BLOCK SEQUENCES OF STRONG M–BASES IN BANACH SPACES

by

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SUMMARY.

Let (x_n) be a strong M-basis of a Banach space, then:
in general the block sequences of (x_n) are not strong M-basic, also if (x_n) is uniformly minimal; moreover, if all the block sequences of (x_n) are strong M-basic, in general (x_n) is neither uniformly minimal nor basic with brackets.

§ INTRODUCCION.

B is a Banach space, (x_n) a sequence of B, $[x_n] = \overline{\text{span}}\,(x_n)$, moreover we say that (y_n) is a *block sequence* of (x_n) if there exists an increasing sequence (q_n) of natural numbers so that, setting $q_0 = 0$,

$$y_m \in \text{span}\,(x_n)_{n=q_{m-1}+1}^{q_m} \qquad \text{for every m.}$$

Some standard definitions:

Let $(x_n) \subset B$ and $(f_n) \subset B^*$ (the dual of B), (x_n, f_n) is *biorthogonal* if

$$f_m(x_n) = \begin{cases} 1 & \text{if } m=n \\ 0 & \text{if } m\neq n \end{cases}, \text{for every m and n;}$$

this is the same as saying that (x_n) is *minimal* ($x_m \notin [x_n]_{n\neq m}$, for every m).

Let (x_n, f_n) be biorthogonal with $[x_n] = B$, (x_n) is said to be

i_1) *uniformly minimal* if $(\|x_n\| \, \|f_n\|)$ is bounded (which is equivalent to $\inf_m \operatorname{dist}(x_m/\|x_m\|, [x_n]_{n \neq m}) > 0$);

i_2) *M-Basis* of B if $[f_n]$ is *total on* B, that is $[f_n]^\perp$ $(= \{ x \in B;\ f_n(x) = 0$ for every $n \}) = \{0\}$;

i_3) *strong M-basis* of B if $[x_{n_k}] = [f_{n'_k}]^\perp$, for every $(n_k) \cup (n'_k) = (n)$, $(n_k) \cap (n'_k) = \phi$;

i_4) *basis with brackets* of B if there exists an increasing sequence (q_n) of natural numbers so that, setting $q_0 = 0$,

$$x = \sum_{m=0}^{\infty} \left(\sum_{n=q_m+1}^{q_{m+1}} f_n(x)\, x_n \right) \quad \text{for every } x \text{ of } B;$$

i_5) *basis* of B if we have i_4) with $q_n = n$ for every n.

Finally (x_n) is said to be *M-basic* *(strong M-basic)* *(basic with brackets)* *(basic)* if it is M-basis (strong M-basis) (basis with brackets) (basis) of $[x_n]$.

The Note concerns the general theory of the Banach espace B, in particular the research of the best sequence wich can represent a separable B. Two famous problems in this direction were stated already in the Banach book [1] (1932): the existence of the basis and the existence of the uniformly minimal M-basis. In spite of many efforts these two problems were solved only recently; precisely the existence of the basis had a negative answer (Enflo [2] 1973) and the existence of the uniformly minimal M-basis a positive answer (Ovsepian–Pelczynski [4] 1975). After this "bracket" of results the most important open question in this direction seems to be the existence of the strong M-basis. Then it appears necessary to know the structure and the properties of the strong M-basis. These sequences have been characterized by Plans and Reyes in [5] and [6].

In a mathematical meeting at Zaragoza (november 1982) the following questions were raised by A. Plans and A. Reyes:

Question 1. Let (x_n) be strong M-basic, are all the block sequences of (x_n) strong M-basic?

Question 2. If every block sequence of (x_n) is strong M-basic, is (x_n) uniformly minimal?

We add

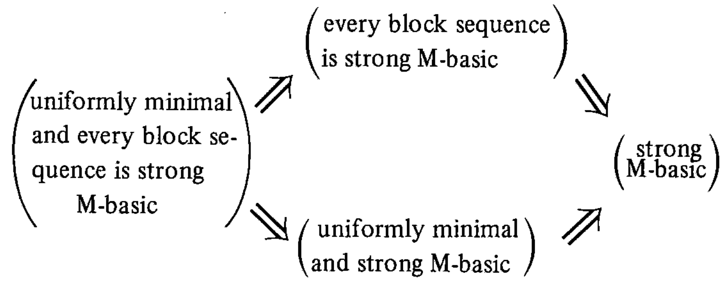
Question 3. Question 1 with the further hypothesis of (x_n) uniformly minimal.

Question 4. If (x_n) is uniformly minimal and every block sequence of (x_n) is strong M-basic, is (x_n) basic with brackets?

The Note answers these questions, precisely:

In § 1 we characterize the strong M-bases by means of two fixed complementary subsequences. After, by means of the idea of [8], in § 2 we are able to construct an example of a uniformly minimal strong M-basis $(x_n) \cup (y_n)$ such that not all the block sequences are strong M-basic (hence questions 1 and 3 have negative answers); moreover all the block sequences of (x_n) are strong M-basic, while (x_n) is not basic with bracktes (hence question 4 has a negative answer). So proceeding in §3, always by means of the idea of [8], we give an example of a sequence (x_n) which is not uniformly minimal, while all the block sequences of (x_n) are strong M-basic (hence question 2 has negative answer too).

Therefore the following implications are strict (that is the inverse implications do not hold):



§ 1. *A characterization of the strong M-bases.*

Next proposition gives a characterization of the strong M-bases by means of properties of two subsequences; this will be used in following paragraphs.

Firstly we recall ([7] p. 243)

I*. (x_n) *strong M-basic* $\Longleftrightarrow$ $(x_{n_k} + [x_{n'_i}])$ *M-basic, for every* $(n_k) \cup (n'_k)$

$$= (n), (n_k) \cap (n'_k) = \phi.$$

Proposition I. *Let* (x_n) *and* (y_n) *be sequences of B, then*

$$(x_n) \cap (y_n) \text{ is strong M-basic} \Longleftrightarrow \begin{cases} (x_n + [y_k]) & \text{is strong M-basic;} \\ (y_n + [x_{m_k}]) & \text{is strong M-basic for every } (m_k) \subseteq (n) \end{cases}$$

Proof. Let us prove $\Longrightarrow$:

It is evident, indeed let

(1) $$(m_k) \cup (m'_k) = (n) \quad , \quad (m_k) \cap (m'_k) = \phi,$$

by th.I* it is sufficient to see that $(y_n + [x_{m_k}])$ is strong M-basic.

Indeed otherwise there exist two subsequences (m''_k) and (m'''_k) of (n) so that

(2) $(m''_k) \cup (m'''_k) = (n)$; $(m''_k) \cap (m'''_k) = \phi$, $(y_{m''_n} + [(x_{m_k}) \cup (y_{m'''_k})])$ is not M-basic.

Hence by (1) and (2) it would follow that

$(y_{m''_n} + [(x_{m_k}) \cup (y_{m'''_k})] \cup (x_{m'_n} + [(x_{m_k}) \cup (y_{m'''_k})])$ is not M-basic,

therefore by th.I* $(x_n) \cup (y_n)$ would not be strong M-basic.

Let us prove $\Longleftarrow$:

Let (m_k) and (m'_k) be the sequence of (1), set

(3) $$Z = [(x_{m'_k}) \cup (y_{m'_k})]$$

(we can consider a suitable permutation of (y_n)), by th. I* it is sufficient to prove that

(4) $$(x_{m_n} + Z) \cup (y_{m_n} + Z) \text{ is M-basic.}$$

By hypothesis $(y_n + [x_{m'_k}])$ is strong M-basic; hence , by th.I* and by (3),

(5) $$(y_{m_n} + Z) \text{ is M-basic.}$$

Moreover by hypothesis $(y_n + [x_k])$ is M-basic, hence there exist $(\tilde{H}_n) \subset (B/[x_k])^*$ so that

(6) $$(y_n + [x_k] \ , \ \tilde{H}_n) \text{ is biorthogonal.}$$

Set

(7) $$h_n(x) = \tilde{H}_n (x + [x_k]) \text{ for every x of B, for every n.}$$

By (1), (6) and (7) it follows that

$$[(x_k) \cup (y_{m'_k})] \subset [h_{m_n}]^{\perp};$$

hence by (3)

(8) $$Z + [x_{m_k}] \subset [h_{m_n}]^{\perp}.$$

By (8) we can set

$$H_{m_n} (x + Z) = h_{m_n} (x) \text{ for every x of B and for every n.}$$

By (5), (6), (7) and (9) (9)

$$(y_{m_n} + Z, H_{m_n}) \text{ is biorthogonal, with } [H_{m_n}] \text{ total on } [y_{m_n} + Z]. \qquad (10)$$

Moreover, by (8) and (9),

$$[x_{m_n} + Z] \subset [H_{m_n}]^{\perp}. \qquad (11)$$

By hypothesis $(x_n + [y_k])$ is strong M–basic; hence, by th.I* and by (3), $(x_{m_n} + \overline{Z + [y_{m_k}]})$ is M–basic; that is there exists $(\tilde{F}_{m_n}) \subset (B/\overline{Z + [y_{m_k}]})^*$ so that

$$(x_{m_n} + \overline{Z + [y_{m_k}]}, \tilde{F}_{m_n}) \text{ is biorthogonal, } [\tilde{F}_{m_n}] \text{ is total on } [x_{m_n} + \overline{Z + [y_{m_k}]}]. \tag{12}$$

Set

$$F_{m_n}(x + Z) = \tilde{F}_{m_n}(x + \overline{Z + [y_{m_k}]}) \text{ for every } x \text{ of } B \text{ and for every } n. \tag{13}$$

By (12) and (13) it follows that

$$(x_{m_n} + Z, F_{m_n}) \text{ is biorthogonal, } [y_{m_n} + Z] \subset [F_{m_n}]^{\perp}. \tag{14}$$

Finally by (10), (11) and (14) $(x_{m_n} + Z, F_{m_n}) \cup (y_{m_n} + Z, H_{m_n})$ is biorthogonal; hence let $\bar{x} \in B$ so that

$$\bar{x} \in [(x_n) \cup (y_n)] \;,\; F_{m_n}(\bar{x} + Z) = H_{m_n}(\bar{x} + Z) = 0 \text{ for every } n; \tag{15}$$

in order to have (4) it is sufficient to prove that

$$\bar{x} \in Z.$$

By (15) $\bar{x} + \overline{Z + [y_{m_k}]} \in [x_{m_n} + \overline{Z + [y_{m_k}]}]$; moreover by (13) and (15) $\tilde{F}_{m_n}(\bar{x} + \overline{Z + [y_{m_k}]}) = 0$ for every n; hence by (12) $\bar{x} \in \overline{Z + [y_{m_k}]}$; that is $\bar{x} + Z \in [y_{m_n} + Z]$; on the other hand by (15) $H_{m_n}(\bar{x} + Z) = 0$ for every n, hence by (10) $\bar{x} \in Z$; which completes the proof of prop. I.

§ 2. BLOCK SEQUENCES OF UNIFORMLY MINIMAL STRONG M–BASES.

Next proposition answers question 3 (hence question 1) and question 4.

Proposition II. *There exists a Banach space B_o with two sequences (x_n) and (y_n) so that*

(i) *$(x_n) \cup (y_n)$ is uniformly minimal strong M–basis of B_o;*
(ii) *not all the block sequences of $(x_n) \cup (y_n)$ are strong M–basic;*
(iii) *all the block sequences of (x_n) are strong M–basic but (x_n) is not basic with brackets.*

Proof: Let $(x_n) \cup (z_n)$ be a linearly independent sequence of vectors of a linear space. Set

$$r_o = 0; r_1 = 1, r_2 = 2; r_{2m+1} = r_{2m}(2^m + 1), r_{2(m+1)} = r_{2m+1} + 2^m r_{2m}$$

$$\text{for every } m \geqslant 1. \tag{16}$$

Firstly we define a norm on a particular subspace of span (x_n):

$$u_p = \sum_{m=1}^{p-1} \frac{1}{2^{m-1}} \sum_{n=r_{2(m-1)}+1}^{r_{2m}} x_n + \frac{1}{2^{p-1}} \left(\sum_{n=r_{2(p-1)}+1}^{r_{2p-1}} x_n - \sum_{n=r_{2p-1}+1}^{r_{2p}} x_n \right)$$

$$\text{for every } p \geqslant 1; \tag{17}$$

$$\left\| \sum_{n=1}^{m} a_n x_n \right\| = \max\left\{ |a_n|; 1 \leqslant n \leqslant m \right\} \text{ for every } \sum_{n=1}^{m} a_n x_n \in \text{span}(u_n).$$

Now we define the norm on span (x_n):

$$I\{x, u\} = \sum_{n=1}^{m} |a_n| + \|u\| \text{ for } x \in \text{span}(x_n),\ u \in \text{span}(u_n),$$

$$x - u = \sum_{n=1}^{m} a_n x_n; \tag{18}$$

$$\|x\| = \inf\left\{ I\{x, u\};\ u \in \text{span}(u_n) \right\} \text{ for every } x \text{ of span}(x_n).$$

Then we define the norm on span (x_n) + span (z_n) and we consider the completion:

$$\|x + \sum_{n=1}^{m} a_n z_n\| = \|x\| + \max\{|a_n|; 1 \leqslant n \leqslant m\} \text{ for every } x \text{ of span } (x_n)$$

$$\text{and for every } (a_n)_{n=1}^{m}; \tag{19}$$

$$B_o = \text{completion of span } (x_n) + \text{span } (z_n).$$

Finally choose the sequence (y_n):

$$y_n = x_n + z_n \text{ for every } n. \tag{20}$$

We pass to prove that (x_n) and (y_n) satisfy thesis.

We affirm that

$$\|\sum_{n=1}^{m} a_n x_n\| \geqslant \max\{|a_n|; 1 \leqslant n \leqslant m\}, \text{ for every } (a_n)_{n=1}^{m}. \tag{21}$$

Indeed, setting $a_n = 0$ for $m + 1 \leqslant n \leqslant r_{2m}$, by (16), (17) and (18) it follows that

$$\|\sum_{n=1}^{m} a_n x_n\| = \|\sum_{n=1}^{r_{2m}} a_n x_n\| = \inf\left\{\sum_{n=1}^{r_{2m}} |a_n - b_n| + \sum_{n=r_{2m}+1}^{r_{2(m+p)}} |b_n| + \max\left\{|b_n|; 1 \leqslant n \leqslant r_{2(m+p)}\right\}; \sum_{n=1}^{r_{2(m+p)}} b_n x_n \in \text{span } (u_n)\right\} = \min\left\{\sum_{n=1}^{r_{2m}} |a_n - b_n| + \sum_{n=r_{2m}+1}^{r_{2(m+1)}} |b_n| + \max\left\{|b_n|; 1 \leqslant n \leqslant r_{2(m+1)}\right\};\right.$$

$$\sum_{n=1}^{r_{2(m+1)}} b_n x_n \in \operatorname{span}(u_n)_{n=1}^{m+1} \Big\} .$$

On the other hand

$$\sum_{n=1}^{r_{2m}} |a_n - b_n| + \sum_{n=r_{2m}+1}^{r_{2(m+1)}} |b_n| + \max\left\{|b_n|; 1 \leqslant n \leqslant r_{2(m+1)}\right\}$$

$$\geqslant \sum_{n=1}^{r_{2m}} |a_n - b_n| + \max\left\{|b_n|; 1 \leqslant n \leqslant r_{2m}\right\} \geqslant \max_{1 \leqslant n \leqslant r_{2m}}$$

$$|a_n - b_n| + \max\left\{|b_n|; 1 \leqslant n \leqslant r_{2m}\right\} \geqslant \max\left\{|a_n - b_n| + |b_n|;\right.$$

$$\left. 1 \leqslant n \leqslant r_{2m}\right\} \geqslant \max\left\{|a_n|; 1 \leqslant n \leqslant r_{2m}\right\} = \max\left\{|a_n|;\right.$$

$$\left. 1 \leqslant n \leqslant m\right\},$$

which completes proof of (21).

By (17), (18), (19) and (20) if follows that

$$\| x_n \| = 1 \text{ and } \| y_n \| = 2 \text{ for every } n. \tag{22}$$

Moreover, for every $(a_n)_{n=1}^{p} \cup (b_n)_{n=1}^{p}$ of numbers and for every m, by (19), (20) and (21) it follows that

$$\left\| x_m + \sum_{n=1, n \neq m}^{p} a_n x_n + \sum_{n=1}^{p} b_n y_n \right\| =$$

$$\left\| x_m (1 + b_m) + \sum_{n=1, n \neq m}^{p} (a_n + b_n) x_n + \sum_{n=1}^{p} b_n z_n \right\| =$$

$$\left\| x_m (1 + b_m) + \sum_{n=1, n\neq m}^{p} (a_n + b_n) x_n \right\| + \left\| \sum_{n=1}^{p} b_n z_n \right\| \geqslant |1 + b_m| + \max \{ |b_n| ; 1 \leqslant n \leqslant p \} \geqslant$$

$$|1 + b_m| + |b_m| \geqslant 1 ;$$

$$\left\| y_m + \sum_{n=1}^{p} a_n x_n + \sum_{n=1, n\neq m}^{p} b_n y_n \right\| =$$

$$\left\| z_m + x_m (1 + a_m) + \sum_{n=1, n\neq m}^{p} (a_n + b_n) x_n + \sum_{n=1, n\neq m}^{p} b_n z_n \right\| = \left\| x_m (i + a_m) + \sum_{n=1, n\neq m}^{p} (a_n + b_n) x_n \right\| + \left\| z_m + \sum_{n=1, n\neq m}^{p} b_n z_n \right\| \geqslant 1.$$

Hence by (22) there exist (f_n) and (h_n) in B* so that

$$\text{(since } \|y_m - x_m\| = 1) \tag{23}$$

$(x_n, f_n) \cup (y_n, h_n)$ is biorthogonal, $\| f_n \| = 1$ and $\| h_n \| = 4$ for every n.

In what follows we use a known characterization of the M-basic sequences ([3]; see also [7] p. 225 Rem. 8.3) :

$$(w_n) \text{ minimal and } \cap_{m=1}^{\infty} [w_n]_{n \geqslant m} = \{0\} \Longleftrightarrow (w_n) \text{ M-basic.} \tag{24}$$

Fix (n_k), (m_k) and (m_k') so that

$$(n_k) \subseteq (n)\ ,\ (m_k) \cup (m'_k) = (n)\ ,\ (m_k) \cap (m'_k) = \emptyset\ , \qquad (25)$$

We affirm that

$$\cap_{p=1}^{\infty}\ [x_{m_k} + [(x_{m'_i}) \cup (y_{n_i})]]_{k \geqslant p} = \{0\}\ . \qquad (26)$$

Fix $\bar{p}$.

By (25) there exist $\bar{s}, \bar{q}, p', p'', p''', q', q'', q'''$ so that

$$r_{2(\bar{s}-1)} < m_{\bar{p}} \leqslant r_{2\bar{s}}\ ;\ m_{\bar{q}} \leqslant r_{2(\bar{s}+1)} < m_{\bar{q}+1}\ ; \qquad (27)$$

$$(n)_{n=1}^{r_{2\bar{s}}} = (m'_k)_{k=1}^{p'} \cup (m''_k)_{k=1}^{p''} \cup (m'''_k)_{k=1}^{p'''}\ ,\ (n)_{n=r_{2\bar{s}}+1}^{r_{2(\bar{s}+1)}} =$$

$$(m'_k)_{k=p'+1}^{q} \cup (m''_k)_{k=p''+1}^{q''} \cup (m'''_k)_{k=p'''+1}^{q'''}\ ;$$

$$(m''_k)_{k=1}^{q''} \cup (m'''_k)_{k=1}^{q'''} = (m_k)_{k=1}^{\bar{q}}\ ,\ (m''_k)_{k=1}^{q''} = (n_k) \cap (m_k)_{k=1}^{\bar{q}}.$$

Fix $(\bar{a}_{m_k})_{k=1}^{\bar{p}}$.

For every $(b_{m'_k})_{k=1}^{q'} \cup (c_{m''_k})_{k=1}^{q''} \cup (d_k)_{k=1}^{r_{2(\bar{s}+1)}}$ of numbers, with

$$\sum_{k=1}^{r_{2(\bar{s}+1)}} d_k x_k \in \operatorname{span}(u_n)\ ,$$

by (17), (18) and (27) it follows that

$$\mathrm{I}\left\{ \sum_{k=1}^{\bar{p}} \bar{a}_{m_k} x_{m_k} + \sum_{k=1}^{q} b_{m'_k} x_{m'_k} + \sum_{k=1}^{q''} c_{m''_k} x_{m''_k}\ ,\right.$$

$$\left. \sum_{k=1}^{r_{2(\bar{s}+1)}} d_k x_k \right\} = \sum_{k=1}^{p'''} \left| \bar{a}_{m'''_k} - d_{m'''_k} \right| +$$

$$\sum_{k=1}^{p''} \left| \bar{a}_{m_k''} + c_{m_k''} + d_{m_k''} \right| + \sum_{k=1}^{q'} \left| b_{m_k'} - d_{m_k'} \right| +$$

$$\sum_{k=p'''+1}^{q'''} \left| d_{m_k'''} \right| + \sum_{k=p''+1}^{q''} \left| c_{m_k''} - d_{m_k''} \right| +$$

$$\max \left\{ |d_k| \; ; \; 1 \leqslant k \leqslant r_{2(\bar{s}+1)} \right\}.$$

Hence by (17), (18), (19), (20) and (27) it is possible to verify that

$$\left\| \sum_{k=1}^{\bar{p}} \bar{a}_{m_k} x_{m_k} + [(x_{m_i}) \cup (y_{n_i})] \right\| = \left\| \sum_{k=1}^{\bar{p}} \bar{a}_{m_k} x_{m_k} + \right.$$

$$\left. [x_{m_i'}]_{i=1}^{q'} + [y_{m_i''}]_{i=1}^{q''} \right\|.$$

Consequently there exists $(\bar{b}_{m_k'})_{k=1}^{q} \cup (\bar{c}_{m_k''})_{k=1}^{q''} \cup (\bar{d}_k)_{k=1}^{r_{2(\bar{s}+1)}}$ of numbers so that

$$\left\| \sum_{k=1}^{\bar{p}} \bar{a}_{m_k} x_{m_k} + [(x_{m_i}) \cup (y_{n_i})] \right\| = \sum_{k=1}^{p'''} \left| \bar{a}_{m_k'''} - \bar{d}_{m_k'''} \right| +$$

$$\sum_{k=1}^{p''} \left| \bar{a}_{m_k''} + \bar{c}_{m_k''} + \bar{d}_{m_k''} \right| + \sum_{k=1}^{q'} \left| \bar{b}_{m_k'} - \bar{d}_{m_k'} \right| +$$

$$\sum_{k=p'''+1}^{q'''} \left| \bar{d}_{m_k'''} \right| + \sum_{k=p''+1}^{q''} \left| \bar{c}_{m_k''} - \bar{d}_{m_k''} \right| + \max\left\{ \left| \bar{d}_k \right| \; ;\right.$$

$$1 \leqslant k \leqslant r_{2(\bar{s}+1)} \Big\} + \max\Big\{ \big| \bar{c}_{m_k''} \big| \ ; 1 \leqslant k \leqslant q'' \Big\}.$$

Therefore it is easy to see, by (27), that

$$\Big\| \sum_{k=1}^{\bar{p}} \bar{a}_{m_k} x_{m_k} + x + [(x_{m_i'}) \cup (y_{n_i})] \Big\| \geqslant$$

$$\Big\| \sum_{k=1}^{\bar{p}} \bar{a}_{m_k} x_{m_k} + [(x_{m_i'}) \cup (y_{n_i})] \Big\| ,$$

for every x of $[x_{m_k}]_{k > \bar{q}}$.

That is we have (26); hence, by (23) and (24), it follows that

$$(x_{m_k} + [(x_{m_i'}) \cup (y_{n_i})]) \text{ is M-basic ;}$$

therefore, by th. I*,

$$(x_n + [y_{n_i}]) \text{ is strong M-basic.} \tag{28}$$

On the other hand, by (19) and (20), for every $(a_n)_{n=1}^{m}$ of numbers,

$$\Big\| \sum_{n=1}^{m} a_n y_n + [x_k] \Big\| = \Big\| \sum_{n=1}^{m} a_n z_n + \sum_{n=1}^{m} a_n x_n + [x_k] \Big\| =$$

$$\Big\| \sum_{n=1}^{m} a_n z_n + [x_k] \Big\| = \Big\| \sum_{n=1}^{m} a_n z_n \Big\| = \max\{|a_n| \ ; 1 \leqslant n \leqslant m\} ;$$

that is $(y_n + [x_k])$ is equivalent to the natural basis of c_o, hence is strong M-basic; therefore, by (28) and by prop. I, $(x_n) \cup (y_n)$ is strong M-basic ; which, by (23), completes proof of (i).

Moreover by (17) it follows that

$$u_{p+q} - u_p = \frac{1}{2^{p-2}} \sum_{n=r_{2p-1}+1}^{r_{2p}} x_n + \sum_{m=p+1}^{p+q-1} \frac{1}{2^{m-1}} \sum_{n=r_{2(m-1)}}^{r_{2m}} x_n + \frac{1}{2^{p+q-1}} \left(\sum_{n=r_{2(p+q-1)}+1}^{r_{2(p+q)-1}} x_n - \sum_{n=r_{2(p+q)-1}+1}^{r_{2(p+q)}} x_n \right), \text{ for every p and q.}$$

That is, by (17),

$$\left\| u_{p+q} - u_p \right\| = \frac{1}{2^{p-2}} \quad \text{for every p and q;}$$

hence (u_p) is Cauchy sequence, that is there exists $\bar{u}$ of B_0 so that

$$\bar{u} = \lim_{p \to \infty} u_p \, , \left\| \bar{u} \right\| = 1 \, . \tag{29}$$

Set

$$v_m = \sum_{n=r_{2(m-1)}+1}^{r_{2m}} x_n \text{ for every } m \geqslant 1, V = [v_n]. \tag{30}$$

Let us consider

$$x = \sum_{m=1}^{2p} a_m \sum_{n=r_{m-1}+1}^{r_m} x_n = \sum_{m=1}^{2p}$$

$$\sum_{n=r_{m-1}+1}^{r_m} b_n x_n ;$$

by (16), (17) and (30) it follows that

$$\sum_{m=1}^{p} \left| \sum_{n=r_{2(m-1)}+1}^{r_{2m}} b_n \right| \begin{cases} = \sum_{n=1}^{r_{2p}} | b_n | \text{ if } x \in \text{span}(v_n) , \\ < \frac{1}{2} \sum_{n=1}^{r_{2p}} | b_n | \text{ if } x \in \text{span}(u_n) \end{cases}$$

Therefore by (16), (17), (18) and (30) it is possible to verify that

$$\| v_n \| > 2^n r_{2n-2} \text{ for every } n ; \left\| \sum_{n=1}^{m} a_n v_n \right\| = \sum_{n=1}^{m} | a_n | \| v_n \|$$

for every (31)

$$(a_n)_{n=1}^{m} ; \| u + V \| = \| u \| \text{ for every } u \text{ of } [u_n].$$

By (17), (23), (29) and (30) we have that

$$\bar{u} = \lim_{p \to \infty} \Big(\sum_{m=1}^{p-1} \sum_{n=r_{2(m-1)}+1}^{r_{2m}} f_n(\bar{u}) x_n +$$

$$\sum_{n=r_{2(p-1)}+1}^{r_{2p}} \bar{a}_n x_n \Big) = \lim_{p \to \infty} \Big(\sum_{m=1}^{p-1} \frac{1}{2^{m-1}} v_m +$$

$$\sum_{n=r_{2(p-1)}+1}^{r_{2p}} \bar{a}_n x_n \;),$$

$$\text{with } \bar{a}_n = \begin{cases} \dfrac{1}{2^{p-1}} & \text{for } r_{2(p-1)}+1 \leqslant n \leqslant r_{2p-1}, \\ \\ -\dfrac{1}{2^{p-1}} & \text{for } r_{2p-1}+1 \leqslant n \leqslant r_{2p}. \end{cases}$$

That is, by (17), (18), (29) and (31), $\bar{u}$ is not representable by a series with brackets by means of the elements of (x_n) ; hence 2^{nd} part of (iii) is proved. Moreover, for every p, by (17), (20) and (30) we have that

$$\sum_{n=r_{2(p-1)}+1}^{r_{2p-1}} \frac{y_n}{2^{p-1}} - \sum_{n=r_{2p-1}+1}^{r_{2p}} \frac{y_n}{2^{p-1}} + V =$$

$$(u_p + V) + (\sum_{n=r_{2(p-1)}+1}^{r_{2p-1}} \frac{z_n}{2^{p-1}} - \sum_{n=r_{2p-1}+1}^{r_{2p}} \frac{z_n}{2^{p-1}} + V \;) \;;$$

hence by (17), (19), (29) and (31),

$$\Bigg\| (\sum_{n=r_{2(p-1)}+1}^{r_{2p-1}} \frac{y_n}{2^{p-1}} - \sum_{n=r_{2p-1}+1}^{r_{2p}} \frac{y_n}{2^{p-1}} + V) -$$

$$(\bar{u} + V) \Bigg\| = \Bigg\| (u_p - \bar{u}) + V \Bigg\| + \Bigg\| \sum_{n=r_{2(p-1)}+1}^{r_{2p-1}} \frac{z_n}{2^{p-1}} -$$

$$\left\| \sum_{n=r_{2p-1}+1}^{r_{2p}} \frac{z_n}{2^{p-1}} \right\| = \left\| (u_p - \bar{u}) + V \right\| + \frac{1}{2^{p-1}} =$$

$$\left\| u_p - \bar{u} \right\| + \frac{1}{2^{p-1}} = \frac{1}{2^{p-2}} + \frac{1}{2^{p-1}} .$$

On the other hand by (29) and (31) $\| \bar{u} + V \| = 1$, hence by (24) $(y_n + V)$ is not M-basic; therefore, by (30) and by th. I*, $(y_n) \cup (v_n)$ is a block sequence of $(y_n) \cup (x_n)$ which is not strong M-basic; which proves (ii).

Let (w_n) be a block sequence of (x_n), that is there exists an increasing sequence (t_n) of natural numbers so that, setting $t_o = 0$,

$$w_m = \sum_{k=t_{m-1}+1}^{t_m} a_k' x_k \ , \text{ for every m.} \tag{32}$$

Let

$$(m_k) \cup (m_k') = (n), (m_k) \cap (m_k') = \emptyset, (x_{n_k}) =$$

$$\cup_{k=1}^{\infty} (x_i)_{i=t_{m_k-1}+1}^{t_{m_k}}, W = [w_{m_k'}]. \tag{33}$$

In order to complete proof of (iii) it is sufficient, by th. I* and by (32) and (33), to prove that $(x_{n_k} + W)$ is M-basic, since a block sequence of an M-basic sequence is M-basic too.

We shall proceed as for proof of (26).

Fix $(\bar{a}_{n_k})_{k=1}^{\bar{p}}$.

By (32) and (33) there exist numbers $\bar{s}$, p', $\bar{q}$ and $\bar{m}$ so that

$$r_{2(\bar{s}-1)} < n_{\bar{p}} \leqslant r_{2\bar{s}}, n_{p'-1} < r_{2(\bar{s}+1)} \leqslant n_{p'},$$

$$t_{m_{\bar{q}-1}'} < r_{2(\bar{s}+1)} \leqslant t_{m_{\bar{q}}'}\ ,\ \bar{m} = \max\left\{n_{p'}\ ,\ t_{m_{\bar{q}}'}\right\}. \tag{34}$$

Then for every $(b_k)_{k=1}^{\bar{q}} \cup (c_k)_{k=1}^{\bar{m}}$ of numbers, with

$$\sum_{k=1}^{\bar{m}} c_k\, x_k \in \operatorname{span}(u_n),$$

by (17), (18), (32), (33) and (34) it follows that

$$I\left\{\sum_{k=1}^{\bar{p}} \bar{a}_{n_k} x_{n_k} + \sum_{k=1}^{\bar{q}} b_k\, w_{m_k'}\ ,\ \sum_{k=1}^{\bar{m}} c_k\, x_k\right\} =$$

$$\sum_{k=1}^{\bar{p}} |\bar{a}_{n_k} - c_{n_k}| + \sum_{k=\bar{p}+1}^{p} |c_{n_k}| + \sum_{k=1}^{\bar{q}}$$

$$\sum_{i=t_{m_{k-1}'}+1}^{t_{m_k'}} |b_k\, a_i' - c_i| + \max\left\{|c_k|\ ;\ 1 \leqslant k \leqslant \bar{m}\right\}$$

Hence by (17), (18), (32), (33) and (34) it is possible to verify that

$$\left\|\sum_{k=1}^{\bar{p}} \bar{a}_{n_k} x_{n_k} + W\right\| = \left\|\sum_{k=1}^{\bar{p}} \bar{a}_{n_k} x_{n_k} + [w_{m_k'}]_{k=1}^{\bar{q}}\right\|$$

Consequently there exists $(b_k')_{k=1}^{\bar{q}} \cup (c_k')_{k=1}^{\bar{m}}$ of numbers so that

$$\left\|\sum_{k=1}^{\bar{p}} \bar{a}_{n_k} x_{n_k} + W\right\| = \sum_{k=1}^{\bar{p}} \left|\bar{a}_{n_k} - c_{n_k}'\right| +$$

$$\sum_{k=\bar{p}+1}^{p} \left|c_{n_k}\right| + \sum_{k=1}^{\bar{q}} \sum_{i=t_{m_{k-1}'}+1}^{t_{m_k'}} \left|b_k'\, a_i' - c_i'\right| +$$

$$\max \left\{ |c'_k| \; ; \; 1 \leqslant k \leqslant \bar{m} \right\}.$$

Then by (34) it is easy to see that

$$\left\| \sum_{k=1}^{\bar{p}} \bar{a}_{n_k} x_{n_k} + x + W \right\| \geqslant \left\| \sum_{k=1}^{\bar{p}} \bar{a}_{n_k} x_{n_k} + W \right\| , \text{ for every}$$

$$x \text{ of } [x_{n_k}]_{k > p'} .$$

That is by (23) and (24) $(x_{n_k} + W)$ is M-basic, hence (iii) is proved. This completes the proof of prop. II.

§ 3 . STRONG M-BASES WHICH ARE NOT UNIFORMLY MINIMAL.

Next proposition answers question 2.

Proposition III. *There exists a Banach space B_1 with a sequence (x_n) so that (i) all the block sequences of (x_n) are strong M-basic; (ii) (x_n) is not uniformly minimal.*

Proof. Let (x_n) be a linearly independent sequence of vectors of a linear space and let $(r_n)_{n \geqslant 0}$ be the sequence of (16), we set

$$u_m = \sum_{n=1}^{r_{2(m-1)+1}} x_n - \sum_{n=r_{2(m-1)+1}+1}^{r_{2m}} x_n \text{ , for every } m \geqslant 1 \; ;$$

$$\left\| \sum_{n=1}^{m} a_n x_n \right\| = \sum_{n=1}^{m} \frac{|a_n|}{2^n} \quad \text{for} \quad \sum_{n=1}^{m} a_n x_n \in \operatorname{span}(u_n) . \tag{35}$$

Now we define the norm on span (x_n) :

$$I\{x,u\} = \sum_{n=1}^{m} |a_n|\left(1-\frac{1}{2^n}\right) + \|u\|, \text{for } x \in \text{span}(x_n), u \in \text{span}$$

(u_n)

$$\text{and } x - u = \sum_{n=1}^{m} a_n x_n ; \tag{36}$$

$\|x\| = \inf\left\{ I\{x, u\} ; u \in \text{span}(u_n)\right\}$, for every x of span (x_n) ; B_1 = completion of span (x_n).

Fix $(\bar{a}_n)_{n=1}^{m}$ of numbers and let $(b_n)_{n=1}^{m+p}$ be another sequence of numbers such that

$$\sum_{n=1}^{m+p} b_n x_n \in \text{span}(u_n) ;$$

then by (35) and (36) it follows that

$$I\left\{\sum_{n=1}^{m} \bar{a}_n x_n , \sum_{n=1}^{m+p} b_n x_n\right\} = \sum_{n=1}^{m} |\bar{a}_n - b_n|\left(1-\frac{1}{2^n}\right) +$$

$$\sum_{n=1}^{m} \frac{|b_n|}{2^n} + \sum_{n=m+1}^{m+p} |b_n|\left(1-\frac{1}{2^n}\right) + \sum_{n=m+1}^{m+p} \frac{|b_n|}{2^n} =$$

$$\sum_{n=1}^{m} \left(|\bar{a}_n - b_n|\left(1-\frac{1}{2^n}\right) + \frac{|b_n|}{2^n}\right) + \sum_{n=m+1}^{m+p} |b_n| \geqslant$$

$$\sum_{n=1}^{m} \left(|\bar{a}_n - b_n|\left(1-\frac{1}{2^n}\right) + \frac{|b_n|}{2^n}\right) \geqslant$$

$$\sum_{n=1}^{m} \frac{|\bar{a}_n - b_n + |b_n|}{2^n} \geqslant \sum_{n=1}^{m} \frac{|\bar{a}_n|}{2^n}.$$

Hence by (36) we have that

$$\left\| \sum_{n=1}^{m} \bar{a}_n x_n \right\| \geqslant \sum_{n=1}^{m} \frac{|\bar{a}_n|}{2^n} ;$$

therefore there exists (f_n) of B_1^* so that

$$(x_n , f_n) \text{ is biorthogonal.} \tag{37}$$

By (37) proof of (i) is similar to proof of (iii) of prop. II; hence we pass to (ii). By (35) for every m and p we have that

$$u_m - u_{m+p} = -2 \sum_{n=r_{2(m-1)+1}+1}^{r_{2m}} x_n - \sum_{n=r_{2m}+1}^{r_{2(m+p-1)+1}} x_n + \sum_{n=r_{2(m+p-1)+1}+1}^{r_{2(m+p)}} x_n \text{ ; hence}$$

$$\| u_m - u_{m+p} \| = \sum_{n=r_{2(m-1)+1}+1}^{r_{2m}} \frac{1}{2^{n-1}} + \sum_{n=r_{2m}+1}^{r_{2(m+p)}} \frac{1}{2^n} < \frac{1}{2^{r_{2(m-1)+1}-1}}$$

Therefore by (35) and (37) we have that

$$\lim_{m \to \infty} u_m = \bar{u} \text{ , with } f_n(\bar{u}) = 1 \text{ for every n.}$$

On the other hand by (35) and (36) $\| x_n \| = 1$ for every n; hence by [3] (see also [7] p. 167) $(\| f_n \|)$ is not bounded, which proves (ii). This completes the proof of prop. III.

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