BASES IN THE SPACE OF ENTIRE DIRICHLET FUNCTIONS OF TWO COMPLEX VARIABLES

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ABSTRACT

In this paper, we consider the space X of all Entire functions defined by Dirichlet series of two complex variables, we endow X with two equivalente topologies. The main result is concerned with finding the necessary and sufficient conditions for a base in X to become a proper base.

1. INTRODUCTION.

Let X denote the space of all entire functions defined by Dirichlet series of two complex variables $s_1, s_2 \in \mathbb{C}$ (where $\mathbb{C}$ is the complex plane equipped with the usual topology). When $f \in X$, it means that $f : \mathbb{C}^2 \to \mathbb{C}$ such that

$$f(s_1, s_2) = \sum_{m,n=o}^{\infty} a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2) \tag{1.1}$$

where

$$0 = \lambda_o < \lambda_1 < \lambda_2 < \ldots \qquad < \lambda_m \to \infty \text{ with } m$$

$$0 = \mu_o < \mu_1 < \mu_2 < \ldots \qquad < \mu_n \to \infty \text{ with } n$$

and further (see [1])

$$\limsup_{m+n\to\infty} \frac{\log |a_{m,n}|}{\lambda_m + \mu_n} = -\infty \tag{1.2}$$

$$\limsup_{m+n\to\infty} \frac{\log (m+n)}{\lambda_m + \mu_n} < \infty . \tag{1.3}$$

For each $f \in X$, and σ_1, σ_2 real, we define the family $\{M(f,\sigma_1,\sigma_2) : \sigma_1, \sigma_2 > o\}$ of semi-norms on X, where

$$M(f;\sigma_1,\sigma_2) = \sup_{-\infty < t_1, t_2 < \infty} |f(\sigma_1 + it_1, \sigma_2 + it_2)|$$

Let for each $f \in X$, and σ_1, σ_2 real, define

$$\| f;\sigma_1,\sigma_2 \| = \sum_{m,n=o}^{\infty} |a_{m,n}| \exp(\lambda_m \sigma_1 + \mu_n \sigma_2)$$

Then $\{ \| f;\sigma_1,\sigma_2 \| ; \sigma_1,\sigma_2 > o \}$ again defines a family of semi-norms on X One has proved earlier (see [1]) that for some $\alpha > o$.

$$M(f;\sigma_1,\sigma_2) \leqslant \| f;\sigma_1,\sigma_2 \| \leqslant C(\alpha) M(f;\sigma_1+\alpha, \sigma_2+\alpha) \tag{1.4}$$

$C(\alpha)$ being a constant depending on α only. Hence one finds that the topology generated by the family of semi-norms $\{M(f;\sigma_1,\sigma_2):\sigma_1,\sigma_2 > o\}$ and the topology generated by the family of semi-norms $\{\| f;\sigma_1,\sigma_2 \| :\sigma_1,\sigma_2 > o\}$ are equivalent.

We note that the single variable case already dealth with P.K.Kamthan and S.K.Singh Gautam [2].

2. Bases in X.

Définition 2.1: A sequence $\{f_{m,n} : m, n \geqslant o\} \subset X$ is said to be a base for X, if for each $f \in X$, there exists a unique sequence $\{a_{m,n} : m, n \geqslant o\} \subset \mathbb{C}$, such that

$$f = \sum_{m,n=o}^{\infty} a_{m,n} f_{m,n}$$

where the convergence of this double series is with respect to the topology on X The members $a_{m,n}$ are called the base functions.

In view of this definition, we find that $\{\delta_{m,n}\}$ (where $\delta_{m,n}(s_1,s_2) = \exp$

$(\lambda_m s_1 + \mu_n s_2))$ is a base for X and moreover, for this base, the base functions satisfy the following condition:

$$\limsup_{m+n\to\infty} \frac{\log |a_{m,n}|}{\lambda_m + \mu_n} = -\infty \tag{2.1}$$

However, for all bases in X, the corresponding coefficients do not necessarily satisfy (2.1). For instance, consider $\{ f_{m,n} \}$, where

$$f_{m,n}(s_1, s_2) = \exp(m s_1 + n s_2) / (m + n/2)!$$

is in X and forms a base in X. Now

$$\exp(\exp s_1 + \exp s_2) = \sum_{m,n=o}^{\infty} a_{m,n} f_{m,n}(s_1, s_2)$$

and so $a_{m,n} = 1$, for all $m, n \geqslant o$; $a_{m,n} = o$ for $m \neq n$, thus

$$\limsup_{m+n\to\infty} \frac{\log |a_{m,n}|}{\lambda_m + \mu_n} = o$$

consequently there are two types of base in X, for which (2.1) is true or not true.

Definition 2.2. A sequence $\{ f_{m,n} : m, n \geqslant o \} \subset X$ will be called a genuine base for X if the corresponding base functions satisfy (2.1).

Definition 2.3. A sequence $\{ f_{m,n} : m, n \geqslant o \} \subset X$ will be called an absolute base for X, if each $f \in X$ can be uniquely expressed as $\Sigma\, a_{m,n} f_{m,n}$ where the double series is absolutely convergent with respect to the topology on X.

Definition 2.4. A sequence $\{ f_{m,n} : m, n \geqslant o \} \subset X$ will be called proper base for X if it is a genuine as well as an absolute base for X.

3. CHARACTERISATION OF PROPER BASES

Our discussion on this direction will require a number of intermediary results, first of all, we have

Lemma 3.1. *Let* $\{\phi_{m,n}\} \subset X$, *and suppose that* $\Sigma\, \phi_{m,n}$ *converges absolutely with respect to the topology on* X, *ie* $\Sigma\, M(\phi_{m,n}\,;\sigma_1,\sigma_2)$ *converges for every real* σ_1, σ_2 . *Then given* $\alpha > o$ *and* $\sigma_1, \sigma_2 > o$, *there corresponds an integer* N, *such that for all* $m + n \geqslant N$, *we have*

$$\log M(\phi_{m,n}\,;\sigma_1,\sigma_2) < \alpha\,(\lambda_m + \mu_n)$$

Proof. The proof is straight forward. Indeed, let the conclusion of the Lemma be false. Then we may find two increasing sequences $\{m_k\}$, $\{n_\ell\}$ such that

$$\log M(\phi_{m_k,n_\ell}\,;\sigma_1,\sigma_2) > \alpha\,(\lambda_{m_k} + \mu_{n_\ell})$$

Therefore

$$\sum_{m,n=o}^{\infty} M(\phi_{m,n};\sigma_1,\sigma_2) > \sum_{k,\ell=o}^{\infty} \exp \alpha\,(\lambda_{m_k}+\mu_{n_\ell})$$

and this contradicts the hypothesis of the lemma.

Theorem 3.2. *Let* $\{\alpha_{m,n}\,;m,n \geqslant o\} \subset X$. *Suppose* $\{C_{m,n}\}$ *be an arbitrary sequence contained in* $\mathbb{C}$, *such that*

$$\limsup_{m+n\to\infty} \frac{\log |C_{m,n}|}{\lambda_m + \mu_n} = -\infty \tag{3.1}$$

Then the series $\Sigma\, M(\alpha_{m,n}\, C_{m,n}\,;\sigma_1,\sigma_2)$ *converges if and only if*

$$\limsup_{m+n\to\infty} \frac{\log M(\alpha_{m,n};\sigma_1,\sigma_2)}{\lambda_m + \mu_n} < \infty, \text{ for each } \sigma_1,\sigma_2 \tag{3.2}$$

Proof (Necessity): Let (3.1) hold good. Suppose (3.2) is not true. Hence for some $\sigma_1, \sigma_2 > o$, there corresponds sequences $\{m_k\}$, $\{n_\ell\}$ such that

$$\text{Log}\, M(\alpha_{m_k,n_\ell}\,;\sigma_1,\sigma_2) > (k+\ell)\,(\lambda_{m_k} + \mu_{n_\ell})\ ,\quad k,\ell \geqslant o \tag{3.3}$$

Define $\{C_{m,n}\} \subset \mathbb{C}$ as follows

$$\log |C_{m,n}| = \begin{cases} \lambda_m + \mu_n - \text{Log}\, M(\alpha_{m,n};\sigma_1,\sigma_2) & \text{for } m = m_k\ ;\ n = n_\ell \\ -\infty & m \neq m_k\ ,\ n \neq n \end{cases}$$

Then from (3.3), we have

$$\limsup_{m+n\to\infty} \frac{\log |C_{m,n}|}{\lambda_m + \mu_n} = -\infty$$

and (3.1) holds. But,

$$M(C_{m_k,n_\ell}\alpha_{m_k,n_\ell};\sigma_1,\sigma_2) = C_{m_k,n_\ell} M(\alpha_{m_k,n_\ell};\sigma_1,\sigma_2)$$
$$= \exp(\lambda_{m_k} + \mu_{n_\ell})$$
$$M(C_{m,n}\alpha_{m,n};\sigma_1,\sigma_2) = o \text{ , for } m \neq m_k \text{ , } n \neq n_\ell.$$

and this contradicts Lemma 3.1.

(Sufficiency). Let (3.2) be satisfied. Then for each σ_1, $\sigma_2 > o$, there exists a constant $\epsilon = \epsilon(\sigma_1, \sigma_2)$, such that

$$\log M(\alpha_{m,n};\sigma_1,\sigma_2) < \epsilon(\lambda_m + \mu_n) \qquad \text{for } m+n \geqslant N_o(\epsilon) \tag{3.4}$$

Let $\epsilon_1 > \epsilon$. Then there exists $N_1 = N_1(\epsilon_1)$, such that

$$|C_{m,n}| \leqslant \exp -\epsilon_1(\lambda_m + \mu_n) \text{ ; } m+n \geqslant N_1 \tag{3.5}$$

We get from (3.4) and (3.5)

$$M(C_{m,n}\alpha_{m,n};\sigma_1,\sigma_2) = |C_{m,n}| M(\alpha_{m,n};\sigma_1,\sigma_2)$$
$$\leqslant \exp(\epsilon - \epsilon_1)(\lambda_m + \mu_n) \text{ , } m+n \geqslant N = \max(N_o, N_1)$$

hence

$$\sum_{m\,n=o}^{\infty} M(C_{m,n}\alpha_{m,n};\sigma_1,\sigma_2)$$

converges for every σ_1, σ_2 (in view of lemma 1 [1]).

Lemma 3.3. *Let* $\{\alpha_{m,n}; m, n \geqslant o\} \subset X$ *and* $\{C_{m,n}; m, n \geqslant o\}$ *be an arbitrary sequence in* $\mathbb{C}$ *, such that*

$$\sum_{m,n=o}^{\infty} M(\alpha_{m,n} C_{m,n};\sigma_1,\sigma_2)$$

converge, then

$$\limsup_{m+n\to\infty} \frac{\log |C_{m,n}|}{\lambda_m + \mu_n} = -\infty \tag{3.6}$$

if and only if

$$\lim_{\sigma_1,\sigma_2\to\infty} \left\{ \liminf_{m+n\to\infty} \frac{\log M(\alpha_{m,n};\sigma_1,\sigma_2)}{\lambda_m + \mu_n} \right\} = +\infty \tag{3.7}$$

Proof (Necessity). Let (3.6) be true and suppose that (3.7) is false. Then for each $\sigma_1, \sigma_2 > o$ and some $\beta > o$.

$$\lim_{\sigma_1,\sigma_2\to\infty} \left\{ \liminf_{m+n\to\infty} \frac{\log M(\alpha_{m,n};\sigma_1,\sigma_2)}{\lambda_m + \mu_n} \right\} < \beta < +\infty$$

Since $M(\alpha_{m,n};\sigma_1,\sigma_2)$ is monotonocally increasing in $\sigma_1, \sigma_2 > o$ for each fixed pair (m, n). Then there exist sequences $\{m_k\}$, $\{n_\ell\}$ such that

$$\operatorname{Log} M(\alpha_{m_k, n_\ell};\sigma_1,\sigma_2) < \beta(\lambda_{m_k} + \mu_{n_\ell})$$

Define $\{C_{m,n}\} \subset \mathbb{C}$ as follows

$$\log |C_{m,n}| = \begin{cases} -2\beta(\lambda_m + \mu_n) & , \quad m = m_k \ , \ n = n_\ell \\ -\infty & , \quad m \neq m_k \ , \ n \neq n_\ell \end{cases}$$

Then for a given σ_1, σ_2

$$\sum_{m,n=o}^{\infty} |C_{m,n}| M(\alpha_{m,n};\sigma_1,\sigma_2) \leqslant \sum_{m,n=o}^{\infty} \exp -\beta(\lambda_m + \mu_n) < +\infty$$

and so $\sum_{m,n=o}^{\infty} M(C_{m,n}\,\alpha_{m,n};\sigma_1,\sigma_2)$ converge for each real σ_1, σ_2. Consequently from (3.6), $\limsup_{m+n\to\infty} \frac{\log |C_{m,n}|}{\lambda_m + \mu_n} = -\infty$. But it is not true. Since

$$\limsup_{m+n\to\infty} \frac{\log |C_{m,n}|}{\lambda_m + \mu_n} = -2\beta$$

(Sufficiency). Suppose (3.7) is true and (3.6) is not true. Thus $\sum_{m,n=o}^{\infty} M(C_{m,n}\alpha_{m,n}; \sigma_1, \sigma_2)$ converges for each σ_1, σ_2, but

$$\limsup_{m+n\to\infty} \frac{\log |C_{m,n}|}{\lambda_m + \mu_n} \neq -\infty$$

There exist sequences $\{m_k\}$, $\{n_\ell\}$, such that

$$\log |C_{m_k, n_\ell}| > \alpha(\lambda_{m_k} + \mu_{n_\ell}) \quad ; \alpha > -\infty$$

By (3.7), one may find σ_1, σ_2, such that

$$\liminf_{m+n\to\infty} \frac{\log M(\alpha_{m,n}; \sigma_1, \sigma_2)}{\lambda_m + \mu_n} > 2 - \alpha$$

Therefore

$$\frac{\log M(C_{m_k, n_\ell}\alpha_{m_k, n_\ell}; \sigma_1, \sigma_2)}{\lambda_{m_k} + \mu_{n_\ell}} > 2 \quad , \quad k, \ell \geqslant o$$

and this contradicts lemma 3.1 . The proof is now complete.

Theorem 3.4. *Let* $\{\alpha_{m,n} : m, n \geqslant o\}$ *be an absolute base in X. Then* $\{\alpha_{m,n}\}$ *is propre base if and only if* (3.2) *and* (3.7) *hold good. Theorem 3.4. is the main result of this paper, follows by combining theorem 3.2 and lemma 3.3.*

REFERENCES

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[2] P.K. KAMTHAN and S.K. SINGH GAUTAM: Bases in the space of analytic Dirichlet transformations. Collec. Math. (1972), 9 - 16.

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