathcal{D}_{(aX_1, X_2)} Y = \nabla_{X_2} \nabla_{aX_1} Y - \nabla_{\nabla_{X_2}(aX_1)} Y = \nabla_{X_2}(a\nabla_{X_1} Y) -$$
$$- \nabla_{a\nabla_{X_2} X_1 + X_2(a) X_1} Y = a\nabla_{X_2} \nabla_{X_1} Y + X_2(a)\nabla_{X_1} Y -$$

$$- \nabla_{a\nabla_{X_2} X_1} Y - \nabla_{X_2(a) X_1} Y = a\nabla_{X_2}\nabla_{X_1} Y + X_2(a)\nabla_{X_1} Y -$$

$$a\nabla_{\nabla_{X_2} X_1} Y - X_2(a)\nabla_{X_1} Y = a\mathfrak{D}_{(X_1, \dot{X}_2)} Y.$$

$$\mathfrak{D}_{(X_1, aX_2)} Y = \nabla_{aX_2}\nabla_{X_1} Y \quad \nabla_{\nabla_{aX_2} X_1} Y = a\nabla_{X_2}\nabla_{X_1} Y$$

$$- \nabla_{a\nabla_{X_2} X_1} Y = a\mathfrak{D}_{(X_1, X_2)} Y.$$

$$(C_1)\ \mathcal{C}_{(aX_1, X_2)} Y = (aX_1) \otimes \nabla_{X_2} Y + X_2 \otimes \nabla_{aX_1} Y \quad (\nabla_{X_2}(aX_1)) \otimes Y =$$

$$= aX_1 \otimes \nabla_{X_2} Y + aX_2 \otimes \nabla_{X_1} Y \quad (a\nabla_{X_2} X_1 + X_2(a) X_1) \otimes Y =$$

$$= a\mathcal{C}_{(X_1, X_2)} Y \quad X_2(a) X_1 \otimes Y.$$

$$(C_2)\ \mathcal{C}_{(X_1, aX_2)} Y = X_1 \otimes \nabla_{aX_2} Y + (aX_2) \otimes \nabla_{X_1} Y \quad (\nabla_{aX_2} X_1) \otimes Y =$$

$$- a\mathcal{C}_{(X_1, \dot{X}_2)} Y. \ \square$$

3. DERIVATIONS OF ORDER n

Definition 3.1

Let $\{\bar{\Delta}^{mn}\}_{o \leqslant m \leqslant n}$, $n \geqslant 2$, where each $\bar{\Delta}^{mn}$ is a multiadditive map $A^m \times M \longrightarrow \mathcal{A}(M^n, M)$, verifying the conditions:

(Δ_0) $\bar{\Delta}^{mn}(a_1, \ldots, a_n, X) - X \otimes \sum_{\sigma \in S_n} (da_{\sigma(1)} \otimes \ldots \otimes da_{\sigma(n)})$, where S_n is the symmetric group of order n.

(Δ_1) $\bar{\Delta}^{mn}(a_{\sigma(1)}, \ldots, a_{\sigma(m)}, X) = \bar{\Delta}^{mn}(a_1, \ldots, a_m, X)$; $o \leqslant m \leqslant n$. $o \in S_n$.

(Δ_2) $\bar{\Delta}^{mn}(a_1, \ldots, a_m, bX) - b\,\bar{\Delta}^{mn}(a_1, \ldots, a_m, X) + \bar{\Delta}^{m+1,n}(a_1, \ldots, a_m, b, X)$;
  $o \leqslant m \leqslant n \quad 1$.

We shall say that the $(n + 1)$-tuple $\{\bar{\Delta}^{mn}\}_{o \leqslant m \leqslant n}$ defines a *differentiation of order n* $\bar{\Delta}^n = \bar{\Delta}^{on}$ on M, and that $\bar{\Delta}^n X$ is the *differential of order n* of $X \in M$.

Definition 3.2

If $\mathcal{H}(M^n, M)$ is the A-module of multiadditive maps and $\mathcal{L}(M^n, M)$ the A-module of A-multilinear maps of M^n into M, we shall say that a differentiation of order n defined by the $(n + 1)$-tuple $\{\bar{\Delta}^{mn}\}_{o \leqslant m \leqslant n}$ is *multiadditive* if $\bar{\Delta}^{mn}$ is a map $A^m \times M \longrightarrow \mathcal{H}(M^n, M)$, $o \leqslant m \leqslant n$, and it is *covariant* or a *connection of order n* if $\bar{\Delta}^{mn}$ is a map $A^m \times M \longrightarrow \mathcal{L}(M^n, M)$, $o \leqslant m \leqslant n$.

Definition 3.3

If $\overline{\Delta}^n = \overline{\Lambda}^{on}$ is a differentiation of order n defined by the (n + 1)-tuple $\{\overline{\Delta}^{mn}\}_{0\leqslant m\leqslant n}$, we shall call *derivative of order n* with respect to $(X_1, ..., X_n)\epsilon M^n$ the following map:

$$\Delta^n(X_1, ..., X_n) : M \longrightarrow M$$
$$Y \longrightarrow (\overline{\Delta}^n Y)(X_1, ..., X_n),$$

and the next map, *derivation of order n*:

$$\Lambda^n : (X_1, ..., X_n) \longrightarrow \Lambda^n(X_1, ..., X_n).$$

If the differentiation is multiadditive or covariant, we shall say that the derivation is *multiadditive* or *covariant*, respectively.

4. GRATUATED EXTENSIONS OF AN ADDITIVE MAP

Definition 4.1

Let A be a unitary commutative ring such that it has no zero divisors and its characteristic is zero. If P and Q are two A-modules and $F : P \longrightarrow Q$ is an additive map, we shall call *graduated extensions* of F the maps $F^{(n)} : A^n \times P \longrightarrow Q$, $n\epsilon N$, which are defined iteratively by: $F^{(0)} = F$, and if $n \geqslant 1$: $F^{(n)}(a_1, ..., a_n, p) = F^{(n-1)}(a_2, ..., a_n, a_1 p) - a_1 F^{(n-1)}(a_2, ..., a_n, p)$.

It is easy to verify that $F^{(n)}$ is a multiadditive map, for all $n\epsilon N$.

Proposition 4.1

$$F^{(n)}(a_1, ..., a_n, p) = \sum_{k=0}^{n} \sum_{\sigma\epsilon S_n} \frac{(-1)^{n-k}}{k!(n-k)!} a_{\sigma(k+1)} \cdots a_{\sigma(n)} F(a_{\sigma(1)} \cdots a_{\sigma(k)} p) \quad (4.1).$$

Proof

It can be seen in [6].

Corollary 4.1

$F^{(n)}$ verifies the following properties:

1) $F^{(n)}(a_1, ..., a_n, p) = F^{(n)}(a_{\tau(1)}, ..., a_{\tau(n)}, p)$, $\tau\epsilon S_n$.

2) $F^{(n)}(a_1, ..., \overset{(i)}{1}, ..., a_n, p) = 0$, $1 \leqslant i \leqslant n$.

Proof

1) is a consequence of Proposition 4.1.

Let us prove 2). It follows from 1) that we can suppose that 1 is on the first place and it is easy to prove that $F^{(n)}(1, \ldots, \hat{a}_i, \ldots, a_n, p) = 0$. □

Proposition 4.2

If F and G are two additive maps of P into Q, then:

1) $F^{(1)} = G^{(1)} \Leftrightarrow F - G$ is A-linear.
2) $(F + G)^{(n)} = F^{(n)} + G^{(n)}$.
3) $(aF)^{(n)} = aF^{(n)}$.

Proof

1) If $F^{(1)} = G^{(1)}$, as F and G are additive maps, $F - G$ is additive. Because of $F(ap) - aF(p) = G(ap) - aG(p)$, $F - G$ is A-linear.

Conversely, if $F - G$ is A-linear, then $F(ap) - G(ap) = aF(p) - aG(p)$, therefore $F^{(1)} = G^{(1)}$.

If we develop $(F + G)^{(n)}(a_1, \ldots, a_n, p)$ and $(aF)^{(n)}(a_1, \ldots, a_n, p)$ according to (4.1), 2) and 3) are easily proved. □

5. ADDITIVE MAPS OF DEGREE n

Definition 5.1

An additive map $F : P \longrightarrow Q$ is said to be *of degree n* when $F^{(n)} \neq 0$ and $F^{(n+1)} = 0$.

Proposition 5.1

If F is an additive map of degree n, then $F^{(r)} \neq 0$ for any $r \leqslant n$ and $F^{(s)} = 0$ for any $s > n$.

Proof

$F^{(n+1)}(a_1, \ldots, a_{n+1}, p) = F^{(n)}(a_2, \ldots, a_{n+1}, a_1 p) - a_1 F^{(n)}(a_2, \ldots, a_{n+1}, p) =$
$= 0$, and analogously $F^{(s)} = 0$ if $s > n$.

$F^{(r)} \neq 0$ for any $r \leqslant n$, since if $F^{(r)} = 0$ with $r < n$, then $F^{(r+1)} = 0$. □

As an application of Proposition 4.2, we obtain:

Proposition 5.2

1) Every additive map of degree zero is a non-zero A-linear map, and conversely.

2) If F and G are two additive maps of degree n, then F + G is an additive map of degree $\leq$ n.

3) If F is an additive map of degree n and $a \in A - \{0\}$, then aF is an additive map of degree n.

Proof

1) If F is of degree zero, then $F^{(1)}(a, p) = F(ap) - aF(p) = 0$, therefore F is A-linear, and as $F^{(0)} = F$, F is non-zero. Conversely, if F is A-linear and $F \neq 0$, then $F^{(0)} \neq 0$ and $F(ap) = aF(p)$, therefore $F^{(1)}(a, p) = 0$; that is, F is of degree zero.

2) If F and G are two additive maps of degree n, F + G is additive and $(F + G)^{(n+1)} = F^{(n+1)} + G^{(n+1)} = 0$. As $(F + G)^{(n)} = F^{(n)} + G^{(n)}$, if $F^{(n)} = -G^{(n)}$, then F + G is of a less degree than n, and if $F^{(n)} \neq -G^{(n)}$, then it is of degree n.

3) The proof is trivial because $(aF)^{(n)} = aF^{(n)}$ and A has not zero divisors. □

6. DERIVATIVES OF DEGREE n

Again let M be the A-module of the vectors of A and $(X_1, \ldots, X_n) \in M^n$.

Definition 6.1

If $n \geq 1$, any additive map of degree n $D_{(X_1, \ldots, X_n)}: M \longrightarrow M$, such that:

$$D_{(X_1, \ldots, X_n)}^{(n)}(a_1, \ldots, a_n, Y) = \sum_{\sigma \in S_n} X_1(a_{\sigma(1)}) \ldots X_n(a_{\sigma(n)})Y \qquad (6.1),$$

is called a *derivative of degree n* with respect to $(X_1, \ldots, X_n)$.

Remarks

1) It is equivalent to define a derivative of degree n as an additive map $D_{(X_1, \ldots, X_n)}: M \longrightarrow M$ verifying $D_{(X_1, \ldots, X_n)}^{(n)} \neq 0$ and the relation (6.1).

In fact, then $D_{(X_1, \ldots, X_n)}^{(n+1)}(a_0, a_1, \ldots, a_n, Y) =$

$$= D_{(X_1, \ldots, X_n)}^{(n)}(a_1, \ldots, a_n, a_0Y) - a_0 D_{(X_1, \ldots, X_n)}^{(n)}(a_1, \ldots, a_n, Y) =$$

$$= \sum_{\sigma \in S_n} X_1(a_{\sigma(1)}) \ldots X_n(a_{\sigma(n)})a_0Y - a_0 \sum_{\sigma \in S_n} X_1(a_{\sigma(1)}) \ldots X_n(a_{\sigma(n)})Y = 0;$$

that is, $D_{(X_1, \ldots, X_n)}$ is of degree n.

2) If $D(X_1, ..., X_n)$ is a derivative of degree n, then $X_i \neq 0$ for all i = 1, ..., n; since if $X_i = 0$ existed, then $D(X_1, ..., X_n)^{(n)} = 0$ because of (6.1), in contradiction to $D(X_1, ..., X_n)$ being a map of degree n.

3) By (6.1) we have that $D(X_1, ..., X_n)$ is also a derivative with respect to $(X_{\tau(1)}, ..., X_{\tau(n)})$, for all $\tau \epsilon S_n$.

Definition 6.2

We shall call every additive map of degree zero *derivative of degree zero*. We shall denote by $\mathfrak{D}^0(M)$ the set of derivatives of degree zero.

Definition 6.3

Let $\mathfrak{D}^n(M)$ be the set of derivatives of degree $n \geq 1$. The map $D : M^n \longrightarrow \mathfrak{D}^n(M)$, with $D : (X_1, ..., X_n) \longrightarrow D(X_1, ..., X_n)$ if $X_i \neq 0$, is called *derivation of degree n.* If $D(X_1, ..., X_i + X_i', ..., X_n) = D(X_1, ..., X_i, ..., X_n) + D(X_1, ..., X_i', ..., X_n)$, $1 \leq i \leq n$, the derivation of degree n is said *multiadditive.* If D is A-multilinear, that is, multiadditive and $D(X_1, ..., aX_i, ..., X_n) = aD(X_1, ..., X_i, ..., X_n)$, $1 \leq i \leq n$, the derivation of degree n is said *covariant.* A covariant derivation of degree n is also called a *connection of degree n.*

Proposition 6.1

The derivatives and derivations of degree $n \geq 1$ verify the following properties:

1) $D(X_1, ..., X_i, ..., X_n) + D'(X_1, ..., X_i', ..., X_n)$ is a derivative with respect to $(X_1, ..., X_i + X_i', ..., X_n)$.

2) If D is multiadditive, then $D(X_1, ..., X_i + X_i', ..., X_n)^{(r)} = D(X_1, ..., X_i, ..., X_n)^{(r)} + D(X_1, ..., X_i', ..., X_n)^{(r)}$, $o \leq r \leq n$.

3) If D is covariant, then $D(X_1, ..., aX_i, ..., X_n)^{(r)} = aD(X_1, ..., X_i, ..., X_n)^{(r)}$, $o \leq r \leq n$.

Proof

From $(D(X_1, ..., X_i, ..., X_n) + D'(X_1, ..., X_i', ..., X_n))^{(n)} = D(X_1, ..., X_i, ..., X_n)^{(n)} + D'(X_1, ..., X_i', ..., X_n)^{(n)}$ 1) is deduced. To prove 2) and 3) it is sufficient to express $D(X_1, ..., X_i + X_i', ..., X_n)^{(r)}$ and

$D_{(X_1, ..., aX_i, ..., X_n)}^{(r)}$, respectively, in function of $D_{(X_1, ..., X_i + X_i', ..., X_n)}$ and $D_{(X_1, ..., aX_i, ..., X_n)}$ according to (4.1). □

Definition 6.4

If $n \geq 1$ and $X \in M$, the map $\bar{D}X : M^n \longrightarrow M$ is called *differential of degree n*, when if $(Y_1, ..., Y_n) \in M^n$, $Y_i \neq 0$, then $(\bar{D}X)(Y_1, ..., Y_n) = D_{(Y_1, ..., Y_n)}X$ is a derivative of degree n of X with respect to $(Y_1, ..., Y_n)$. The map $\bar{D} : M \longrightarrow \mathcal{A}(M^n, M)$ such that $X \longrightarrow \bar{D}X$, is called *differentiation of degree n.*

Since every derivative of degree n $D_{(Y_1, ..., Y_n)}$ is an additive map, then: $\bar{D}(X + Y)(Y_1, ..., Y_n) = D_{(Y_1, ..., Y_n)}(X + Y) = (\bar{D}X)(Y_1, ..., Y_n) + (\bar{D}Y)(Y_1, ..., Y_n)$; that is, $\bar{D}(X + Y) = \bar{D}X + \bar{D}Y$; therefore the differentiation of degree n is an additive map. The following result relates the graduated extension of the differentiation and one of the derivative.

Proposition 6.2

$$(\bar{D}^{(r)}(a_1, ..., a_r, X))(Y_1, ..., Y_n) = D_{(Y_1, ..., Y_n)}^{(r)}(a_1, ..., a_r, X).$$

Proof

Apply Proposition 4.1. □

Proposition 6.3

Every differentiation of degree n is an additive map of degree n $\bar{D} : M \longrightarrow \mathcal{A}(M^n, M)$ verifying:

$$\bar{D}^{(n)}(a_1, ..., a_n, X) = X \otimes \Big(\sum_{\sigma \in S_n} da_{\sigma(1)} \otimes ... \otimes da_{\sigma(n)}\Big) \qquad (6.3).$$

Proof

$(X \otimes (\sum_{\sigma \in S_n} da_{\sigma(1)} \otimes ... \otimes da_{\sigma(n)}))(Y_1, ..., Y_n) = \sum_{\sigma \in S_n} Y_1(a_{\sigma(1)}) ... Y_n(a_{\sigma(n)})X = D_{(Y_1, ..., Y_n)}^{(n)}(a_1, ..., a_n, X) = (\bar{D}^{(n)}(a_1, ..., a_n, X))(Y_1, ..., Y_n)$; this proves the validity of (6.3).

$\bar{D}^{(n+1)}(a_0, a_1, ..., a_n, X) = \bar{D}^{(n)}(a_1, ..., a_n, a_0X) - a_0\bar{D}^{(n)}(a_1, ..., a_n, X) = 0$ because of (6.3), therefore $\bar{D}^{(n+1)} = 0$.

Finally $\bar{D}^{(n)} \neq 0$, since $D_{(Y_1, ..., Y_n)}^{(n)} \neq 0$ for all $(Y_1, ..., Y_n)$, and so there will exist $(a_1, ..., a_n, X)$ such that $D_{(Y_1, ..., Y_n)}^{(n)}(a_1, ..., a_n, X) \neq 0$; that is, $(\bar{D}^{(n)}(a_1, ..., a_n, X))(Y_1, ..., Y_n) \neq 0$. □

The following results are easily proved:

Proposition 6.4

If the derivation is multiadditive, the differentiation verifies:

$$(\bar{D}X)(Y_1, \ldots, Y_i + Y_i', \ldots, Y_n) = (\bar{D}X)(Y_1, \ldots, Y_i, \ldots, Y_n) + (\bar{D}X)(Y_1, \ldots, Y_i', \ldots, Y_n) \quad (6.4)$$

Then we shall say that $\bar{D}$ is *multiadditive.*

If the derivation is covariant, the differentiation verifies also:

$$(\bar{D}X)(Y_1, \ldots, aY_i, \ldots, Y_n) = a(\bar{D}X)(Y_1, \ldots, Y_i, \ldots, Y_n) \quad (6.4')$$

Then we shall say that $\bar{D}$ is *covariant.*

Proposition 6.5

If $\bar{D} : M \longrightarrow \mathcal{A}(M^n, M)$ is an additive map of degree n verifying (6.3) and we define, for all $(Y_1, \ldots, Y_n)$, a map $D_{Y_1, \ldots, Y_n} : M \longrightarrow M$ such that $D_{(Y_1, \ldots, Y_n)}X = (\bar{D}X)(Y_1, \ldots, Y_n)$, for all $X \in M$; then $D_{Y_1, \ldots, Y_n}$ is a map of degree n. Moreover, if $\bar{D}$ satisfies (6.4), then D is multiadditive, and if $\bar{D}$ verifies (6.4′), then D is covariant.

Therefore it is indifferent to define in the first place the derivative and to deduce from it the differential or to define first the differential and to deduce the derivative from it.

Proposition 6.6

We can define a connection of degree $n \geqslant 1$, indistinctly, by means of a covariant derivation of degree n $D : M^n \longrightarrow \mathfrak{D}^n(M)$ or by means of a covariant differentiation of degree n $\bar{D} : M \longrightarrow \mathcal{C}_n^1 \approx \mathcal{L}(M^n, M)$. Both are related by: $D_{(X_1, \ldots, X_n)}Y = (\bar{D}Y)(X_1, \ldots, X_n)$.

7. Comparison between the derivatives of degree n and other derivatives

Proposition 7.1

The definitions of derivative of degree zero and of non-zero banal derivative are equivalent.

Proof

If D_0 is a non-zero banal derivative, it is an additive map of M into M, and $D_0 = D_0^{(0)} \neq 0$. As D_0 is A-linear, $D_0^{(1)}(a, X) = D_0(aX) - aD_0X = 0$ for all $(a, X) \in A \times M$. Hence D_0 is a derivative of degree zero. Analogously, the converse can be proved easily.

Consequently, we have the following relation between the set $\mathfrak{D}^0(M)$ of derivatives of degree zero and the set $\mathfrak{D}_0(M)$ of banal derivatives: $\mathfrak{D}^0(M) = \mathfrak{D}_0(M) - \{0\}$. □

Proposition 7.2

The definition of derivative of degree one and of non-banal primary derivative are equivalent.

Proof

If D_X is a derivative of degree one, it is an additive map, therefore it verifies the axiom (c) of the primary derivatives. According to (6.1), $D_X^{(1)}(a, Y) = X(a)Y$; consequently $D_X(aY) - aD_XY = X(a)Y$, and therefore (d) is proved. According to the remark 2) of Definition 6. 1, $X \neq 0$, therefore the primary derivative is not banal.

Conversely, if D_X is a non-banal primary derivative, it is additive because of (c), and as $D_X^{(1)}(a, Y) = D_X(aY) - aD_XY = X(a)Y$, (6.1) is satisfied. As $X \neq 0$, it exists $a \in A$ such that $X(a) \neq 0$, and if $Y \neq 0$, then $D_X^{(1)}(a, Y) = X(a)Y \neq 0$, since it exists $b \in A$ such that $Y(b) \neq 0$, and $X(a)Y(b) \neq 0$, because A has no zero divisors: therefore $D_X^{(1)} \neq 0$. Finally, $D_X^{(2)}(a_1, a_2, Y) = D_X^{(1)}(a_2, a_1 Y) - a_1 D_X^{(1)}(a_2, Y) = X(a_2)a_1 Y - a_1 X(a_2)Y = 0$, for all $(a_1, a_2, Y) \in A^2 \times M$, and consequently $D_X^{(2)} = 0$. □

Evidently, we have the next relation: $\mathfrak{D}^1(M) = \mathfrak{D}(M) - \mathfrak{D}_0(M)$.

Corollary 7.1

If $X \neq 0$, the definitions of additive and covariant primary derivations and additive and covariant derivations of degree one are equivalent.

Corollary 7.2

If $X \neq 0$, the definitions of differential, additive differentiation and covariant differentiation, all three being primary and of degree one, are equivalent.

Proposition 7.3

If $X_1 \neq 0 \neq X_2$, every Bompiani derivative is a derivative of degree two.

Proof

Let D be a Bompiani derivation defined by the pair $(\mathfrak{C}, \mathfrak{D})$. The Bompiani derivative $D_{(X_1, X_2)}: M \longrightarrow M$ verifies: $D_{(X_1, X_2)} Y = \mathfrak{D}_{(X_1, X_2)} Y \in \mathcal{H}(M, M)$, for all $Y \in M$, therefore it is additive.

From (C_4) we have: $D_{(X_1, X_2)}{}^{(1)}(a, Y) = D_{(X_1, X_2)}(aY) \quad aD_{(X_1, X_2)}Y = = \mathfrak{D}_{(X_1, X_2)}(aY) \quad a\mathfrak{D}_{(X_1, X_2)}Y = c_1^1(da \otimes \mathfrak{C}_{(X_1, X_2)}Y) + X_2(X_1(a))Y$, and because of (C_3): $D_{(X_1, X_2)}{}^{(2)}(a_1, a_2, Y) = c_1^1(da_2 \otimes \mathfrak{C}_{(X_1, X_2)}(a_1 Z)) - - a_1 c_1^1(da_2 \otimes \mathfrak{C}_{(X_1, X_2)}Y) = c_1^1(da_2 \otimes (X_1(a_1)X_2 \otimes Y + X_2(a_1)X_1 \otimes Y)) = = \sum_{\sigma \in S_2} X_1(a_{\sigma(1)})X_2(a_{\sigma(2)})Y$.

Moreover $D_{(X_1, X_2)}{}^{(2)} \neq 0$. In fact:

Let us suppose that $\nexists\, a \in A / X_1(a) \neq 0,\ X_2(a) \neq 0$. Then, as $X_1 \neq 0$, it exists $a_1 \in A$ such that $X_1(a_1) \neq 0$, therefore $X_2(a_1) = 0$. As $X_2 \neq 0$, there is $a_2 \in A$ such that $X_2(a_2) \neq 0$; consequently $X_1(a_2) = 0$. If $Y \neq 0$, we have:
$D_{(X_1, X_2)}{}^{(2)}(a_1, a_2, Y) = X_1(a_1)X_2(a_2)Y \neq 0$.

If, on the contrary, $\exists\, a \in A / X_1(a) \neq 0,\ X_2(a) \neq 0$, let us take $a_1 = a_2 = a$, $Y \neq 0$. We obtain: $D_{(X_1, X_2)}{}^{(2)}(a_1, a_2, Y) = 2X_1(a)X_2(a)Y \neq 0$. According to the remark 1) of Definition 6.1, we have that $D_{(X_1, X_2)}{}^{(3)} = 0$, therefore $D_{(X_1, X_2)}$ is a map of degree two. □

Corollary 7.3

If $X_1 \neq 0 \neq X_2$, every Bompiani biadditive derivation (respectively, Bompiani covariant derivation) is a biadditive derivation (respectively, a covariant derivation) of degree two.

Corollary 7.4

If $X_1 \neq 0 \neq X_2$, every differential, biadditive differentiation and covariant differentiation of Bompiani is, respectively, a differential, a biadditive differentiation and a covariant differentiation of degree two.

Proposition 7.4

If $n \geqslant 2$ and $X_i \neq 0$, $1 \leqslant i \leqslant n$, the definitions of differentiation of order n and differentiation of degree n are equivalent.

Proof

Let $\bar{\Delta}^n = \bar{\Delta}^{on}$ be a differentiation of order n defined by the multiadditive maps $\{\bar{\Delta}^{mn}\}_{o \leqslant m \leqslant n}$. Let us see that $\bar{\Delta}^n$ is a differentiation of degree n.

The map $\bar{\Delta}^n$: $M \longrightarrow \mathcal{A}(M^n, M)$ is additive, and from (Δ_2), we have: $\bar{\Delta}^{m+1,n}(a_1, ..., a_m, b, X) = \bar{\Delta}^{mn}(a_1, ..., a_m, bX) \quad b\,\bar{\Delta}^{mn}(a_1, ..., a_m, X)$. Because of (Δ_1), we obtain: $\bar{\Delta}^{m+1,n}(b, a_1, ..., a_m, X) = \bar{\Delta}^{mn}(a_1, ..., a_m, bX)$ $\cdot\ b\,\bar{\Delta}^{mn}(a_1, ..., a_m, X)$; therefore $\bar{\Delta}^{m+1,n}: A^{m+1} \times M \longrightarrow M$, $o \leqslant m \leqslant n - 1$, is the $(m + 1)$-th graduated extension of the additive man $\bar{\Delta}^n$, and by (Δ_0), (6.3) is proved.

Also $\overline{\Delta}^{nn} \neq 0$. In fact, if $X \neq 0$, $a \in A$ exists such that $X(a) \neq 0$, and consequently $\overline{\Delta}^{nn}(a, ..., a, X) \neq 0$, since $(\overline{\Delta}^{nn}(a, ...a, X))(X, ..., X) =$ $= (X\otimes(da\otimes\overset{(n)}{\ldots}da + \overset{(n!)}{\ldots} + da\otimes\overset{(n)}{\ldots}\otimes da))(X, ..., X) = n!\,(X\otimes(da\otimes\overset{(n)}{\ldots}\otimes da))$ $(X, ..., X) = n!X(a)^n X \neq 0$, because it exists $a \in A$, such that $(n!X(a)^n X)(a)$ $= n!X(a)^{n+1} \neq 0$.

Finally, $\overline{\Delta}^{n+1,n} = 0$, since we have $\overline{\Delta}^{n+1\,n}(a_0, a_1, ..., a_n, X) =$ $= \overline{\Delta}^{nn}(a_1, ..., a_n, a_0X) - a_0\overline{\Delta}^{nn}(a_1, ..., a_n, X) = 0$.

Conversely, let $\overline{D}$ be a diferentiation of degree n. As $\overline{D}: M \longrightarrow \mathcal{A}(M^n, M)$ is additive, its graduated extensions are multiadditive. Let us prove that $\overline{D} = \overline{D}^{(o)}$ is a diferentiation of order n defined by its graduated extensions $\{\overline{D}^{(m)}: A^m \times M \longrightarrow \mathcal{A}(M^n, M)\}_{o\leqslant m\leqslant n}$.

By 1) of Corollary 4. 1, (Δ_1) is satisfied, and consequently, from it and from the definition of graduated extension, we obtain: $\overline{D}^{(m+1)}(a_1, ..., a_m, b, X) =$ $= \overline{D}^{(m+1)}(b, a_1, ..., a_m, X) = \overline{D}^{(m)}(a_1, ..., a_m, bX) - b\overline{D}^{(m)}(a_1, ..., a_m, X)$, therefore (Δ_2) is proved.

Finally, (Δ_0) is verified because of (6.3). □

Proposition 7.5

If $n \geqslant 2$ and $X_i \neq 0$, $1 \leqslant i \leqslant n$, the definitions of multiadditive and covariant differentiation of order n and of degree n, are equivalent.

Proof

If $\overline{\Delta}^n = \overline{\Delta}^{0\,n}$ is multiadditive, $\overline{\Delta}^{mn}$ is a map of $A^m \times M$ into $\mathcal{H}(M^n, M)$, therefore $\overline{\Delta}^n X \in \mathcal{H}(M^n, M)$ for all $X \in M$, and consequently (6.4) is satisfied. Conversely, if $\overline{D}$ verifies (6.4), $\overline{D}X \in \mathcal{H}(M^n, M)$ for all $X \in M$, and because of Proposition 4.1 which allows to express every extension $\overline{D}^{(m)}$ in terms of $\overline{D}$, it is deduced that also $\overline{D}^{(m)}(a_1, ..., a_m, X) \in \mathcal{H}(M^n, M)$, for all $(a_1, ..., a_m, X) \in A^m \times M$.

Analogously the proposition is proved if the differentiation is covariant. □

If we repeat the same reasoning with derivatives, we obtain the following result:

Corollary 7.5

If $n \geqslant 2$ and $X_i \neq 0$, $1 \leqslant i \leqslant n$, the definitions of derivative, multiadditive derivation and covariant derivation of order n and of degree n, are equivalent.

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