GENERALIZING NORMALITY FOR OPERATORS ON BANACH SPACES: HYPONORMALITY. I.

by

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0. INTRODUCTION.

The class of normal operators on Hilbert spaces played a very special role in the theory of linear operators. This is due to the fact that for this class there is an almost complete theory. Thus, attempts to extend one or another property of normal operators is quite natural. In particular, to define the notion of hermitian operator (or hermitian element of a Banach algebra) it is of fundamental importance to extend directly in the theory known for the Hilberts space case. The first successful attempt to define the notion of hermitian element of a Banach algebra was made by I. Vidav (1956). Shorthly after this G. Lumer (1961) introduced the very useful notion of a semiinner product which permits to define this notion of hermitian operator in much the same way as for the case of Hilbert spaces. The purpose of the present paper is to point out several possibilities to define the closs of hyponormal operators in the case of Banach algebras (as well as for the special case of A = L(X), the Banach algebra of all bounded linear operators on a complex Banach space). As it is known the class of hyponormal operators on Hilbert space has some very interesing properties (and this is clear, to some extent, from the numerous papers dedicated to the stüdy of this class). It is worth mentioning that this class was introduced independently by the american mathematician P.R. Halmos and by the societ-russian mathematician A.T. Taldykin as a result of very different investigations in operator theory. P.R. Halmos was led to consider this class from the problems connected with some researches of F. Riesz, J. Radon, J. Schauder and S. Nikolskiĭ.

In what follows we propose several definitions of hyponormality, which are, to some extent, the analogue of the hyponormality notion as it known for Hilbert space operators. The first definition we suggest is for the case of Banach algebras elements and uses the notion of hermitian element introduced by I. Vidav. Next we consider the above definition in the case of the algebra $A = L(X)$ of all bounded linear operators on a complex Banach spaces. There we obtain, under some assumptions, results which are similar to the case of Hilbert space operators.

Anhoter definition we propose is suggested by a conjecture of S. Friedland 1982) related to some functions associated with operators in L(X), namely the following one : if T is in L(X) then for each $x \neq o$ we can define (in the usual way) the element $e^{tT}x$. Then we consider the function

$$g_{T,x}(t) = \|e^{tT}x\|.$$

The basic results of S. Friedland is that for operators on Hilbert space a necessary and sufficient condition for the normality of T is that the family of functions

$$(g_{T,x}(t),\ g_{T*x}(t))$$

must be a family of convex functions (on R). The conjecture of S. Friedland is taht T is hyponormal if and only if

$$(g_{T,x}(t))$$

consists of convex functions. We prove this conjecture using some of Berberian's constructions, which are, as it is pointed out in the paper of W. Luxemburg (11) somehow nonstandard. Further we discuss other possible definitions.

1. Hyponormal elements of Banach algebras.

Let us consider A a complex Banach algebra with identity (which is denoted by 1). Following I. Vidav we say an element h of A is hermitian if for all t in R,

$$\|e^{ith}\| = 1.$$

Using the hermitian elements of A we introduce the following class of elements.

Definition 1.1. The element a in A is said to be descomposable if

$$a = h + ik$$

were h,k are hermitian elements.

The class of normal elements in Banach algebras can be defined in two distinct ways.

Definition 1.2. The element n of A is said to be normal if the following conditions are satisfied:

1. n is decomposable, $n = h + ik$,
2. $hk = kh$.

Definition 1.3. The element n of A is said to be normal if the following conditions are satisfied:

1. n is descomposable, $n = h + ik$,
2. for all integers p and q the elements $h^p k^q$ are hermitian and

$$h^p k^q = k^q h^p.$$

The fact that the bove definition introduces two different classes of elements follows from the example of G. Lumer of a hermitian operator h with the property that h^2 is not hermitian.

Now we are ready to introduce the notion of hyponormal element in a Banach algebra.

Definition 1.4. The elements a is said to be hyponormal if the following conditions are satisfied:

1. a is decomposable, $a = h + ik$,
2. the element

$$i\,(hk\text{-}kh)$$

has the spectrum in $[0, \infty)$.

If $A = L(H)$ then the above definition reduces to the well known definition of hyponormal operators be cause of the fact that the above element is supposed to be with the spectrum in $[0, \infty)$ is equivalent with its positivity.

Definition 1.5. The element a of A is said to be co-hyponormal if the following conditions are satisfied:

1. a is decomposable, $a = h + ik$,
2. the element $\bar{a} = h\text{-}ik$ is hyponormal.

From just the definition we get easily the following.

Proposition 1.6. If a is a hyponormal element of the Banach algebra A then

$$ua + v$$

is hyponormal for all complex numbers u, v.

The following proposition gives a method to obtain new hyponormal elements from the known ones.

Proposition 1.7. Let a be a hyponormal element of A ,$a = h + ik$. Suppose further thak k^{-1} exists and is hermitian. Then the element

$$b = h\text{-}ik^{-1}$$

is hyponormal.

Proof. First we compute the commutador which defines the hyponormality:

$$i(h(-k^{-1}) - (-k^{-1})h) = k^{-1}\,(i(hk\text{-}kh))k^{-1}.$$

This is a hermitian element and we must show that its spectrum is in $[0, \infty]$.

Let us consider s in $(-\infty, o)$ and then it is in the resolvent set of i(hk-kh). Thus we have,

$$k^{-1}i(hk\text{-}kh)\,)k^{-1} = k^{-1}(i(hk\text{-}kh) - s + s\text{-}sk^2)k^{-1} =$$
$$= k^{-1}(\,(i(hk\text{-}kh)-s)\,)\,(1 + (i(hk\text{-}kh)-s)^{-1}s(1\text{-}k^2)\,)k^{-1}.$$

Now the element i(hk-kh)–s is hermitian so we have a growth condition satisfied (11). This condition is of the form

$$k_1/|s|$$

and since we may suppose without loss of generality that $1\text{-}k^2$ has a norm smaller than the (universal) constant k_1 we obtain finally that

$$\|(i(hk\text{-}kh)-s)^{-1}\ \ s\ \ (1\text{-}k^2)\| < 1.$$

This implies clearly that s is the resolvent of the above element ant the assertion is proved.

Remark 1.8. If L(X) = A, where X is a Hilbert space then the proof of the above assertion is almost trivial. Of course the condition k^{-1} is automatically satisfied. The following example shows that it is a necessary hypothesis.

Example 1.9. Let us consider the Banach algebra of all 3 x 3 matrices say A with the norm is defined as follows. We consider each matrix acting as an operator on the C^3 equiped with the following norm:

$$\|(x,y,z)\| = \sup_{t \in R} |e^{it}x + y + e^{-it}z|.$$

The matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is hermitian (with respect to the norm defined by the norm on C^3 on the algebra A. The operator with the matrix

$$\begin{pmatrix} -3/2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

is clearly hermitian and invertible. The inverse is not hermitian since its spectral radius is strictly smaller than the norm. We present now a type of convergence for elements in Banach algebras.

Definition 1.10. Let (a_n) be a sequence of decomposble elements of the Banach algebra A. We say that (a_n) is bounded–*– convergent to the element a of A if the following conditions are satisfied:

1. (a_n) is a bounded sequence, a is decomposable,
2. for each state φ of A,

$$\lim \varphi(a_n) = \varphi(a),$$
$$\lim \varphi(\bar{a}_n) = \varphi(\bar{a}).$$

Then we have the following result.

Proposition 1.11. The set of all hyponormal elements of A is closed for the bounded– * – convergence.

Remark 1.12. If may be of interest to know the analogue of the strong convergence from the space of operators on a Banach space in the case of Banach algebras.

It is a well known fact that if T is a hyponormal operator on a Hilbert space then the following inequality holds: for any z in the resolvent set

$$\|(z-T)^{-1}\| = 1/d(z,\sigma(T))$$

where d( ) is the distance between z and the set $\sigma(T)$. The class of hermitian elements of the algebra of all bounded operators on a Hilbert space can be characterized using the functions

$$\|(T-z)^{-1}\|$$

for z on the imaginary axis. This result can be extended to hermitian elements of Banach algebras.

Proposition 1.13. The element h of the Banach algebra A is hermitian if and only if for each complex number of the form z = it the following inequality holds:

$$\|(h-z)^{-1}\| = d(z, \sigma(h)).$$

Proof. Let φ be a state of A and thus if h is a hermitian element we have

$$|\varphi(h-z)| = |\varphi(h) - z| \geqq |t|$$

and thus

$$\|(h-z)^{-1}\| = 1/d(z, \sigma(h)).$$

The converse assertion follows exactly as in (5) and thus we omit it.

Remark 1.14. Some improvements of the results in (5) are given in (14), (13).

Remark 1.15. From the example 1.9 we see that the G_1-property is not valid for hermitian operators. It is of interest to know if G_1-property holds for hyponormal (or at least for hermitian operators) in the case ofsome usefull algebras, such as the algebra of all bounded linear operators on uniformly convex spaces.

2. Hyponormality for operators on Banach spaces.

Let us consider X be a complex Banch space and A = L(X), be the Banach algebra of all bounded linear operators on X. An element T in L(X) is said to be hermitian if the following equivalent conditions hold:

1. for all x in X and t in R, $\|e^{itT}x\| = \|x\|$,

2. if [,] is them semi-inner product on X (in the sense of G. Lumer (1961)) then

$$[Tx,x]$$

is a real-valued function on X,

3. for $t \in R$, $t \to 0$

$$\| I + itT \| = 1 + 0(t).$$

We say that an element T in L(X) is decomposable if it is of the form

$$T = h + ik$$

where h,k are hermitian elements of L(X).

Remark 2.1. There exist Banach spaces for whith the set of all hermitian element in L(X) reduces to the trivial family $(tI)_{t \in R}$. Such an example is the space of all lipschitz functions of order $\alpha \in (0,1)$.

The class of hyponormal operators is defined as follows.

Definition 2.2. The element T in L(X) is a hyponormal operator on X if the following conditions are satisfied:

1. T is decomposable, $T = h + ik$,
2. for each x in X,

$$[i(hk-kh)x,x] \geqq 0.$$

If T in L(X) is decomposable we say that $\overline{T} = h\text{-}ik$ is the X-adjoint of T

Definition 2.3. We say that the sequence of decomposable operators (T_n) is X-strongly $-*-$convergent to the decoposable operator T if the following conditions are satisfied:

1. $\sup \|T_n\| < \infty$,
2. for each x in X,

$$\lim T_n x = Tx \ , \lim \overline{T}_n x = \overline{T}x.$$

Obviously we have the following result.

Proposition 2.4 If (T_n) is X-strongly $-*-$convergent sequence of hyponormal operators then T is a hyponormal operator.

For the following result we suppose that the Banach space X has the following property:

(H^2) if T is a hermitian operator then $[T^2x,x] = 0$ implies that $Tx = 0$.

We conjecture that this hypothesis is satisfied of all Banach paces.

Theorem 2.5. If the Banach space has the (H^2)-property then for any hyponormal operator T in L(X) from $Tx = 0$ it follows that $\bar{T}x = 0$.

Proof. Obviously we have that $\bar{T}Tx = 0$ and thus

$$[\bar{T}Tx,x] = 0$$

which implies that

$$[(h^2 + k^2)x,x] - [(-h^2 + k^2)x,x] = 0$$

or

$$[k^2(x),x] = 0.$$

This gives that $kx = 0$. Since $Tx = h(x) + ik(x) = 0 = h(x)$ we get that $h(x) = 0$.

This obviously implies that $\bar{T}x = 0$.

The result just obtained may be formulated using the following notion.

Definition 2.6. We say that the closed linear subspace X_1 of X is reducing for the decomposable operator T of L(X) if

1. X_1 is invariant under T,
2. X_1 is invariant under $\bar{T}$.

Corollary 2.7. Let T be a hyponormal operator on the Banach space X satisfying the condition (H^2). Then for all z in the complex plane C,

$$N(T\text{-}z)$$

reduces T.

Another very important consequence of the above result is the following one.

Theorem 2.8. Let $T \in L(X)$ be a hyponormal operator on the Banach space X satisfying the property (H^2).

Then

$$\text{Re}\ \sigma_p(T) \subseteq \sigma(\text{Re}\ T).$$

(If T is a decomposable operator $T = h + ik$ then $h = \text{Re}\ T, k = \text{Im}\ T$).

Conjecture 2.9. If T is hyponormal then

$$\text{Re}\ \sigma_{ap}(T) \subseteq \sigma(\text{Re}\ T).$$

Remark 2.10. We can prove this conjecture for the following class of Banach spaces, namely those for which the Berberian's space associated to X satisfies again the (H^2) property. We consider this Berberian space in some detail.

Another related conjecture is suggested by the the so called Weyl's theorem.

Conjecture 2.11. For the hyponormal operator defined on the Banach space X satisfying the condition (H^2) the Weyl's theorem holds. This means the following equality:

$$w(T) = \sigma(T) - \pi_{\infty}(T)$$

where w(T) is the Weyl spectrum of T, $\sigma(T)$ is the spectrum of T and $\pi_{oo}(T)$ denotes the set of all isolated eigenvalues of T which are of finite multiplicity.

3. GROUPS OF OPERATORS AND HYPONORMALITY.

Let us suppose that X is a complex Hilbert space and let T be in L(X). We consider the group of operators

$$(e^{tT})_{t \in R}$$

defined in the usual way. For each $x \neq 0$ in X we define the function (on R) by the formula

$$t \to g_{T,x}(t) = \|e^{tT}x\|.$$

These functions have some interesting properties first noted by S. Friedland (1982). For the proof of the results below we refer to Friedland's paper.

Lemma 3.1. (S. Friedland) If $T \in L(X)$ is hyponormal then

$$g_{T,x}(t)$$

is a convex function.

Related to this result S. Friedland gives the following conjecture.

Conjecture 3.2. (S. Friedland) If $T \in L(X)$ then T is hyponormal if and only if

$$(+) \qquad d^2/dt^2(\log(g_{T,x}(t))(0) \geq 0$$

for all x in X.

From the Lemma 3.1. it is clear that the above condition is sufficient since, as remarked by S. Friedland (+) is equivalent to the inequality

$$d^2/dt^2(\log g_{T,x}(t))(0) = 1/2\ \|x\|^{-2} \left((\langle T^2 + T^2 + 2T\ T)x\ ,x\ \rangle - \right.$$
$$\left. - \langle (T + T)x,x \rangle^2 \right) \geqq 0$$

which is equivalent to

$$\langle (T + T)x,x\rangle^2 \geqq \langle (T^2 + T^2 + 2TT)x,x \rangle \|x\|2$$

We mention now another equivalent form of (+).

Lemma 3.3. (S. Friedland) (Lemma 4 in (3)) If $T = P + iQ$ with P,Q hermitian operators then (+) is equivalent to the inaquality

$$(++) \qquad i/2(QP\text{-}PQ) \leqq (P\text{-}s)^2$$

for all real numbers s.

Another useful result for us in what follows is the following.

Lemma 3.4. (S. Friedland) (Lemma 5 in (3)). If T isatisfies (++) and $Px = sx$ then

$$PQx = QPx.$$

Now we are in the position to prove the conjecture 3.2. First we note that we may suppose without loss of generality that $o \leqq P \leqq 1$. Now since P is a hermitian operator ,$\sigma(P)$ coincides with the approximate point spectrum of P, $\sigma_{ap}(P)$.

Now we can use the Berberian's construction (1), (6) and thus we may assume without loss of generality that the spectrum of P coincides with the point spectrum. For the reader's convenience we indicate very shorthly the construction of Berberian. If X is a complex Banach space then we consider the space m(X) of all sequences $x = (x_n)$,x_n in X and $(\|x_n\|)$ a bounded sequence. If we consider a fixed generalized Banach limit, denoted by glim on m, the space of all bounded sequences of complex numbers, then we set

$$[x,y] = \text{glim}\,(\langle x_n,y_n \rangle).$$

This is clearly a bilinear form on m(X) and

$$N = (x, [x,x] = 0)$$

is an linear subspace of m(X) and, we can consider the quotient space m(X)/N which can be clearly organized as a pre-hilbert space. The completion of this spaces is denoted by h(X). Now every operator T in L(X) induces an operator in L(X)) in a natural way. The corresponding operator is denoted by $\overline{T}$. The mapping

$$T \to \overline{T}$$

is a faithful $*$ –representation and preserves the spectrum. Also the following important property holds: if z is in the approximate point spectrum of T,z is in the point spectrum of $\overline{T}$.

Since the operators in our inequality are hermitian, in order to prove the inequality (+) it suffices to prove it for the corresponding operator in L(h(X)).

Now we can decompose the space h(X) as follows:

$$h(x) = H_1 \oplus H_2$$

where

$H_1 = (x, x \in h(X), Px = sx$ for some s in $\sigma(P)$). It is clear that for all u in H_1 we have

$$\langle i/2(\bar{P}\bar{Q}-\bar{Q}\bar{P})\, u,u \rangle = 0$$

since for each x with the property that Px = sx we have

$$\bar{P}\bar{Q}x = \bar{Q}\bar{P}x$$

according to lemma 3.4. Now if u is in the ortogonal complement of H_1 we have that Pu = 0 and for s = 0 in (++) we get that

$$\langle 1/2(PQ-QP)u,u \rangle \leqq 0.$$

Thus the conjeture 3.2 is proved.

The above conjecture suggests the consideration of a class of operator on Banach spaces which we call again hyponormal and is defined as follows.

Definition 3.5. Let X be a complex Banach space and T be in L(X). We say that T is hyponormal if for each nonzero x in X the function (on R)

$$t \to \|e^{tT}x\|$$

is convex.

The operator T is said to be co-hyponormal if T is hyponormal. From the result given above we see that above classes reduce to the classes of hyponormal, respectively co-hyponormal operators when X is a Hilbert space.

Remark 3.6. In the above definition of hyponormal operators the decomposability is not required.

Remark 3.7. We conjecture that this class of hyponormal operators is quite large compared with the class of hyponormal operators defined using hermitian elements. In particular there may exist Banach spaces having not nontrivial hyponormal operators in the sense of Definition 3.5. (i.e these operators are not of the form zI,z a complex number). Examples of such spaces may be the space of all continuous complex-valued function in $(z,|z| \leqq 1)$ and holomorphic in $(z,|z| < 1)$ or the spaces H^p for $(z,|z| \leqq 1)$ of course for p in $(1,\infty)$, $p \neq 2)$.

4. OTHER POSSIBLE DEFINITIONS OF HYPONORMALITY FOR OPERATORS ON BANACH SPACES.

Suppose that X is a complex Hilbert space and T is a bounded linear operator. If T is hyponormal then the following inequalities hold:

1. $\|T^*x\| \leqq \|Tx\|, \quad x \in X$
2. $\|Tx\|^2 \leqq \|T^2x\| \, \|x\|, \quad x \in X$
3. $\|Tx\|^n \leqq \|T^nx\| \, \|x\|^{n-1} \quad n > 1, x \in X.$

The inequality in 1 may be used to define the class of operators on Banach spaces which are decomposbale and the inequalities in 2 and 3 may be used to define classes of operators on arbitrary Banach spaces. For some results concerning the classes of operators defined using the inequalities in 2 and 3 see (6).

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