AN EXISTENCE THEOREM FOR AN IMPLICIT NONLINEAR EVOLUTION EQUATION

by

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ABSTRACT.

In this paper we prove the existence of a solution for a non linear evolution equation of the form:

$$\frac{d}{dt} B(u(t)) + A(t, u(t)) \ni f(t)$$

Where A and B are nonlinear operators, possibly multivalued.

The proof is based on implicit discretization in time and passing to the limit as the time step goes to zero.

An application to a Stefan problem, arising from the solidification of a metal in a mould, is given.

1. INTRODUCTION.

Let V be a reflexive separable Banach space and H a Hilbert space such that V is densely included in H.

Let us denote by $\| \ \|$ (respectively $| \ |$) the norm in V (resp. in H) and by V' the topological dual of V. As usually, let $\| \ \|_*$ be the dual norm in V'. By identifying H to its topological dual space H', we have $V \subset H \subset V'$.

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Let us consider φ_B a proper convex lower semicontinuous function from H to $(-\infty, \infty]$ which is supposed to be continuous at some point of V. Its subdifferential $\partial \Phi_B$ is the maximal monotone operator given by for example Brezis [4]):

$$v \in \partial \Phi_B (u) \text{ if and only if } (v, w-u) \leqslant \Phi_B (w) - \Phi_B (u) \text{ for all } w \in H. \tag{1.1}$$

Let B be the multivalued operator from V to V′ defined by:

$$B = i^* \circ \partial \Phi_B \circ i \tag{1.2}$$

Where $i : V \to H$ denotes the injection from V into H and i^* its dual map.
Finally, let $A(t, \cdot)$ be a family of operators from V to V′.
This paper deals with the following nonlinear evolution equation:

$$\frac{d}{dt} (B(u(t)) + A(t, u(t)) \ni f(t) \tag{1.3}$$

Where f is a given function from $[0, T]$ to V'

The purpose is to prove the existence of a solution for the initial value problem:

Find two functions u and v such that:

$$\frac{d}{dt} v(t) + A(t, u(t)) \ni f(t) \qquad \text{a. e. on } [0, T] \tag{1.4}$$

$$v(t) \in B(u(t)) \qquad \text{a. e. on } [0, T] \tag{1.5}$$

$$v(0) = v_0 \tag{1.6}$$

Where v_0 is arbitrarily given in the range of B.

For this, it is assumed that i is a compact map, $\partial \Phi_B$ is a strongly monotone bounded operator on V and $A(t, \cdot)$ are uniformly bounded weakly coercive operators which define a map from $L^p(0, T; V)$ to $L^{p'}(0, T; V')$, $\frac{1}{p} + \frac{1}{p'} = 1$, pseudomonotone on the space $W(0, T)$ (*) (Lions [10]).

Implicit evolution equations like (1.3) have been considered in many articles (Grange and Mignot [9], Showalter [14], Calvert [5], Barbu [5], Di Benedetto and Showalter [6], Bermudez and Saguez [3]).

(*) $W(0,T) = \left\{ u \in L^p(0, T; V) \,;\, \frac{du}{dt} \in L^{p'}(0, T; V') \right\}$

The existence of a solution is proved in Grange and Mignot [9] and Barbu [1], essentially assuming that A is the subdifferential of a coercive continuous convex function on V and then, in particular, independent of t.

More recently, Di Benedetto and Showalter [6] have obtained the existence for a more general case in which A is also independent of t and A and B are maximal monotone operators, but only one of them has to be a subdifferential.

On the other hand, in Bermudez and Saguez [3], A is taken to be the sum of a subdifferential and another time dependent operator which is dominated by A. Existence of a solution for (1.4) - (1.6) is proved by relaxing the coerciveness of A but assuming $\partial \Phi_B$ is strongly monotone.

The method used in this paper is similar to that used in Grange and Mignot [9] or in Bermudez and Saguez [3]. It consists on introducing an implicit time-discretized problem and then passing to the limit as the time step goes to zero.

By not requiring A to be a maximal monotone operator, new applications to nonlinear partial differential equations can be obtained. As an example, a Stefan problem in a nonhomegeneous medium, with thermal parameters depending on the temperature and convective heat transfer in the liquid phase, is considered.

2. THE MAIN RESULT

We do the following assumptions:

The injection of V into H is compact. (A1)

$\partial \Phi_B \cdot i$ is bounded. (A2)

1. $\partial \Phi_B$ is strongly monotone on i (V) in the following sense: there exists a positive constant β and a real number p, $1 < p < \infty$, such that:

$$(v_1 - v_2, u_1 - u_2) \geq \beta \mid u_1 - u_2 \mid^p, v_j \in (\partial \phi_B \cdot i)(u_j), u_j \in V, j = 1, 2. \quad (A3)$$

2. There exists a positive constant M and a real number r, $1 \leq r < \infty$, such that.

$$\| (\partial \Phi_B)^{-1} (v_1) - (\partial \Phi_B)^{-1} (v_2) \|_* \leq M \| v_1 - v_2 \|_*^r, \text{with } v_1, v_2 \in H$$

There exist two positive constants M_1 and M_2 such that

$$||A(t,v)||_* \leqslant M_1 ||v||^{p-1} + M_2 \tag{A4}$$

The operator A defined by (Au) (t) = A(t,u (t)) a. e. on [0, T], maps the space L^p (0, T; V) into the space $L^{p'}$ (0, T; V') and has the following continuity property:

$$\text{if } u_j \to u \text{ in } L^p(0,T;V) \text{ weak} \tag{A5}$$

$$u_j' \to u' \text{ in } L^{p'/r}(0,T;V') \text{ weak}$$

$$\text{and } \limsup_{j\to\infty} \int_0^T (A(t,u_j(t)), u_j(t) - u(t))\, dt \leqslant 0$$

$$\text{then } \liminf_{j\to\infty} \int_0^T (A(t,u_j(t)), u_j(t) - v(t))\, dt \geqslant$$

$$\geqslant \int_0^T (A(t,u(t)), u(t) - v(t))\, dt \qquad \text{for all } v \in L^p(0,T;V)$$

A is weakly coercive in the following sense: (A6)

There exists a constant ω such that :

$$\liminf_{||u||_{L^p(0,T;V)} \to \infty} \frac{\int_0^T [\omega |u(t)|^p + (A(t,u(t)), u(t))]\, dt}{||u||^p_{L^p(0,T;V)}} > 0$$

Remark 2.1.

Since φ_B is assumed to be continuous at some point of i(V), it follows from Ekeland and Teman [8] taht $B = \partial(\varphi_B \cdot i)$. Hence A2 implies dom (B) = V and φ_B is finite and continuous on i(V). In particular, it can be supposed that $\varphi_B(0) = 0$.

Remark 2.2

Assumptions A1 and A2 show that B is compact, i.e., maps bounded sets of V into relatively compact sets of V′.

Remark 2.3.

Assumption A4 implies that A is bounded.

The main result of this paper is the following:

Theorem 1. *Under the assumptions (A1) – (A6), for every f given in* $L^{p'}(0, T; V')$ *and* v_0 *given in the range of B, there exist* $u \in L^p(0, T; V)$ *and* $v \in L^\infty(0, T; H')$*, with* $\frac{du}{dt} \in L^{p'/r}(0, T; V')$, $\frac{dv}{dt} \in L^{p'}(0, T; V')$*, such that:*

$$\begin{cases} \dfrac{d}{dt} v(t) + A(t, u(t)) = f(t) & \text{a. e. } t \in [0, T] \quad (2.1) \\ v(t) \in B(u(t)) & \text{a. e. } t \in [0, T] \quad (2.2) \\ v(0) = v_0 & \quad (2.3) \end{cases}$$

Remark 2.4

The equality (2.3) makes sense in V′ because $v \in C([0, T]; V')$.

3. PROOF OF THEOREM 1.

Let N be a positive integer and $k = \frac{T}{N}$. For $(a^0, \ldots, a^N) \in E^{N+1}$, E being a Banach space, we denote by $\pi_k(a)$ the step function on [0, T] defined by:

$$\pi_k(a)(t) = a^{n+1} \text{ if } nk < t \leqslant (n+1)k, \quad n = 0, \ldots, N\text{-}1 \tag{3.1}$$

$$\pi_k(a)(0) = a^0 \tag{3.2}$$

and by $\Lambda_k(a)$ the continuous function from [0, T] to E, linear on each interval $[nk, (n+1)k]$ and such that:

$$\Lambda_k(a)(nk) = a^n \qquad n = 0, \ldots, N \tag{3.3}$$

Finally, we denote by $\nabla_k \pi_k(a)$ the step funcion $\frac{d}{dt} \nabla_k(a)$.

Consider the following descretized problem:

To find $(u_k^0, \ldots, u_k^N) \in V^{N+1}$ such that:

$$\begin{cases} \dfrac{v_k^{n+1} - v_k^n}{k} + A_k^n u_k^{n+1} = f_k^n \; ; \; v_k^{n+1} \in B u_k^{n+1}, \quad n=0, \ldots, N-1 & (3.4) \\ v_k^0 = v_0 & (3.5) \end{cases}$$

Where

$$f_k^n = \frac{1}{k} \int_{nk}^{(n+1)k} f(t)\, dt, \quad , \qquad f_k^n \in V' \tag{3.6}$$

and A_k^n the operator from V to V′ defined by:

$$A_k^n u = \frac{1}{k} \int_{nk}^{(n+1)k} A(t, u)\, dt. \tag{3.7}$$

In order to prove the existence of a solution for this discretized problem, we state the following lemmas:

Lemma 1. *The operator* A_k^n *defined by (3.7) verifies:*

(i) $$\| A_k^n u \|_* \leq M_1 \| u \|^{p-1} + M_2$$

(ii) A_k^n *is pseudomonotone from* V to V′, i. e. (Lions [10]):

If $u_j \rightharpoonup u$ in V weak,

and $$\limsup_{j\to\infty} (A_k^n u_j, u_j - u) \leq 0$$

then $\lim_{j\to\infty} \inf (A_k^n u_j, u_j - v) \geq (A_k^n u, u - v)$, for all $v \in V$.

(iii) $$\liminf_{\|u\|\to\infty} \frac{\omega |u|^p + (A_k^n u, u)}{\|u\|^p} \geqslant \alpha > 0.$$

Proof: Firstly i) and iii) follows immediately from A4 and A6.

To prove ii) observe that, from the compactness assumption A1, we have $|u_j - u| < k$ for j large enough.

Moreover, we may suppose that u_j is different from u for all j. These facts allow to define the following functions:

$$\tilde{u}_j(t) = \begin{cases} u_j - \dfrac{t - nk}{|u_j - u|}(u - u_j) & \text{if } nk - |u_j - u| \leqslant t \leqslant nk \qquad (3.8) \\ u_j & \text{if } nk \leqslant t \leqslant (n+1)k \\ u_j - \dfrac{t - (n+1)k}{|u_j - u|}(u_j - u) & \text{if } (n+1)k \leqslant t \leqslant (n+1)k + |u_j - u| \\ u & \text{otherwise.} \end{cases}$$

Let $\tilde{u}$ be defined by $\tilde{u}$ (t) = u for all $t \epsilon [0, t]$.

Then it is not difficult to see that $\tilde{u}_j \longrightarrow \tilde{u}$ in $L^\infty(0, T; V)$ weak star and $\tilde{u}'_j \longrightarrow \tilde{u}'$ in $L^q(0, T; V')$ for arbitrary $q \epsilon (1, \infty)$. In particular,

$$\tilde{u}_j \rightharpoonup \tilde{u} \text{ in } L^p(0, T; V) \text{ weak.} \qquad (3.9)$$

and

$$\tilde{u}'_j \rightharpoonup \tilde{u}' \text{ in } L^{p'/r}(0, T; V') \text{ weak.} \qquad (3.10)$$

Moreover, by using the assumption A4, we obtain:

$$\limsup_{j \to \infty} \int_0^T A(\tilde{u}_j, \tilde{u}_j - \tilde{u})\, dt \leqslant \qquad (3.11)$$

$$\leqslant \limsup_{j\to\infty} \int_{nk-|u_j-u|}^{nk} (A(t,\tilde{u}_j(t)), \tilde{u}_j(t) - u_j)\, dt \; +$$

$$+ \limsup_{j\to\infty} \int_{nk}^{(n+1)k} (A(t,u_j), u_j - u)\, dt \; +$$

$$+ \limsup_{j\to\infty} \int_{(n+1)k}^{(n+1)k+|u_j-u|} (A(t,\tilde{u}_j(t), \tilde{u}_j(t) - u)\, dt \leqslant 0$$

Now, the assumption A5 implies:

$$\liminf_{j\to\infty} \int_0^T (A(t,\tilde{u}_j(t)), \tilde{u}_j(t) - \tilde{v}(t))\, dt \tag{3.12}$$

$$\geqslant \int_0^T (A(t,\tilde{u}(t)), \tilde{u}(t) - \tilde{v}(t)\, dt \text{ for all } \tilde{v} \in L^p(0,T;V).$$

For every v in V, let $\tilde{v}$ be defined by:

$$\tilde{v}(t) = \begin{cases} v & \text{if } nk \leqslant t \leqslant (n+1)k \\ u & \text{otherwise} \end{cases} \tag{3.13}$$

By putting this $\tilde{v}$ in (3.12) we obtain:

$$\liminf_{j\to\infty} (A_k^n u_j\, u_j - v) \geqslant (A_k^n u, u - v) \tag{3.14}$$

Which completes the proof.

Remark 3.1.

From i) and iii) in the lemma 1 we can deduce the existence of a constant γ, independent of k and n, such that:

$$\omega\, |u|^p + (A_R^n u, u) \geqslant \alpha\, \|u\|^p - \gamma$$

Lemma 2. *The discretized problem (3.4) (3.5) has a solution, for k sufficiently small.*

Proof. We shall prove that the problem (3.4) has a solution, for v_k^n and f_k^n given in V' and k sufficiently small.

For this observe that u_k^{n+1} is solution of (3.4) if and only if it is solution of the following variational inequality:

To find u_k^{n+1} in V such that:

$$(A_k^n u_k^{n+1}, v-u_k^{n+1}) + \frac{1}{k} \Phi_B (i(v)) - \frac{1}{k} \Phi_B (i(u_k^{n+1}))$$

$$\geqslant (f_k^n + \frac{1}{k} v_k^n, v-u_k^{n+1}) \quad \text{for all } v \epsilon V. \tag{3.15}$$

The assumption A3 implies that if $z \epsilon V$ and $w \epsilon \partial \Phi_B(z)$ then (see Rockafellar [13])

$$(\Phi_B \circ i)(z') \geqslant (\Phi_B \circ i)(z) + (w, z'-z) + \frac{1}{2} \beta |z'-z|^p \tag{3.16}$$

and the lemma 1 gives:

$$\lim_{||u|| \to \infty} \frac{(A_k^n u, u) + \frac{1}{k} (\Phi_B \circ i)(u)}{||u||} = \infty \tag{3.17}$$

The existence of a solution of (3.15) is now deduced from Lions [10, th. 8.5 II].

4. A PRIORI ESTIMATES.

In this paragraph the following a priori estimates will be obtained:

$$|u_k^n| \leqslant C_1 \tag{4.1}$$

$$k \sum_{n=0}^{s} \| u_k^{n+1} \|^p \leqslant C_2 \qquad , 0 \leqslant s \leqslant N-1 \tag{4.2}$$

$$k \sum_{n=0}^{s} \left\| \frac{u_k^{n+1} - u_k^n}{k} \right\|_*^{p/r} \leqslant C_3 \qquad 0 \leqslant s \leqslant N-1 \tag{4.3}$$

were, C_1, C_2 and C_3 denote some positive constants independent of k, n and s.

Multiplying (3.4) by u_k^{n+1} we obtain:

$$\frac{1}{k}(v_k^{n+1} - v_k^n \;, u_k^{n+1}) + (A_k^n u_k^{n+1}, u_k^{n+1}) = (f_k^n \;, u_k^{n+1}). \tag{4.4}$$

Moreover, we have:

$$\sum_{n=0}^{s} (v_k^{n+1} - v_k^n \;, u^{n+1}) \geqslant \Phi_B^* (v_k^{s+1}) - \Phi_B^* (v^0) \tag{4.5}$$

because $v_k^n \in Bu_k^n$, n = 0, 1, . . ., N.

This inequality implies:

$$\sum_{n=0}^{s} (v_k^{n+1} - v_k^n \,, u_k^{n+1}) \geqslant \frac{\beta}{2} | u_k^{s+1} |^p - \Phi_B^* (v^0) \tag{4.6}$$

because (Ekeland and Teman [8])

$$\Phi_B^* (v_k^{s+1}) + \Phi_B (u_k^{s+1}) = (v_k^{s+1}, u_k^{s+1}) \tag{4.7}$$

and, on the other hand, from (3.16),

$$(v_k^{s+1}, u_k^{s+1}) \geqslant \frac{\beta}{2} | u_k^{s+1} |^p + \Phi_B (u_k^{s+1}) \tag{4.8}$$

Thus, by adding (4.4) from n = 0 to n = s, we obtain:

$$\frac{\beta}{2} | u_k^{s+1} |^p + k \sum_{n=0}^{s} (A_k^n u_k^{n+1}, u_k^{n+1}) \leqslant k \sum_{n=0}^{s} (f_k^n, u_k^{n+1}) + \Phi_B^* \;(v^0) \tag{4.9}$$

from which it follows that:

$$|u_k^n| \leqslant C_1 \tag{4.10}$$

$$k \sum_{0}^{s} ||u_k^{n+1}||^p \leqslant C_2 \tag{4.11}$$

for $k \in (0, \bar{k}]$, $\bar{k} < \frac{\beta}{2\omega}$, by using the discrete Gronwall's lemma, the lemma 1 and the remark 3.1.

Moreover, from (4.11) and the remark 2.3 we obtain:

$$k \sum_{n=0}^{s} \left\| \frac{v_k^{n+1} - v_k^n}{k} \right\|_*^{p'} \leqslant C_5 \tag{4.12}$$

By using (A3) we finally deduce the estimate (4.3).

5. Passing to the limit

From the a priori estimates obtained in the previous paragraph we deduce the existence of subsequences, still denoted u_k, v_k, such that:

$\pi_k u_k \longrightarrow u$ in $L^p(0,T;V)$ weak and $L^p(0,T;H)$ strong (5.1)

$\pi_k v_k \longrightarrow v$ in $L^p(0,T;H)$ weak star and $L^{p'}(0,T;V')$ strong (5.2)

$\nabla_k \pi_k u_k \longrightarrow \frac{du}{dt}$ in $L^{p'/r}(0,T;V')$ weak (5.3)

$\nabla_k \pi_k v_k \longrightarrow \frac{dv}{dt}$ in $L^{p'}(0,T;V')$ weak (5.4)

$\pi_k A_k u_k \longrightarrow X$ in $L^{p'}(0,T;V')$ weak (5.5)

$\Lambda_k v_k \longrightarrow v$ in $C([0,T];V')$ (5.6)

because A and $\partial \Phi_B \circ i$ are bounded and i is compact. In particular, (5.6) follows from the Ascoli's theorem.

By passing to the limit in (3.4) we obtain

$$\frac{dv}{dt} + X = f \tag{5.7}$$

The next step is to prove that $v(t) \in B(u(t))$ and $X(t) = A(t, u(t))$ a.e. on (O, T).

For this, first note that from (5.1) and (5.2) we have

$$\lim_{k \to 0} \int_0^T (\pi_k v_k, \pi_k u_k)\, dt = \int_0^T (v(t), u(t))\, dt, \tag{5.8}$$

which implies $v(t) \in B(u(t))$, by using lemmas 2 and 3 of Grange and Mignot [9].

Multiplying (3.4) by u_k^{n+1} and then adding from $n = 0$ to $n = N\text{-}1$ we obtain:

$$(\Phi_B \circ i)^* (v_k^N) + \int_0^T (A(t, \pi_k u_k), \pi_k u_k)\, dt \tag{5.9}$$

$$\leqslant (\Phi_B \circ i)^* (v_0) + \int_0^T (f, \pi_k u_k)\, dt.$$

and letting $k \longrightarrow 0$, (5.9) gives

$$(\Phi_B \circ i)^* v(T) + \limsup_{k \to 0} \int_0^T (A(t, \pi_k u_k), \pi_k u_k)\, dt \leqslant \tag{5.10}$$

$$\leqslant (\Phi_B \circ i)^* (v_0) + \int_0^T (f, u)\, dt$$

On the other hand, we are going to show the equality:

$$(\Phi_B \circ i)^* v(T) - (\Phi_B \circ i)^* (v_0) = \int_0^T \left(\frac{dv}{dt}, u\right) dt \tag{5.11}$$

We have

$$v \in B u \iff u \in \partial (\Phi_B \circ i)^* (v) \tag{5.12}$$

and by using the chain rule we deduce

$$\frac{d}{dt}(\Phi_B \circ i)^*(v) = \left(\frac{dv}{dt}, \partial(\Phi_B \circ i)^*(v)\right)_{V'V} = \left(\frac{dv}{dt}, u\right)_{V'V} \quad (5.13)$$

Finally, integrating (5.13) between 0 and T we obtain (5.11).

By replacing the inequality (5.11) in (5.10) it follows that

$$\limsup_{k \to 0} \int_0^T (A(t, \pi_k u_k), \pi_k u_k)\, dt \leqslant \int_0^T -\left(\frac{dv}{dt}, u\right) dt + \int_0^T (f, u) dt = \int_0^T (X, u)\, dt$$

which allows to conclude that $X = A(t, u)$, by using the assumption A5.

6. Application

Let Ω be an open bounded set in $\mathbb{R}^N$ with a smooth boundary τ. We consider the following parabolic equation:

$$\frac{\partial}{\partial t} v(x,t) - \sum_{j=1}^{N} \frac{\partial}{\partial x_j} (A_j(x, t, u, \nabla u)) = h(x,t) \quad (6.1)$$

a. e. in $Q = \Omega \times]0, T[$

$$v(x,t) \in \beta(x, u(x,t)) \quad (6.2)$$

a. e. in Q

with the boundary condition:

$$\sum_{j=1}^{N} A_j(x, t, u, \nabla u)\, \upsilon_j + a(x,t)\, u(x,t) = g(x,t) \text{ on } \Sigma = L^{p'}\,]0, T[\quad (6.3)$$

and the initial condition:

$$v(x, 0) = v_0(x) \qquad \text{in } \Omega. \quad (6.4)$$

A particular case of (6.1) - (6.4) arises, for example, when looking for a variational formulation of a Stefan problem, modelling heat transfer during solidification in a nonhomogeneous medium with heat capacity and thermal conductivities dependent on the temperature, and with convection in the liquid phase (see Bermúdez and Durany [2] and Durany [7]).

More precisely, this problem corresponds to the following choices:

β (x, u): Specific enthalpy per unit of volume at the point x, as a multivalued function of the temperature.

$A_j (x, t, u, \nabla u) = \sum_{i=1}^{N} k_{ji} (x, u) \frac{\partial u}{\partial x_i} + w_i (x, t) (u-u_M)^+$ where k_{ji} represents the thermal conductivity tensor, $\vec{w}$ is the velocity field, supposed to be divergence free and u_M denotes the melting point.

In order to apply the theorem 1 to (6.1) - (6.4), we suppose the following assumptions:

a) β is the subdifferential of a normal convex integrand on $\Omega \times \mathbb{R}$ (Rockafellar [12]), i. e. $\beta (x, u) = \partial_u \Psi(x, u)$ a. e. in Ω.

b) $|v| \leq a_1 |w|^{p-1} + a_2$, for all $w \in \mathbb{R}$, $v \in \beta (x, w)$ for some constants a_1, a_2 and $1<p<\infty$.

c) There exists a constant $b > 0$ such that:

$$(v_1 - v_2)(u_1 - u_2) \geq b |u_1 - u_2|^p$$

for all $u_i \in R$, $v_i \in \beta (x, u_i)$, $i = 1, 2$.

d) The functions

$$A_j: (x, t, \eta, \xi) \in \Omega \times]0, T[\times \mathbb{R} \times \mathbb{R}^N \longrightarrow A_j (x, t, \eta, \xi) \in \mathbb{R}$$

are measurable in x and t, and continuous in η and ξ a. e. in Q.

e) $|A_j (x, t, \eta, \xi)| \leq [|\eta^{p-1}| + |\xi|^{p-1} + k(x, t)] c$

where c is a positive constant and $k \in L^{p'} (Q)$.

f) $\liminf_{|\xi| \to \infty} \dfrac{\sum_{j=1}^{N} A_j(x, t, \eta, \xi)\, \xi_j}{|\xi|^p} > 0.$

g) $\sum_j (A_j(x, t, \eta, \xi) - A_j(x, t, \eta, \xi^*))(\xi - \xi^*) > 0$

for all $\xi, \xi^* \in \mathbb{R}^N$, $\xi \neq \xi^*$. a.e. in Q.

h) $a \in L^\infty(\Sigma)$, $a \geqslant 0$ a.e. on Σ

$$h \in L^\infty(0, T; L^{p'}(\Omega)); \frac{dh}{dt} \in L^{p'}(0, T; L^{p'}(\Omega))$$

$$g \in L^\infty(0, T; L^{p'}(\tau)); g' \in L^{p'}(0, T; L^{p'}(\tau))$$

$$v_0 \in L^{p'}(\Omega).$$

Taking $V = W^{1,p}(\Omega)$ and $H = L^p(\Omega)$, it can be shown (Lions [10]) that the operators B defined by:

$$B(u)(x) = \beta(x, u(x)) \qquad \text{a.e. in } \Omega \qquad (6.5)$$

and A given by:

$$(Au, v) = \sum_{j=1}^{N} \int_Q A_j(x, t, u, \nabla u) \frac{\partial v}{\partial x_j} \, dxdt + \int_\Phi \Sigma a(x, t)\, u\, v\, d\Sigma \quad (6.6)$$

satisfy the assumptions A2 to A6 and hence the theorem 1 gives the existence of the solution of (6.1) - (6.4).

Remark 6. 1. A problem similar to (6.1) - (6.4) has been recently considered by Niezgodka and Pawlow [11] who have proved an existence theorem for it. However their result cannot be applied to our situation because they assume that A_j is C^2 in x and this hypothesis is not generally satisfied by Stefan problems in *nonhomogeneous* media. On the other hand, in [11] A_j is supposed to be independent on ∇u and consequently convective heat transfer cannot be considered.

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