LOCALITZATION AND DUALITY OF TOPOLOGICAL TENSOR–PRODUCTS

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SUMMARY:

Using the localization results in [5] for compact subsets of Schwartz' ϵ–product, pairs (G, F) of quasicomplete locally convex spaces with the property that the duality equations

$$(\epsilon') \quad (G \epsilon F)'_{co} = G'_{co} \tilde{\otimes}_\pi F'_{co}$$

$$(\pi') \quad (G'_{co} \tilde{\otimes}_\pi F'_{co})'_{co} = G \epsilon F$$

hold true (co points at the topology of uniform convergence on compact sets) will be characterized.

1. INTRODUCTION.

From the definition of neighbourhoods for the projective and injective topologies π and ϵ on $G \otimes F$ by

$$\Gamma(U \otimes V) \quad \text{and} \quad (U^\circ \otimes V^\circ)^\circ = (\Gamma(U^\circ \otimes V^\circ))^\circ$$

one suspects immediately that some duality relations of the kind

$$(G \otimes_\pi F)' = G' \tilde{\otimes}_\epsilon F' \quad (G \otimes_\epsilon F)' = G' \tilde{\otimes}_\pi F'$$

*) The results were presented by the second author at the conference on Functional Analysis, Holomorphy, and Approximation Theory held in Rio de Janeiro in 1982 in order to honour Professor Leopoldo Nachbin on the occasion of his 60th bithday.

do hold between these topologies. It turns out, however, that this problem is rather involved. For the Banach-space setting the situation is treated in Diestel–Uhl's book on vector measures. For locally convex spaces and one of the spaces being nuclear there are many results in Grothendieck's thesis for pairs of Fréchet–spaces and pairs of (DF)–spaces – the duals being equipped with the strong topology. In the absence of nuclearity the first result seems to be due to Buchwalter [2] for Fréchet–spaces and (DF)–spaces; he equipped the dual spaces with the topology of uniform convergence on compact sets and assumed that one of the spaces has the approximation property. Bierstedt and Meise [1], however, observed that the proper setting for Buchwalter's results is to look at the duality of the projective tensor product and Schwartz' ϵ–product; they extended one of Buchwalter's results in this way, Köthe the other one in the second volume of his book. Recently the Radon–Nikodym–techniques used for the duality in Banach–spaces (strong topologies) were extended to pairs of Fréchet–spaces and pairs of (DF) –spaces by Collins–Ruess [3] and Defant [4]. In [5] we noticed that the notion of e–localization in the ϵ–product (see also below) is intimately related to the duality problems; namely: Every compact set in $G \epsilon F$ localizes if and only if the natural map

$$I: G'_{co} \otimes_\pi F'_{co} \hookrightarrow (G \epsilon F)'_{co}$$

is continuous. This simple observation is basic for this paper, the purpose of which is to apply the localization results of [5] to the duality relations (ϵ') and (π') stated in the summary.

2. Localization.

For (real or complex) locally convex spaces E (which are supposed to be Hausdorff throughout this paper) $\mathfrak{U}_E$ (o) stands for the set of all zero–neighbourhoods in E. Denote by

b = set of all bounded sets in E ,

co = set of subsets of all absolutely convex compact subsets in E ,

e = set of all equicontinuous sets in E (if E is a dual space).

For two locally convex spaces L(E, F) is the set of all linear continuous operators of E into F; if Σ is any (filtrating) set of bounded subsets of E, the symbol L_Σ (E, F) stands for L (E, F) equipped with the topology of uniform convergence on all $A \in \Sigma$; in particular $L_\Sigma (E, \mathbb{K}) = : E'_\Sigma$. The notation for bilinear forms $E \times F \to \mathbb{K}$ is as follows: B (E, F) stands for all continuous bilinear mappings and B^{Σ_1, Σ_2} (E, F) for all (Σ_1, Σ_2) –hypocontinuous bilinear mappings, provided Σ_1 and Σ_2 are bounded coverings of E and F respectively.

The projective and injective tensorproducts are denoted by $E \otimes_\pi F$ and $E \otimes_\epsilon F$, for their completions the symbols $\tilde{\otimes}_\pi$ and $\tilde{\otimes}_\epsilon$ will be used. For *quasi-complete* locally convex spaces G and F Schwartz' ϵ–product $G \epsilon F$ is defined by

$$G \epsilon F := L_e (G'_{co}, F) = L_e (F'_{co}, G) = B^{e,e} (G'_{co}, F'_{co})$$

–the latter space equipped with the topology of uniform convergence on the cover $e \times e := \{ U^\circ \times V^\circ \mid U \in \mathfrak{U}_G(o) \, , \; V \in \mathfrak{U}_F(o) \}$ of $G'_{co} \times F'_{co}$. Note that $G \epsilon F$ is quasicomplete. Obviously

$$(G'_{co} \tilde{\otimes}_\pi F'_{co})' = B(G'_{co}, F'_{co}) \subset B^{e,e}(G'_{co}, F'_{co}) = G \epsilon F$$

holds.

Definition: *A subset* $D \subset G \epsilon F$ *localizes (is localizable) if there are compact subsets* $K \subset G$ *and* $L \subset F$ *such that*

$$D \subset (K^\circ \otimes L^\circ)^\circ .$$

Here $K^\circ \otimes L^\circ := \{ \varphi \otimes \psi \mid \varphi \in K^\circ \, , \psi \in L^\circ \} \subset G' \otimes F' \subset (G \epsilon F)'$ *by the natural embedding.*

In the notation of [5], section 4, this means exactly that D e–localizes = e–localizes fully in $L(G'_{co}, F) = \mathfrak{T}_e(G'_{co}, F)$; i.e. D is an equicompact set of operators (: = there is a zero–neighbourhood U in G'_{co} such that D (U) is relatively compact in F).

2.1. Proposition: 1. *If* $D \subset G \epsilon F$ *localizes it is relatively compact. The converse is in general not true.*

2. *For every* $D \subset G \epsilon F$ *the following statements are equivalent:*

(a) D *localizes.*

(b) $D \subset B(G'_{co}, F'_{co})$ *is equicontinuous.*

(c) $D \subset (G'_{co} \tilde{\otimes}_\pi F'_{co})'$ *is equicontinuous.*

(d) $D \subset L(G'_{co}, F)$ *is equicompact.*

Proof: (1) is the content of [5], proposition 1, and (2) can be easily checked. ■

Frequently the following result of L. Schwartz will be used:

3. *For every* $D \subset G \epsilon F$ *the following statements are equivalent:*

(a) D *is relatively compact.*

(b) D *is equicontinuous in* $L(G'_{co}, F)$ *and* $L(F'_{co}, G)$.

(c) D *is an equi–*(e, e)*–hypocontinuous set of bilinear forms on* $G'_{co} \times F'_{co}$.

(The equivalence of (a) and (b) is a special case of the global precompactness–lemma in [5], the equivalence of (b) and (c) is easy). Note that localizable subsets of $G \epsilon F$ actually consist of continuous bilinear forms on $G'_{co} \times F'_{co}$.

Corollary: *For every pair* (G, F) *of quasicomplete locally convex spaces the following statements are equivalent:*

1. *Every compact subset of* $G \epsilon F$ *localizes.*
2. *Every equi–*(e, e)*–hypocontinuous set of bilinear forms on* $G'_{co} \times F'_{co}$ *is equicontinuous.*
3. *On* $G'_{co} \otimes F'_{co}$ *the projective and the* (e, e) *–hypocontinuous topologies coincide.*

(For hypocontinuous topologies see e.g. [8], 4.3.). (1) of the corollary implies that all continuous operators of F'_{co} into G are compact:

$$L(F'_{co}, G) = K(F'_{co}, G)$$

–the latter being the space of compact operators.

The question in which cases every compact set in $G \epsilon F$ actually localizes, was treated in [5] as a special case of spaces $\mathfrak{T}_\Sigma(E, F)$ of Σ –precompact, weakly continuous linear operators $E \to F$. The following result will be needed:

2.2. Proposition: *Let* G *and* F *be quasicomplete locally convex spaces. In each of the following cases* (a) – (g) *all compact sets in* $G \epsilon F$ *localize:*

(a) *All equicontinuous sets of* $L(F'_{co}, G)$ *are equibounded and:* G *is a semi–Montel space or* F'_{co} *is a Schwartz –space.*

(b) G *is a Banach –space and* F'_{co} *a Schwartz –space.*

(c) G *and* F *are Fréchet –spaces.*

(d) G *has a countable basis of bounded sets and* F *is an* (LS) *–space.*

(e) G *and* F *have countable basis of compact sets.*

(f) G *is a semi–Montel space and* $F = E'_{co}$ *for a Banach –space* E.

(g) G *is a Fréchet-Montel –space and* $F = E'_{co}$ *for a separated* (LB) *–space* E.

Proof: (a) is just [5], proposition 5 and (b) – (g) are special cases of (a) using proposition 4 of [5]. ■

3. CHARACTERIZATION OF THE DUALITY RELATIONS.

The procedure will be to characterize the validity of (ϵ') and (π')

$$(\epsilon') \quad (G \,\epsilon\, F)'_{co} = G'_{co} \tilde{\otimes}_\pi F'_{co}$$

$$(\pi') \quad (G'_{co} \tilde{\otimes}_\pi F'_{co})'_{co} = G \,\epsilon\, F$$

separately and then to dualize the results with the aid of the following simple remark (G and F are supposed to be always quasicomplete).

3.1. Lemma:

(1) *If* (ϵ') *holds then* (π') *holds iff* co $\subset$ e *in* $(G \,\epsilon\, F)'_{co}$.

(2) *If* (π') *holds then* (ϵ') *holds iff* co $\subset$ e *in* $(G'_{co} \tilde{\otimes}_\pi F'_{co})'_{co}$.

Note that e $\subset$ co holds always by the precompactness–lemma (e.g. [9]) and co $\subset$ e is true in H'_{co} provided H is a Mackey–space.

Clearly, the isomorphisms identifying both sides of (ϵ') and (π') are the "natural" mappings. Let η be the inclusion mapping

$$\eta : (G'_{co} \tilde{\otimes}_\pi F'_{co})'_{co} = B(G'_{co}, F'_{co}) \hookrightarrow B^{e,e}(G'_{co}, F'_{co}) = G \,\epsilon\, F .$$

Since range (η) = K $(G'_{co}$, F) – the space of compact operators –, the following result is obvious:

3.2. Proposition: *The following are equivalent:*

(1) η *is onto, i.e.* $(G'_{co} \tilde{\otimes}_\pi F'_{co})' = G \,\epsilon\, F$ *algebraically.*

(2) *On* $G'_{co} \otimes F'_{co}$ *every* (e,e) *–hypocontinuous bilinear form is continuous.*

(3) *Every continuous linear operator* $G'_{co} \to F$ *is compact.*

(4) *Every one-point set in* G ϵ F *localizes.*

Note that by (4) η is onto in all the cases of proposition 2.2..

G ϵ F induces on B $(G'_{co}$, $F'_{co})$ the topology of uniform convergence on all subsets of e x e and whence on $(G'_{co} \tilde{\otimes}_\pi F'_{co})'$ the topology of uniform convergence on

$$\overline{e \otimes e} := \left\{ D \mid \exists\, U \in \mathcal{U}_G(o), V \in \mathcal{U}_F(o) : D \subset \overline{\Gamma(U^\circ \otimes V^\circ)}^{\,G'_{co} \tilde{\otimes}_\pi F'_{co}} \right\}$$

Since $\overline{e \otimes e} \subset co$ the mapping η is always continuous. On the other hand, the following is obvious:

3.3. ***Proposition:*** *The mapping* η *is a topological isomorphism (in) if and only if* $co \subset \overline{e \otimes e}$ *in* $G'_{co} \tilde{\otimes}_\pi F'_{co}$.

This implies

3.4. ***Theorem:*** *Let* G *and* F *be quasicomplete locally convex spaces. Then*

$$(\pi') \quad (G'_{co} \tilde{\otimes}_\pi F'_{co})'_{co} = G \,\epsilon\, F \qquad (\textit{via } \eta)$$

holds (topologically) if and only if:
(a) *Every one-point set in* $G \,\epsilon\, F$ *localizes.*
(b) $co \subset \overline{e \otimes e}$ *in* $G'_{co} \tilde{\otimes}_\pi F'_{co}$.

3.5. ***Example:*** Take E and H Fréchet–spaces, then $G := E'_{co}$ and $F := H'_{co}$ are complete σ –locally topological (= (gDF) –) spaces with b = co. By 2.2. (e) the condition (a) of the theorem is satisfied and (b) is true since every compact set $D \subset G'_{co} \tilde{\otimes}_\pi F'_{co} = E \tilde{\otimes}_\pi H$ can be *compactly lifted*, i. e. there are compact sets $K \subset E$ and $L \subset H$ with

$$D \subset \overline{\Gamma(K \otimes L)};$$

obviously $K \subset G'$ and $L \subset F'$ are equicontinuous. So (π') holds. Moreover $G'_{co} \tilde{\otimes}_\pi F'_{co} = E \tilde{\otimes}_\pi H$ is Fréchet, whence barrelled, so lemma 3.1. (2) applies and (ϵ') holds, too. This is the *first Buchwalter–theorem,* more precisely Bierstedt and Meise's extension of it ([1], [2], [14]): *let* E *and* H *be two Fréchet–spaces then*

$$(E'_{co} \,\epsilon\, H'_{co})'_{co} = E \tilde{\otimes}_\pi H$$

$$(E \tilde{\otimes}_\pi H)'_{co} = E'_{co} \,\epsilon\, H'_{co}.$$

If E is an infinite–dimensional reflexive Banach–space, $G := (E, \sigma(E, E'))$ and $F := \mathbb{K}$, then

$$(G'_{co} \tilde{\otimes}_\pi F'_{co})'_{co} = (E')'_{co} \neq G = G \,\epsilon\, F,$$

so η is onto, but not open. At the end of section 7 an example will be given where η is open but not onto.

It was pointed out that in

$$\eta : (G'_{co} \tilde{\otimes}_\pi F'_{co})'_{co} \xrightarrow{\equiv} (G'_{co} \tilde{\otimes}_\pi F'_{co})'_{\overline{e \otimes e}} \xleftarrow{\eta_0} G \epsilon F$$

the inclusion η_0 is a topological isomorphism (into). Therefore the Mackey–Arens theorem implies that the dual mapping factors as follows:

$$\eta' : (G \epsilon F)' \xrightarrow{\eta_0'} ((G'_{co} \tilde{\otimes}_\pi F'_{co})'_{\overline{e \otimes e}})' = \cup \overline{e \otimes e} \hookrightarrow G'_{co} \tilde{\otimes}_\pi F'_{co} .$$

In particular, range $(\eta') = \cup \overline{e \otimes e}$ is dense. Moreover, since η transforms equicontinuous sets into compact sets

$$\eta' : (G \epsilon F)'_{co} \longrightarrow G'_{co} \tilde{\otimes}_\pi F'_{co}$$

is continuous.

3.6. Proposition:

(1) *η' is onto if and only if for every $z \in G'_{co} \tilde{\otimes}_\pi F'_{co}$ there are zero–neighbourhoods U in G' and V in F such that*

$$z \in \overline{\Gamma(U^\circ \otimes V^\circ)}^{\,G'_{co} \tilde{\otimes}_\pi F'_{co}}.$$

(2) *η' is injective if and only if $(G'_{co} \tilde{\otimes}_\pi F'_{co})' \subset G \epsilon F$ (via η) is dense.*

Note that by 3.2. the mapping η' is injective in all the cases of 2.2..

To check under which circumstances η' is open (onto its range) define the natural inclusion

$$I : G'_{co} \otimes_\pi F'_{co} \hookrightarrow (G \epsilon F)'_{co} .$$

3.7. Lemma: *$\eta' \circ I$ is the inclusion–map $G'_{co} \otimes_\pi F'_{co} \hookrightarrow G'_{co} \tilde{\otimes}_\pi F'_{co}$.*

Proof: Take $z = \Sigma \varphi_j \otimes \psi_j \in G' \otimes F'$ and $\Phi \in (G'_{co} \tilde{\otimes}_\pi F'_{co})'$ and use the definition of $\eta(\Phi) \in L(G'_{co}, F)$:

$$\langle \Phi, (\eta' \circ I)(z) \rangle = \langle I(z), \eta(\Phi) \rangle = \Sigma \langle I(\varphi_j \otimes \psi_j), \eta(\Phi) \rangle$$

$$= \Sigma \langle \psi_j, \eta(\Phi)(\varphi_j) \rangle = \Sigma \langle \Phi, \varphi_j \otimes \psi_j \rangle = \langle \Phi, z \rangle . \ \blacksquare$$

Now things are prepared to use the basic observation on the continuity of I:

3.8. ***Theorem:*** *For each pair* (G, F) *of quasicomplete locally convex spaces the following are equivalent:*

(1) $\eta' : (G \,\epsilon\, F)'_{co} \to G'_{co} \tilde{\otimes}_\pi F'_{co}$ *is a topological isomorphism (in).*

(2) $I: G'_{co} \otimes_\pi F'_{co} \hookrightarrow (G \,\epsilon\, F)'_{co}$ *is continuous.*

(3) *Every compact subset of* $G \,\epsilon\, F$ *localizes.*

Note that by 3.2. (4) the inclusion η is onto in the case one of these conditions holds. Remember that range $(\eta') = \cup \overline{e \otimes e}$.

Proof: (1) implies (2) since $\eta' \circ I$ is the inclusion–map. The equivalence (2)$\curvearrowright$(3) follows by the definition of localizing and the fact that the sets

$$\left\{ \Gamma(K^\circ \otimes L^\circ) \mid K \subset G \text{ compact}, L \subset F \text{ compact} \right\}$$

form a basis of zero–neighbourhoods in $G'_{co} \otimes_\pi F'_{co}$. (3) $\curvearrowright$ (1): η is onto and whence η' injective. To show that

$$\eta' : (G \,\epsilon\, F)'_{co} \to ((G'_{co} \tilde{\otimes}_\pi F'_{co})'_{co})'_e = G'_{co} \tilde{\otimes}_\pi F'_{co}$$

is open it is enough to note that every compact set in $G \,\epsilon\, F$ is localizable, whence (via η^{-1}) equicontinuous in $(G'_{co} \tilde{\otimes}_\pi F'_{co})'$ by 2.1.. ■

In particular, the last step of the proof showed that co $\subset$ e in $(G'_{co} \tilde{\otimes}_\pi F'_{co})'_{co}$ if η' is a topological isomorphism (in).

The theorem says that in all cases of proposition 2.2.

$$(G \,\epsilon\, F)'_{co} \subset G'_{co} \tilde{\otimes}_\pi F'_{co}$$

as a topological subspace (via η'). But equality does not always hold: take a non–complete Montel–space, $G := E'_{co}$ and $F := \mathbb{K}$, then

$$(G \,\epsilon\, F)'_{co} = E \underset{\neq}{\subset} \tilde{E} = G'_{co} \tilde{\otimes}_\pi F'_{co}.$$

Later on (section 7) a more involved example will be given. Since range (η') is always dense, 3.6. and 3.8. imply

3.9. ***Theorem:*** *For each pair* (G, F) *of quasicomplete locally convex spaces the following statements* (1) – (3) *are equivalent:*

(1) *η' is a topological isomorphism (onto), i.e.*
$(\epsilon)'$ $(G \epsilon F)'_{co} = G'_{co} \tilde{\otimes}_\pi F'_{co}$ *(via η').*

(2) (a) $\cup \overline{e \otimes e} = G'_{co} \tilde{\otimes}_\pi F'_{co}$.
(b) *Every compact set in* G ϵ F *localizes.*

(3) (a) $(G \epsilon F)'_{co}$ *is complete.*
(b) *Every compact set in* G ϵ F *localizes.*

By Lemma 3.7. it is now clear that $(\eta')^{-1} = \tilde{I}$ (the completion of I) if η' is a topological isomorphism (onto).

The most comfortable way to see (3a) being satisfied is when G ϵ F is bornological. If (G, F) is a pair of Köthe–sequence spaces, G an (LB) –space and F a Fréchet–space, then Grothendieck ([10], II théorème 15) showed that for the projective tensorproduct $H = G \tilde{\otimes}_\pi F$ the following statements are equivalent: (1) H *is bornological,* (2) H *is barrelled,* (3) H'_{co} *is quasicomplete,* (4) H'_{co} *is complete* and moreover characterized this in terms of the weight–matrizes of G and F. So, if G or F are nuclear the condition of G ϵ F being bornological is necessary for (ϵ') in many practical situations when G is an (LB) and F a Fréchet–space. This is interesting since there are cases such that all compact sets in G ϵ F localize. Vogt, resuming the results of Grothendieck, characterized in [16], pairs (E, F) of Fréchet–spaces such that every equicontinuous subset of L (E, F) is equibounded. Moreover, he announces an investigation of those Fréchet–spaces E and F such that $E \tilde{\otimes}_\pi \Lambda'_{co}$ and $\Lambda \tilde{\otimes}_\pi F'_{co}$ are bornological for a power–series–space Λ.

3.10. If nuclearity is involved the following alternative characterization of the surjectivity of η' holds:

Proposition: *If* (G, F) *is a pair of quasicomplete convex spaces such that*
(1) G'_{co} *and* F'_{co} *are complete,*
(2) co $\subset$ e *in* G'_{co} *and* F'_{co} ,
(3) F *and* F'_{co} *are nuclear,*
then $\eta' : (G \epsilon F)' \to G'_{co} \tilde{\otimes}_\pi F'_{co}$ *is onto if and only if every continuous operator* $G \to F'_{co}$ *is bounded.*

Proof: The nuclearity assumptions imply

$$\begin{array}{ccc} (G\,\epsilon\,F)' = (G\,\tilde{\otimes}_\pi F)' & = & B(G,F) \\ \downarrow{\scriptstyle \eta'} & & \downarrow{\scriptstyle \varrho} \\ G'_{co}\,\tilde{\otimes}_\pi F'_{co} = G'_{co}\,\epsilon\,F'_{co} & = & L(G, F'_{co}) \end{array} ,$$

where ϱ is the canonical mapping. Using that every bilinear continuous form on G x F is nuclear, it is easily seen that the diagramm commutes. Now the result is immediate. ■

3.11. ***Example:*** *If* G *and* F *are complete* (LB) *–spaces and* F *nuclear,* then by 2.2. (a) condition (3) of theorem 3.8. is satisfied, whence η' is a topological isomorphism (in). Since the assumptions of the proposition are fullfilled, η' is onto and (ϵ') holds. Moreover, since $G\,\epsilon\,F = G\,\tilde{\otimes}_\pi F$ is a barrelled (DF) –space ([13], 15.6.8.), co = e in $(G\,\epsilon\,F)'_{co}$ and, by lemma 3.1., also (π') is valid:

$$(G\,\tilde{\otimes}_\pi F)'_{co} = G'_{co}\,\tilde{\otimes}_\pi F'_{co}$$

$$(G'_{co}\,\tilde{\otimes}_\pi F'_{co})'_{co} = G\,\tilde{\otimes}_\pi F \quad .$$

Grothendieck [10], II p. 76 proved for the strong topologies

$$(G\,\tilde{\otimes}_\pi F)'_b = G'_b\,\tilde{\otimes}_\pi F'_b$$

$$(G'_b\,\tilde{\otimes}_\pi F'_b)'_b = G''_b\,\tilde{\otimes}_\pi F''_b$$

(for pairs (G, F) of (F) – or (DF) –spaces, one being nuclear).

4. COLLECTION OF RESULTS AND EXAMPLES.

The following main result is a direct consequence of 3.1., 3.4. and 3.9.:

4.1. ***Theorem:*** *For each pair* (G, F) *of quasicomplete locally convex spaces the following statements* (1) – (3) *are equivalent:*

(1) *The following duality relations hold true:*

(ϵ') $(G\,\epsilon\,F)'_{co} = G'_{co}\,\tilde{\otimes}_\pi F'_{co}$ *(via* $\eta' = (\tilde{I})^{-1}$*)*

(π') $(G'_{co}\,\tilde{\otimes}_\pi F'_{co})'_{co} = G\,\epsilon\,F$ *(via* η*).*

(2) (a) *Every one –point set in* G ϵ F *localizes.*

(b) $co \subset \overline{e \otimes e}$ *in* $G'_{co} \tilde{\otimes}_\pi F'_{co}$.

(c) $co \subset e$ *in* $(G'_{co} \tilde{\otimes}_\pi F'_{co})'_{co}$.

(3) (a) *Every compact set in* G ϵ F *localizes.*

(b) $(G \epsilon F)_{co}$ *is complete.*

(c) $co \subset e$ *in* $(G \epsilon F)'_{co}$.

Note that if G ϵ F is bornological (whence Mackey) (3b) and (3c) are satisfied.

4.2. The case where $G := E'_{co}$ and $F := H'_{co}$ for Fréchet–spaces E and H was already treated in 3.5.: both, (ϵ') and (π') hold.

Let G *and* F *be Fréchet–spaces.* To apply theorem 4.1. (3) observe first that by 2.2. (c) condition (a) is satisfied; since G ϵ F is bornological also (b) and (c) are fullfilled and whence (ϵ') and (π') do hold. This is the *second Buchwalter–theorem* (due to Buchwalter [2] if one the spaces has the approximation property, for the ϵ –product this result is due to Köthe [14]).

4.3. It is worthwile to note (though this is more or less known) that the two Buchwalter–theorems (see 3.5. and 4.2.) provide the full duality for $(F\overline{S})$ – (= Fréchet–Schwartz) –spaces and (LS) –spaces, namely:

If G *and* F *are both* $(F\overline{S})$ *–spaces (resp. both* (LS) *–spaces) then* G ϵ F *and* $G \tilde{\otimes}_\pi F$ *are* $(F\overline{S})$ *–spaces (resp.* (LS) *–spaces),* $G'_b \epsilon F'_b$ *and* $G'_b \tilde{\otimes}_\pi F'_b$ *are* (LS) *–spaces (resp.* $(F\overline{S})$ *–spaces) and the duality equations*

$$(G \epsilon F)'_b = G'_b \tilde{\otimes}_\pi F'_b \quad , \quad (G'_b \epsilon F'_b)'_b = G \tilde{\otimes}_\pi F$$

$$(G'_b \tilde{\otimes}_\pi F'_b)'_b = G \epsilon F \quad , \quad (G \tilde{\otimes}_\pi F)'_b = G'_b \epsilon F'_b$$

hold true.

For the proof look at the four equations coming from both Buchwalter–theorems, use that the projective tensorproduct and the ϵ –product of $(F\overline{S})$ –spaces are $(F\overline{S})$ –spaces ([13], 15.6.5. and 16.4.3.) and apply the duality results for $(F\overline{S})$ –spaces and (LS) –spaces (see e.g. [7]).

4.4. *Let* G: = *C (X), X a compact space, and* F *be an* (LS) *–space.* Represent $F = \text{ind}\, F_n$ by a compact sequence (F_n) of Banach–spaces, then

$$G \epsilon F = C(X, F) = \underset{n \to}{\text{ind}}\ C(X, F_n)$$

according to a result of Mujica ([15], theorem 1.5.). In particular, $G \epsilon F$ is bornological, whence (3b) and (3c) of theorem 4.1. are satisfied; (3a) follows by 2.2. (b). So also in this case (ϵ') and (π') are valid.

4.5. Using Hollstein's generalization ([12], cor. 4.4.) of Mujica's theorem, this result can be extended to $\mathfrak{L}^\infty$ –spaces G and (LS) –spaces F. But even more is true:

Let G *be an* $\mathfrak{L}^\infty$*–space and* F *a complete bornological space such that* F'_{co} *is Schwartz and* F'_b *has property* (B) *of Pietsch.* By a result of [6], corollary 12 $G \epsilon F$ is bornological, therefore (ϵ') and (π') hold by 2.2. (a) and theorem 4.1. (3). Note that if G is an infinite dimensional $\mathfrak{L}^\infty$ –space and $G \epsilon F$ is bornological, then F'_b has property (B) ([6], corollary 12). Metrizable spaces and σ–locally topological spaces have property (B).

4.6. The external characterization of Schwartz–spaces implies easily ([5], section 6) *that if a quasicomplete locally convex space* F *has the property that for all compact space* X *every one –point set in* C (X) ϵ F = C (X, F) *localizes, then* F'_{co} *is a Schwartz –space.* Since there is a (DFM) –space the dual of which is not Schwartz, the results of 4.4. and 4.5. cannot be improved by taking F a (DFM) –space: *There is a compact space* X *and a* (DFM) *–space* F *such that the map* η *for* G: = C (X) *and* F *is not onto and* η' *is not open onto its range* (see 3.2. and 3.8.).

4.7. So for G and F being complete σ–locally topological (= (gDF)) spaces, the situation is as follows: (ϵ') and (π') hold in the following cases:

(a) G and F semi–Montel spaces (this follows from Buchwalter's theorems since there are Fréchet–spaces E and H with $E'_{co} = F$ and $H'_{co} = G$, see [13], p. 355).

(b) G an $\mathfrak{L}^\infty$–space and F an (LS) –space (4.5.).

(c) G and F bornological, F nuclear (3.11.).

and neither (ϵ') nor (π') holds in general, if G is only a Banach–space and F a semi–Montel space.

4.8. The following notion was already used in 3.5.: A subset $D \subset E \tilde{\otimes}_\pi H$ can be *compactly lifted* if there are compact sets $K \subset E$ and $L \subset H$ such that $D \subset \overline{\Gamma(K \otimes L)}$.

Proposition: *If the following conditions* (1) *and* (2)

(1) E *is a Banach –space and* H *a barrelled* (DF) *–space, or:* E *is a separated* (LB) *–space and* H *a* (DFM) *–space.*

(2) *Every compact set in* $E \tilde{\otimes}_\pi H$ *is compactly liftatle.*

are satisfied, then the duality relations (ϵ') *and* (π') *hold for* $G := H'_{co}$ *and* $F := E'_{co}$.

Proof: In $G \epsilon F$ every compact set localizes according to 2.2. (f) and (g). Moreover, $M := G'_{co} \tilde{\otimes}_\pi F'_{co} = E \tilde{\otimes}_\pi H$ is a barrelled (DF) –space ([13], 15.6.8.). Hence e = co in M'_{co}. Property (2) means just $co \subset \overline{e \otimes e}$ in M, since in G'_{co} and F'_{co} the relation e = co holds. Therefore theorem 4.1. (2) applies. ■

5. LIFTING OF COMPACT SETS IN THE PROJECTIVE TENSORPRODUCT.

It might be interesting to note that this is intimately related to the validity of (π'). Namely, if (π') holds, or even less: if η is a topological isomorphism (in) then (by 3.3.)

$$co \subset \overline{e \otimes e} \text{ in } G'_{co} \tilde{\otimes}_\pi F'_{co} .$$

Since equicontinuous sets in G'_{co} and F'_{co} are in particular relatively compact, this implies that every compact set in $G'_{co} \tilde{\otimes}_\pi F'_{co}$ can be compactly lifted. Conversely, if G and F are Mackey–spaces and every compact set of $G'_{co} \tilde{\otimes}_\pi F'_{co}$ is compactly liftable, then η is a topological isomorphism (in).

5.1. *If* E *and* H *are semi –Montel* σ *–locally topological* (= (gDF)) *spaces*, then there are Fréchet–spaces G and F such that $E = G'_{co}$ and $H = F'_{co}$ (see [13], p. 355). By the second Buchwalter–theorem (4.1.) this means that *all compact sets in* $E \tilde{\otimes}_\pi H$ *can be compactly lifted.* This follows more easily from the lifting of bounded sets in the projective tensorproduct of σ –locally topological spaces ([13], 15.6.2.).

5.2. If E and H are Fréchet–spaces, $G := E'_{co}$ and $F := H'_{co}$ the lifting of compact sets was used in the proof of the first Buchwalter–theorem (3.5.).

5.3. The result in 3.11. implies, *that in* $E \tilde{\otimes}_\pi H$ *all compact sets can be compactly lifted if* H *is a nuclear Fréchet –space and* $E: = G'_{co}$, *where* G *is a complete* (LB) *–space.* Since $E \tilde{\otimes}_\pi H = E \epsilon H$ this implies in particular that all compact sets in $E \epsilon H$ localize.

6. POWER SERIES–SPACES.

Let $\Lambda_1 : = \Lambda_1 (\alpha)$ and $\Lambda_\infty : = \Lambda_\infty(\beta)$ be infinite–dimensional power–series–spaces of finite and infinite order respecticely (see [13] for the notation). They are Fréchet-Schwartz spaces and, by a result of Zahariuta [17], all operators in $L(\Lambda_1, \Lambda_\infty)$ are bounded = compact. It follows with the aid of Grothendieck's factorization theorem for (LF) –spaces ([16], 1.1.) that all equicontinuous subsets of $L(\Lambda_1, \Lambda_\infty)$ are equibounded.

To see consequences of this fact in the present setting, assume that Λ_1 is even *nuclear* and take

$$G: = \Lambda_\infty \quad \text{and} \quad F: = (\Lambda_1)'_{co}.$$

By 2.2. (a) all compact sets in $G \epsilon F$ localize whence η is onto and η' a topological isomorphism (in) by 3.2. and 3.8..

However, there is always an operator

$$G = \Lambda_\infty \longrightarrow \Lambda_1 = F'_{co}$$

which is not bounded ([16], end of section 4.). Therefore, proposition 3.10. implies that η' is not onto:

$$\text{range}(\eta') = \cup \overline{e \otimes e} \underset{\neq}{\subset} G'_{co} \tilde{\otimes}_\pi F'_{co},$$

in particular $co \not\subset \overline{e \otimes e}$ such that η cannot be open by 3.3.. Hence the following facts are settled:

(1) η *is a continuous isomorphism (onto) which is not open:*

$$((\Lambda_\infty)'_{co} \tilde{\otimes}_\pi \Lambda_1)' = \Lambda_\infty \ \epsilon \ (\Lambda_1)'_{co}$$

(as vector spaces; the topology of uniform convergence on compact sets on the left side is strictly finer then the topology on the right side).

(2) η' *is a topological isomorphism which is not surjective:*

$$(\Lambda_\infty \epsilon\ (\Lambda_1)'_{co})'_{co} \subsetneq (\Lambda_\infty)'_{co} \tilde{\otimes}_\pi \Lambda_1.$$

(topological subspace).

Moreover, by 3.6. (1)

(3) *There is a one-point set in* $(\Lambda_\infty)'_{co} \tilde{\otimes}_\pi \Lambda_1$ *which is not compactly liftable.*

Note that it is also possible (by the nuclearity) to look at (2) with G: = $(\Lambda_\infty)'_{co}$ and F: = Λ_1, whence giving an example where η is open but not onto. In particular, these results hold if Λ_1 is replaced by H (D°) –the holomorphic functions on the open unit disk in C – and Λ_∞ by the space H (ℂ) of entire functions.

7. LOCALIZATION AND BARRELLEDNESS.

It was crucial to have good knowledge of those G ϵ F where all compact sets localize; moreover, topological–geometric properties of $G'_{co} \tilde{\otimes}_\pi F'_{co}$ were used frequently. The following remarks show that there are some connections.

7.1. Proposition: *Let* G *and* H *be quasicomplete spaces. If* G'_{co} *and* F'_{co} *are barrelled (resp. quasibarrelled resp. bornological) and all compact sets in* G ϵ F *localize, then*

$$G'_{co} \otimes_\pi F'_{co}$$

is barrelled (resp. quasibarrelled resp. bornological).

Proof: Corollary 2.1. implies that

$$G'_{co} \otimes_{e,e} F'_{co} = G'_{co} \otimes_\pi F'_{co}$$

($\otimes_{e,\ e}$ points at the (e, e) –hypocontinuous topology on the tensor–product). By [6], theorem 5, the (Σ_1, Σ_2) –hypocontinuous tensor–topology of two barrelled (resp. ...) spaces is of the same type. ∎

For the converse look at the following result.

7.2. Proposition: *If* $G'_{co} \otimes_\pi F'_{co}$ *is barrelled, then every equicontinuous subset of the space*

$$K(G'_{co}, F) \subset L(G'_{co}, F) = G \epsilon F$$

of compact operators localizes (equivalently: is equicompact).

Proof: Remember that range $(\eta) = K(G'_{co}, F)$. Take $D \subset K(G'_{co}, F)$ equicontinuous; it is obvious that $\eta^{-1}(D)$ is weak $-*-$ bounded in $(G'_{co} \otimes_\pi F'_{co})'$, whence equicontinuous by the assumption. 2.1. implies that D localizes. ■

The same statement holds for "barrelled" replaced by "quasi–barrelled" – provided every bounded set in $G'_{co} \otimes_\pi F'_{co}$ can be boundedly lifted.

If there is an equicontinuous net $(\varphi_\lambda)_{\lambda \in \Lambda}$ in K (F, F) which converges pointwise to the identity of F, then for every equicontinuous subset $D \subset L(G'_{co}, F) = G \epsilon F$ the set

$$\{\varphi_\lambda \circ T \mid T \in D, \lambda \in \Lambda\}$$

is equicontinuous in $K(G'_{co}, F)$ whence equicompact under the assumptions of the proposition; it is now immediate that D is equicompact as well. The same is true if there is an equicontinuous net in $K(G'_{co}, G'_{co})$ converging pointwise to the identity of G'_{co}. A locally convex space has the *bounded approximation property* if there is an equicontinuous net of finite – dimensional (whence compact) operators converging pointwise to the identity. So the following holds true by 8.1. and 8.2. (recall that a priori the relatively compact subsets of $G \epsilon F$ are those which are equicontinuous in $L(G'_{co}, F)$ *and* $L(F'_{co}, G)$ by Schwartz' theorem 2.1. (3)):

Corollary: *If* G *and* F *are quasicomplete locally convex spaces and* G'_{co} *or* F *has the approximation property then the following statements are equivalent:*

(1) $G'_{co} \otimes_\pi F'_{co}$ *is barrelled.*

(2) G'_{co} *and* F'_{co} *are barrelled and every equicontinuous subset of* $L(G'_{co}, F) = G \epsilon F$ *localizes.*

(3) G'_{co} *and* F'_{co} *are barrelled and every compact subset of* $G \epsilon F$ *localizes.*

Note that (2) implies in particular that all equicontinuous subsets of $L(G'_{co}, F)$ are relatively compact in $G \epsilon F$, i.e. they are equicontinuous in $L(F'_{co}, G)$ as well.

As a consequence of the corollary, the space $\varphi \otimes_\pi \omega$ is not barrelled – a well known result due to Hollstein [11]. More general:

If E *is an infinite–dimensional quasicomplete locally convex space with the bounded approximation property and* co $\subset$ e *in* E'_{co} *then* E $\otimes_\pi$ E'_{co} *is not barrelled.*

(For the proof take G: = E'_{co} and F: = E, observe that the identity operator in L (E, E) = G ϵ F is not compact and apply the corollary).

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