k-REFLEXIVITY AND CANONICAL γ-COMPLETION

by

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1. INTRODUCTION

Depending upon the study of the notion of mixed convergence of a net in [1], it has been shown in [2] that a locally convex space with a Schauder decomposition yields a bi-locally convex space (abbreviated bi-l.c. TVS) which we term as a canonical bi-l.c. TVS and of which the strucutal properties could be related with the types of the Schauder decomposition. This note is a continuation of our investigations on the canonical bi-l.c. TVS. Indeed, we introduce the notion of k-reflexive bi-locally convex spaces and characterize the class of such spaces in terms of the subspaces forming the Schauder decomposition. Further, we identify its γ-completion termed as canonical γ-completion, as a vector-valued sequence space; and finally, having characterized k-reflexivity and boundedly complete Schauder decomposition we establish relationship between k- and γ-reflexivities of a canonical bi-locally convex space.

2. BASIC EXCERPTS

In order to facilitate the reading of the subject matter of this paper, we mention in this section the rudiments from the theory of locally convex spaces, Schauder bases and decompositions, bi-locally convex spaces as introduced in [1] and [2], and the theory of vector-valued sequence spaces. However, we refer to [6], [7], [8], [9], [10], [12], [13], [15] and [16] and several refences given therein for various terms, results and detailed discussions of these topics.

Throughout the sequel, we denote by (X,T) a Hausdorff locally convex space (abbreviated l.c. TVS) X equipped with a locally convex topology T which is generated by the family D_T of all continuous seminorms. Its algebraic, sequential and topological duals are respectively denoted by X', X^+ and X^*. Clearly, $X^* \subseteq X^+ \subseteq X'$. An l.c. TVS (x,T) is said to be a *Mazur space* if $X^+ = X^*$. The symbols $\sigma(X,X^*)$, $\tau(X,X^*)$ and $\beta(X,X^*)$ are used to denote the weak, Mackey and strong topologies on X.

A *Schauder base* in an l.c. TVS (X,T) is a sequence $\{x_i\}$ in X with associated sequence of coefficient functionals $\{f_i\}$ contained in X^* such that each x in X is represented uniquely in the form

$$x = \sum_{i\geq 1} f_i(x)x_i,$$

where the infinite series converges in the topology T of X. A Schauder base $\{x_i; f_i\}$ is known as *shrinking* if $\{f_i\}$ is a Schauder base for $(X^*,\beta(X^*,X))$ and *boundedly complete* if for any sequence $\{\alpha_i\}$ of scalars, the series $\sum_{i\geq 1}\alpha_i x_i$ converges to a point of X whenever $\left\{\sum_{i=1}^{n}\alpha_i x_i, n \geq 1\right\}$ is bounded in X.

For our later use in the paper, let us recall the following from 5 (cf. also 16).

Proposition 2.1: The strong dual $(X^*,\beta(X^*,X))$ of a Mazur space (X,T) is a complete locally convex space.

Proposition 2.2: A barrelled space with a Schauder base is a Mazur space.

A notion more general than that of a Schauder base is of *Schauder decomposition* (S.D.) which is defined as a pair $\{M_n; P_n\}$ of a sequence $\{M_n\}$ of subspaces of X and a sequence $\{P_n\}$ of continuous projections from X onto M_n such that each x in X is expressed uniquely in the form $x = \sum_{i\geq 1} x_i$, $P_i(x) = x_i$, $i \geq 1$. We write $S_n = \sum_{i=1}^{n} P_i$, $n \geq 1$ and denote the adjoint maps of P_n and S_n by P_n^* and S_n^* respectively. Then we have

Definition 2.3: An S.D. $\{M_n; P_n\}$ of an l.c. TVS (X,T) is said to be

(i) an *e-Schauder decomposition* (e-S.D.) if the sequence $\{S_n\}$ is equicontinuous;

(ii) *monotone* if for each $p \in D_T$ and $x = \sum x_i$, $x_i \in M_i$

$$p\left(\sum_{i=1}^{m} x_i\right) \leq p\left(\sum_{i=1}^{n} x_i\right), \text{ for all } m,n \text{ with } m \leq n;$$

(iii) *boundedly complete* if for each sequence $\{x_i\}$ with $x_i \in M_i$, $i \geqslant 1$ and $\left\{\sum_{i=1}^{n} x_i : n \geqslant 1\right\}$ being bounded in X, the series $\sum_{i \geqslant 1} x_i$ is convergent in X; and

(iv) *shrinking* if $\{R(P_i^*), P_i^*\}$ is a S.D. for $(X^*, \beta(X^*,X))$, where $R(P_i^*)$ is the range of P_i^*.

We need the following in the sequel (cf. 15, Theorem 2.9, p. 34; and 11).

Proposition 2.4: Let $\{M_n; P_n\}$ be an e-S.D. for an l.c. TVS (X,T). Then the topology T is equivalent to a locally convex topology $\bar{T}$ on X which is generated by the family $\{\bar{p}_\alpha\}$ of seminorms, where $\bar{p}_\alpha(x) = \sup_{n \geqslant 1} p_\alpha(S_n(x))$ for $P_\alpha \in D_T$; and the S.D. $\{M_n; P_n\}$ is monotone in $(X, \bar{T})$.

Next, we have

Definition 2.5: (i) A linear space X equipped with two Hausdorff locally convex topologies T_1 and T_2 on X such that T_1 is finer than T_2 is said to be a *bi-l.c. TVS* and is denoted by X_b, that is, $X_b = (X, T_1, T_2)$. (ii) A net $\{x_\delta : \delta \in \Lambda\}$ in X_b is said to be *γ-convergent* to x (resp. γ-Cauchy) written as $x_\delta \xrightarrow{\gamma} x$, provided $\{x_\delta\}$ is T_1-bounded ant T_2-convergent to x (resp. T_1-bounded and T_2-Cauchy). A bi-l.c.TVS X_b is said to be (iii) *γ-complete* (resp. *γ-sequentially complete*) if every γ-Cauchy net (resp. γ-Cauchy sequence) in X_b, γ-converges to a point of X, and (iv) *quasinormal* (resp. *normal*) if there exists a family D_1 of seminorms generating T_1 such that for each $p \in D_1$, there exists $q \in D_1$, depending on p such that

$$x_\delta \xrightarrow{T_2} x \Rightarrow p(x) \leqslant \varliminf_{\delta} q(x_\delta)$$

$$(\text{resp. } x_\delta \xrightarrow{T_2} x \Rightarrow p(x) \leqslant \varliminf_{\delta} p(x_\delta), \forall p \in D_1)$$

where $\varliminf_{\delta} p(x_\delta)$ is defined as $\sup_{\alpha} \inf_{\delta > \alpha} p(x_\delta)$. A subset B of a bi-l.c. TVS X_b is said to be *γ-dense* (resp. *γ-sequentially dense*) in X_b provided for each x in X, there is a net $\{x_\delta\}$ (resp. sequence $\{x_n\}$) in B such that $x_\delta \xrightarrow{\gamma} x$ (resp. $x_n \xrightarrow{\gamma} x$).

Definition 2.6: Let $X_b = (X, T_1, T_2)$ be a bi-l.c. TVS. An f in X' is said to be a *γ-continuous linear functional* if $x_\delta \xrightarrow{\gamma} x \to f(x_\delta) \to f(x)$ in $\mathbb{K}$ for each γ-convergent net $\{x_\delta\}$ in X_b. The vector space consisting of all γ-continuous linear functionals on X_b is termed as the *γ-dual* of X_b and is *denoted* by X_γ^*. The symbol X_i^* stands for the topological dual of (X,T_i), $i = 1,2$. A bi-l.c. TVS X_b is said to be *saturated* if $X_\gamma^* = X_1^*$.

Proposition 2.7: Let $X_b \equiv (X,T_1,T_2)$ be a bi-l.c. TVS where (X,T_1) is assumed to be a Mazur space. Then

$$X_2^* \subset X_\gamma^* \subset X_1^*.$$

Further, X_2^* is dense in X_γ^* relative to the topology $\beta(X_1^*,X)$ provided X_b is quasinormal.

Proposition 2.8: Let $X_b \equiv (X,T_1,T_2)$ be a quasinormal space. Then for each $p \in D_1$ there is a $q \in D_1$ such that

$$p(x) \leqslant \sup\left\{|f(x)| : f \in v_q^0 \cap X_2^*\right\}$$

$$\leqslant \sup\left\{|f(x)| : f \in v_q^0 \cap X_\gamma^*\right\},$$

where v_q^0 denotes the polar of $v_q = \{x : q(x) \leqslant 1\}$ in X'. In case X_b is normal, then

$$p(x) = \sup\left\{|f(x)| : f \in v_p^0 \cap X_\gamma^*\right\}.$$

Definition 2.9: Let $X_b \equiv (X,T_1,T_2)$ be a quasinormal bi-l.c. TVS such that (X,T_1) is a Mazur space. If

$$X_{\gamma 1}^{**} = (X_\gamma^*, \beta(X_1^*, X)\,|\,_{X_\gamma^*})^* = (X_2^*, \beta(X_1^*, X)\,|\,_{X_2^*})^* = X_{21}^{**},$$

then there exists a well-defined one-to-one linear map from X to $X_{\gamma 1}^{**}$ defined by

$$J(x)(f) = f(x),\ \forall\ f \in X_\gamma^* \text{ and } x \in X.$$

X_b is said to be *γ-semireflexive* if J is onto and *γ-reflexive* if J is a topological isomorphism from (X,T_1) onto $(X_{21}^{**}, \beta(X_{21}^{**}, X_2^*))$ and from (X,T_2) into $(X_{22}^{**}, \beta(X_{22}^{**}, X_2^*))$ where X_{22}^{**} is the topological dual of X_2^* relative to $\beta(X_2^*,X)$.

Proposition 2.10: A bi-l.c. TVS X_b is saturated and γ-semireflrexive if and only if (X,T_1) is semireflexive where (X,T_1) is a Mazur space.

Concerning canonical bi-l.c. TVS, we recall the following from [2].

Definition 2.11: Let $\{M_n;P_n\}$ be an S.D. in an l.c. TVS (X,T). Then the bi-l.c. TVS $X_b = (X,\bar{T},T^*)$ is known as the *canonical* bi-l.c. TVS where $\bar{T}$ and T^* are the locally convex topologies generated respectively by the families $D_{\bar{T}} = \{\bar{p} : p \in D_T\}$ and $D_{T^*} = \{p^* : p \in D_T\}$ of seminorms $\bar{p}$ and p^* defined for x in X with $x = \sum_{i=1}^{\infty} x_i,\ x_i \in M_i,\ i \geq 1$, by

$$\bar{p}(x) = \sup_{n \geq 1} p\Big(\sum_{i=1}^{n} x_i\Big) \text{ and } p^*(x) = \sum_{i \geq 1} \frac{p(x_i)}{2^i}\ .$$

The topology T^* is known as the *canonical topology*.

Proposition 2.12: If $\{M_n;P_n\}$ is an e-S.D. of an l.c. TVS (X,T), then $(X,\bar{T},T^*)$ is a quasinormal canonical bi-l.c. TVS.

Proposition 2.13: Let (X,T) be a Mazur space containing an e-S.D. $\{M_n;P_n\}$. Then $\{P_n^*(X^*). P_n^*\}$ is an s-S.D. of the γ-dual X_γ^* of the canonical bi-l.c. TVS X_b relative to the topology $\beta(X^*, X) \mid X_\gamma^*$. Consequently, an e-S.D. of a Mazur space (X,T) is shrinking if and only if the corresponding canonical bi-l.c. TVS is saturated.

Proposition 2.14: An e-S.D. $\{M_n;P_n\}$ of an l.c. TVS (X,T) is boundedly complete if $X_b \equiv (X,\bar{T},T^*)$ is γ-sequentially complete. Conversely, X_b is γ-sequentially complete if $\{M_n;P_n\}$ is boundedly complete and each M_n is T-sequentially complete.

Regarding vector-valued sequence spaces (abbreviated VVSS), let us consider a vector space X and denote by $\Phi(X)$ the vector space of all sequences $\{x_i\}$, $x_i \in X$, $i \geq 1$ such that $x_i = 0$ for all except finite indices i. A *vector-valued sequence space* $\Lambda(X)$ is defined as the vector space of sequences from X with respect to the usual pointwise addition and scalar multiplication with $\Phi(X) \subset \Lambda(X)$. We denote an arbitrary member of $\Lambda(X)$ by x, where $x = \{x_i\}$, $x_i \in X$, $i \geq 1$; and for x in X, we write δ_i^x to denote the sequence $\{0, 0, \ldots, 0, x, 0, \ldots\}$, x being placed at the i^{th} coordinate. For $i \geq 1$, set

$$N_i = \left\{ \delta_i^x : x \in X \right\}.$$

Then N_i is a subspace of $\Lambda(X)$ for each $i \geq 1$.

Assume now that X is an l.c. TVS and $\mathcal{F}$ is a Hausdorff locally convex topology on $\Lambda(X)$. Then $(\Lambda(X), \mathcal{F})$ is said to be a GK-*space* if the maps $C_i: \Lambda(X) \to X$, $C_i(x) = x_i$, $i \geq 1$ are continuous.

3. k-Reflexive Bi-Locally Convex Spaces

For a Mazur space (X,T) with an e-S.D. $\{M_n; P_n\}$, the pair $\{P_n(X_1^*), P_n^*\}$ is an e-S.D. for the γ-dual X_γ^* of the canonical bi-l.c. TVS $X_b \equiv (X, T, T^*)$, with respect to the topology $\beta(X_1^*, X)|_{X_\gamma^*}$ where $X_1^* = (X,T)^*$ by virtue of Proposition 2.13. This fact is exploited to introduce the k-conjugate spaces as follows:

Definition 3.1: Let $X_b \equiv (X, T, T^*)$ be the canonical bi-l.c. TVS corresponding to a Mazur space (X,T) with an e-S.D. $\{M_n; P_n\}$. Then the canonical bi-l.c. TVS $(X_\gamma^*, \beta(X_1^*, X)|_{X_\gamma^*}, T_1^*)$ defined by the γ-dual X_γ^* of X_b and its e-S.D. $\{P_n^*(X_1^*), P_n^*\}$ is termed as the *first* k-*conjugate* space of X_b and is *denoted* by k-X_b, where T_1^* is the corresponding canonical topology defined by the family $D_{T_1^*}$ of seminorms defined as:

$$D_{T_1^*} = \left\{ p_B^* : B \text{ varies over T-bounded subsets of X and} \right.$$

$$\left. p_B^*(f) = \sum_{j \geq 1} \frac{p_B(P_j^* f)}{2^j} \text{ for } f \in X_\gamma^* \right\}.$$

In case $(X_\gamma^*, \beta(X_1^*, X)|_{X_\gamma^*})$ is a Mazur space, the canonical bi-l.c. TVS $(X_{\gamma\gamma}^{**}, \beta(X_{\gamma 1}^{**}, X_\gamma^*)|_{X_{\gamma\gamma}^{**}}, T_2^*)$ where $X_{\gamma 1}^{**} = (X_\gamma^*, \beta(X_1^*, X)|_{X_\gamma^*})^*$, defined by the γ-dual $X_{\gamma\gamma}^{**}$ of the first k-conjugate space and its e-S.D. $\{P_n^{**}(X_{\gamma 1}^{**}), P_n^{**}\}$ is known as the *second* k-*conjugate space* and is *denoted* by k^2-X_b. Here T_2^* is the corresponding canonical topology generated by the family $D_{T_2^*}$ of seminorms given by

$$D_{T_2^*} = \Big\{ p_{B^*}^* : B^* \text{ varies over } \beta(X_1^*, X)|_{X_\gamma^*} \text{ bounded subsets of } X_\gamma^* \text{ and}$$

$$p_{B^*}(F) = \sum_{j \geq 1} \frac{p_{B^*}(P_j^{**}(F))}{2^j} \text{ for } F \in X_{\gamma\gamma}^{**} \Big\}.$$

Let us observe that $P_i^{**}(X_{\gamma\gamma}^{**}) = P_i^{**}(X_{\gamma 1}^{**})$, $\forall\, i \geq 1$ and so we *denote* both these subspaces by the common symbol $X_{\gamma\gamma i}^{**}$, $i \geq 1$ for the sake of convenience.

In view of k-conjugate spaces defined above, it is natural to enquire whether we can define an embedding from X to $X_{\gamma\gamma}^{**}$. In this direction, we prove Theorem 3.3. However, let us mention here that we *consider throughout a Mazur space* (X,T) *with an* e-S.D. $\{M_n; P_n\}$ *such that the* γ*-dual* X_γ^* *of the canonical* bi-l.c. TVS $X_b = (X,T,T^*)$ *is Mazur relative to* $\beta(X_1^*, X)|_{X_\gamma^*}$ and J is the map definet from X to $X_{\gamma 1}^{**}$ by the relation

$$(Jx)(f) = f(x), \ \forall\, f \in X_\gamma^*.$$

Then we begin with the following simple

Lemma 3.2: $JM_k \subset X_{\gamma\gamma k}^*$, for each $k \geq 1$.

Proof: For proving the result, we need show that $Jx_k \in X_{\gamma\gamma}^{**}$ and $P_k^{**}(Jx_k) = Jx_k$ for $x_k \in M_k$, $k \geq 1$. However, these are simple to verify and so the proof is omitted.

Restricting (X,T) further, we have

Theorem 3.3: Let (X,T) also be infrabarrelled. Then J maps X into $X_{\gamma\gamma}^{**}$ and is a topological isomorphism from (X,T) onto $(JX, \beta(X_{\gamma 1}^{**}, X_\gamma^*)|_{JX})$ and also from (X,T^*) onto $(JX, T_2^*|_{JX})$.

Proof: In order to show that $Jx \in X^{**}_{\gamma\gamma}$ for $x \in X$, consider a net $\{ f_\delta : \delta \in \Lambda \}$ in X^*_γ such that $f_\delta \xrightarrow{\gamma} 0$ in k-X_b. Then $\{ f_\delta \}$ is $\beta(X^*_1, X)$ -bounded and

$$(*) \qquad f_\delta(x_j) \longrightarrow 0, \ \forall \ x_j \in M_j \text{ and } j \geqslant 1$$

by Lemma 3.2. Since (X,T) is infrabarrelled, there exists a $p \in D_T$ such that

$$(**) \qquad | f_\delta(x) | \leqslant p(x), \ \forall \ x \in X \text{ and } \delta \in \Lambda.$$

Also, for $x = \sum_{j \geqslant 1} x_j$, $x_j \in M_j$, $j \geqslant 1$ and for p in (**), there exists $N_0 \in \mathbb{N}$ with

$$p\Big(\sum_{j \geqslant N_0+1} x_j\Big) < \frac{\epsilon}{2} \ .$$

From (*), there exists $\delta_0 \in \Lambda$ such that

$$| f_\delta(x_j) | < \frac{\epsilon}{2N_0} \ , \text{ for } \delta > \delta_0 \text{ and } j = 1, \cdots, N_0.$$

Hence

$$| f_\delta(x) | < \epsilon, \ \forall \ \delta > \delta_0.$$

Thus $(Jx)(f_\delta) = f_\delta(x) \longrightarrow 0$ and so $Jx \in X^{**}_{\gamma\gamma}$.

For showing the $T - \beta(X^{**}_{\gamma 1}, X^*_\gamma)$ continuity of J, let us consider a net $\{ x_\delta : \delta \in \Lambda \}$ in X such that $x_\delta \longrightarrow 0$ in T. Let v be a $\beta(X^{**}_{\gamma 1}, X^*_\gamma)$ neighborhood of origin. Then there exists a $\beta(X^*_1, X)$-bounded subset B of X^*_γ such that $B^0 \subset v$. As (X,T) is infrabarrelled, B is equicontinuous and so there exists a T-neighborhood u of origin such that $B \subset u^\bullet$, the polar of u relative to the dual pair $\langle X, X^*_\gamma \rangle$. Consequently, considering the polarity relative to the dual pair $\langle X^*_\gamma, X^{**}_{\gamma 1} \rangle$, we have

$$u^{\bullet 0} \subset B^0 \subset v.$$

Also there exists $\delta \in \Lambda$ such that $x_\delta \in u$, $\forall\, \delta \geq \delta_0$. Hence

$$|(Jx_\delta)(f)| = |f(x_\delta)| \leq 1, \forall\, f \in u^\circ \text{ and } \delta \geq \delta_0.$$

Thus $Jx_\delta \in v$ for $\delta \geq \delta_0$ and the required continuity of J follows.

For proving the continuity of J^{-1}, consider a net $\{y_\delta : \delta \in \Lambda\}$ in JX such that $y_\delta \to 0$ in $\beta(X^{**}_{\gamma_1}, X^*_\gamma)$. Write $x_\delta = J^{-1}(y_\delta)$, $\delta \in \Lambda$. By Propositions 2.8 and 2.12, there exists a family D_1 of seminorms generating T such that for each $p \in D_1$, we get a q in D_1 such that for x in X.

$$p(x) \leq \sup\left\{|f(x)| : f \in X^*_\gamma \cap v^0_q\right\} = p_{B^*}(Jx) \tag{3.4}$$

where $B^* = X^*_\gamma \cap v^0_q$, is a $\beta(X^*_1, X)$-bounded subset of X^*_γ. Consequently, $p(x_\delta) \leq p_{B^*}(Jx_\delta) \to 0 \Rightarrow J^{-1}$ is $\beta(X^{**}_{\gamma_1}, X^*_\gamma)|_{JX}$ – T continuous. Thus J is a T – $\beta(X^{**}_{\gamma_1}, X^*_\gamma)|_{JX}$ topological isomorphism.

In order to show that J is a T^* – $T^*_2|_{JX}$ topological isomorphism, consider first a seminorm $p^*_B \in D_{T^*_2}$ and $x \in X$. Now

$$p^*_{B^*}(Jx) = \sum_{j\geq 1} \frac{p_{B^*}(P^{**}_j(Jx))}{2^j}.$$

By Lemma 3.2, $P^{**}_j(Jx) = Jx_j$, $j \geq 1$ and so

$$p^*_{B^*}(Jx) = \sum_{j\geq 1} \frac{p_{B^*}(Jx_j)}{2^j}$$

$$\leq \sum_{j\geq 1} \frac{M\,p(x_j)}{2^j} = M\,p^*(x)$$

for some $p \in D_T$ and $M > 0$ as J is T – $\beta(X^{**}_{\gamma_1}, X^*_\gamma)|_{JX}$ continuous. Hence J is T^* – $T^*_2|_{JX}$ continuous. Similarly, one can establish the continuity of J^{-1} by making use of the inequality (3.4). Hence the result follows.

The above theorem leads us to

Definition 3.5: Let (X,T) be a Mazur infrabarrelled space with an e-S.D. $\{M_n;P_n\}$ such that $\{X^*_\gamma, \beta(X^*_1, X) \mid X^*_\gamma\}$ is a Mazur space. Then the corresponding canonical bi-l.c. TVS $X_b = (X,T,T^*)$ is called *k-reflexive* if the embedding J as defined above from X to $X^{**}_{\gamma\gamma}$ is onto.

Note: In the sequel, unless otherwise specified we shall consider an l.c. TVS (X,T) which is Mazur, infrabarrelled with an e-S.D. $\{M_n;P_n\}$ such that $(X^*_\gamma, \beta(X^*_1, X) \mid X^*_\gamma)$ is a Mazur space.

For our next result, we need the following general result contained in

Lemma 3.6: If R: $(X^*, \beta(X^*, X)) \longrightarrow (Y^*, \beta(Y^*, Y))$ is a topological isomorphism, then so is its adjoint R^*: $(Y^{**}, \beta(Y^{**}, Y^*)) \longrightarrow (X^{**}, \beta(X^{**}, X^*))$ where X^{**} and Y^{**} are respectively the topological duals of $(X^*, \beta(X^*, X))$ and $(Y^*, \beta(Y^*, Y))$.

Proof: For the $\beta(Y^{**}, Y^*)$ $\beta(X^{**}, X^*)$ continuity of R^*, we refer to [6, p. 256]. The one-to-one character of R^* follows from the ontoness of R. For showing that R^* is onto, consider $x^{**} \in X^{**}$. Then the linear functional F on Y^* defined as

$$F(Rf) = x^{**}(f), \ \forall f \in X^*$$

is a member of Y^{**} such that $R^*(F) = x^{**}$. Hence R^* is onto.

Since $(R^*)^{-1} = (R^{-1})^*$, the $\beta(X^{**}, X^*)$ $\beta(Y^{**}, Y^*)$ continuity of $(R^*)^{-1}$ follows and hence the result is established.

Next, we prove

Proposition 3.7: If $\{M_n;P_n\}$, $\{P^*_n(X^*_1); P^*_n\}$ and $\{P^{**}_n(X^{**}_{\gamma\gamma}); P^{**}_n\}$ are respectively e-S.D. of (X,T), $(X^*_\gamma, \beta(X^*_1, X) \mid X^*_\gamma)$ and $(X^{**}_{\gamma\gamma}, \beta(X^{**}_{\gamma 1}, X^*_\gamma) \mid X^{**}_{\gamma\gamma})$, then for each $j \geq 1$ there exists a topological isomorphism

$$\varphi_j: (M_j^{**}, \beta(M_j^{**}, M_j^*)) \longrightarrow (P_j^{**}(X_{\gamma\gamma}^{**}), \beta(X_{\gamma 1}^{**}, X_\gamma^*)\mid_{P_j^{**}(X_{\gamma\gamma}^{**})})$$

such that $\varphi_j \mid_{JM_j}$ is an identity mapping.

Proof: For proving the result we first show that $(M_j^*, \beta(M_j^*, M_j))$ is topologically isomorphic to $(P_j^*(X_1^*), \beta(X_1^*, X)\mid_{P_j^*(X_1^*)})$ for each $j \geq 1$. Fix $j \geq 1$ and define $R: M_j^* \longrightarrow P_j^*(X_1^*)$ by

$$R(f) = P_j^*(\hat{f}),\ f \in M_j^*$$

where $\hat{f}$ is the continuous extension of f from M_j to the whole space X. Clearly R is well-defined, linear and one-to-one. Further, R is $\beta(M_j^*, M_j)$ $\beta(X_1^*, X)\mid_{P_j^*(X_1^*)}$ continuous for if B is $\sigma(X, X_1^*)$-bounded or equivalently T-bounded subset of X, then there exists a $T|_{M_j}$ bounded subset $B_1 = P_j(B)$ of M_j such that

$$p_B(R(f)) = p_{B_1}(f),\ \forall f \in M_j^*.$$

Also, if A is $\sigma(M_j, M_j^*)$-bounded subset of M_j, then it is $\sigma(X, X_1^*)$-bounded (cf. [6], p. 262) and so the equality

$$p_A(R^{-1}(P_j^* f)) = p_A(f\mid_{M_j}),\ \forall f \in X_1^*$$

implies that R^{-1} is $\beta(X_1^*, X)\mid_{P_j^*(X_1^*)}$ $\beta(M_j^*, M_j)$ continuous. Hence

$$(*) \qquad R: (M_j^*, \beta(M_j^*, M_j)) \longrightarrow (P_j^*(X_1^*), \beta(X_1^*, X)\mid_{P_j^*(X_1^*)})$$

is a topological isomorphism. Consequently, by Lemma 3.6

$$R^*: ((P_j^*(X_1^*))^*, \beta((P_j^*(X_1^*))^*, P_j^*(X_1^*)) \longrightarrow (M_j^{**}, \beta(M_j^{**}, M_j^*))$$

is a topolgocial isomorphism where M_j^{**} is the topological dual of $(M_j^*, \beta(M_j^*, M_j))$.

Applying (*) to the S.D. $\{P_j^*(X_1^*); P_j^*\}$ in $(X_\gamma^*, \beta(X_1^*, X) \mid_{X_\gamma^*})$, we can find a topological isomorphism S from the space $((P_j^*(X_1^*))^*, \beta((P_j^*(X_1^*))^*, P_j^*(X_1^*))$ to the space $(P_j^{**}(X_{\gamma\gamma}^{**}), \beta(X_{\gamma 1}^{**}, X_\gamma^*) \mid_{P_j^{**}(X_{\gamma\gamma}^{**})})$. Define

$$\varphi_j = S o (R^*)^{-1}.$$

Them φ_j is the required topological isomorphism and this completes the proof of the first part.

For showing $\varphi_j \mid_{JM_j}$ is an identity mapping, observe that (i) $JM_j \subset P_j^*(X_{\gamma\gamma}^{**})$; (ii) $JM_j \subset (P_j^*(X_1^*))^*$; (iii) $JM_j \subset M_j^{**}$; and (iv) $S \mid_{JM_j}$ and $(R^*)^{-1} \mid_{JM_j}$ are identity mappings.

Indeed, the containment (i) follows from Lemma 3.2. The relations (ii), (iii) and the first part of (iv) can be easyly verified. For the second part of (iv), it suffices to show that $R^* \mid_{JM_j}$ is an identity and this follows from the fact that for $f \in M_j^*$ and $x_j \in M_j$

$$(R^*(Jx_j))(f) = (Jx_j)(P_j^*(f)) = f(x_j) = (Jx_j)(f).$$

Thus $\varphi_j \mid_{JM_j}$ is an identity and the result is now completely proved.

Using Proposition 3.7, we characterize k-reflexive bi-l.c. TVS as follows:

Theorem 3.8: If (X, T) is sequentially complete, then $X_b = (X, T, T^*)$ is k-reflexive if and only if each M_k is reflexive.

Proof: Let us first prove the equivalence of the following two statements, namely,

(i) $J: X \to X_{\gamma\gamma}^{**}$ is onto,

(ii) $J: M_j \to P_j^{**}(X_{\gamma\gamma}^{**})$ is onto for each $j \geqslant 1$.

(i) ⇒ (ii). Let $F \in P_j^{**}(X_{\gamma\gamma}^{**})$. Since $P_j^{**}(X_{\gamma\gamma}^{**}) \subset X_{\gamma\gamma}^{**}$, there exists an $x \in X$ such that $Jx = F$. Therefore, $P_j^{**}(Jx) = Jx$ and so $f(P_j x) = f(x)$, $\forall f \in X_\gamma^*$. Hence $x = P_j x \in M_j$ and (ii) follows.

(ii) $\Rightarrow$ (i). Consider an $F \in X^{**}_{\gamma\gamma}$. Then there exist $x_j \in M_j$ such that $Jx_j = P_j^{**}(F)$, $j \geqslant 1$. Consequently,

$$F = \sum_{j \geqslant 1} P_j^{**}(F) = \sum_{j \geqslant 1} Jx_j$$

where the convergence of the series is being relative to $\beta(\dot{X}^{**}_{\gamma 1}, X^*_\gamma) | X^{**}_\gamma$. By the inequality (3.4) in the proof of Theorem 3.3, it follows that $\left\{ \sum_{j=1}^{n} x_j : n \geqslant 1 \right\}$ is a T-Cauchy sequence and therefore there exists an $x \in X$ such that

$$x = \sum_{j \geqslant 1} x_j \Rightarrow Jx = \sum_{j \geqslant 1} Jx_j.$$

Thus $F = Jx$, for some $x \in X$ and (i) follows.

Now if X_b is k-reflexive, $J: M_j \to X^{**}_{\gamma\gamma j}$ is a $T|_{M_j}$ $\beta(X^{**}_{\gamma 1}, X^*_\gamma)|_{X^{**}_{\gamma\gamma j}}$ topological isomorphism. Since $JM_j = X^{**}_{\gamma\gamma j} = M_j^{**}$ by the preceding result, $J: M_j \to M_j^{**}$ is a $T|_{M_j}$ - $\beta(M_j^{**}, M_j^*)$-topological isomorphism and henece M_j is reflexive for each $j \geqslant 1$.

Conversely, if M_j is reflexive, then the map J maps X onto $X^{**}_{\gamma\gamma}$ by the above arguments. Hence X_b is k-reflexive.

4. Canonical γ-Completion

In this section we construct a VVSS equipped with two locally convex topologies corresponding to a canonical bi-l.c. TVS, that behaves like a γ-completion of a bi-l.c. TVS in the sense of the following

Definition 4.1: Let $X_b \equiv (X, T_1, T_2)$ be a bi-l.c. TVS. If there exists a normal γ-complete bi-l.c. TVS $\hat{X}_b \equiv (\hat{X}, \tau, \tau^*)$ containing X_b as a γ-dense subspace, then $\hat{X}_b$ is known as a *γ-completion* of X_b. We call the γ-completion of a canonical bi-l.c. TVS as the *canonical γ-completion.*

Recalling the map $J: X \to X^{**}_{\gamma\gamma}$ as well as the restrictions on the space (X, T, T^*) and its γ-dual from the preceding section, namely, (X, T) is a Mazur, infrabarrelled space with an e-S.D. $\{M_n; P_n\}$ such that the γ-dual X^*_γ of the corresponding canonical bi-l.c. TVS is Mazur relative to $\beta(X^*_1, X)|_{X^*_\gamma}$, we first prove

Proposition 4.2: For a given $x_j \in M_j$, $j \geq 1$, the series $\sum_{j\geq 1} Jx_j$ is $\sigma(X^{**}_{\gamma 1}, X^*_\gamma)$-convergent provided $\left\{\sum_{j=1}^{n} Jx_j; n \geq 1\right\}$ is $\beta(X^*_1, X)|_{X^*_\gamma}$ equicontinuous.

Proof: Write $G_n = \sum_{j=1}^{n} Jx_j$, $n \geq 1$. Since $\{G_n : n \geq 1\}$ is $\beta(X^*_1, X_1)|_{X^*_\gamma}$ continuous, there exists a T-bounded set B in X such that

$$(*) \qquad |\langle f, G_n\rangle| \leq p_B(f), \ \forall\, n \geq 1 \text{ and } f \in X^*_\gamma.$$

If $Y = \text{sp} \bigcup_j P^*_j(X^*_1)$, then $G_n(g)$ donverges for all g in Y and so we can define a linear map $F: Y \to \mathbb{K}$ as follows:

$$(**) \qquad F(g) = \lim_{n\to\infty} G_n(g), \ \forall\, g \in Y.$$

By $(*)$, F is $\beta(X^*_1, X)|_Y$ continuous. Therefore, we can extend F to a $\beta(X^*_1, X)|_{X^*_\gamma}$ continuous linear functional $\hat{F}$ on X^*_γ.

For showing $\sigma(X^{**}_{\gamma 1}, X^*_\gamma)$-convergence of $\sum_{j\geq 1} Jx_j$, consider $f \in X^*_\gamma$. Then for B in $(*)$, there exists n_0 in $\mathbb{N}$ such that

$$p_B\left(f - \sum_{i=1}^{n_0} P^*_i f\right) < \frac{\epsilon}{3}.$$

Also by $(**)$, for $g \in Y$, there exists m_0 in $\mathbb{N}$ such that

$$\left|\hat{F}\left(\sum_{j=1}^{n_0} P^*_j f\right) - G_n\left(\sum_{j=1}^{n_0} P^*_j f\right)\right| < \frac{\epsilon}{3}, \ \forall\, n \geq m_0.$$

Consequently,

$$|\hat{F}(f) - G_n(f)| \leqslant p_B(f - \sum_{j=1}^{n_0} P_j^* f) + |(G_n - \hat{F})(\sum_{j=1}^{n_0} P_j^* f)| + p_B(\sum_{j=1}^{n_0} P_j^* f - f)$$

$$\leqslant \epsilon, \ \forall\, n \geqslant m_0.$$

This completes the proof.

Note: Let us note that if $(X_\gamma^*, \beta(X_1^*, X) | _{X_\gamma^*})$ is infrabarrelled, then the convergence of the series $\sum_{j\geqslant 1} Jx_j$ in the topology $\sigma(X_{\gamma 1}^{**}, X_\gamma^*)$ implies the $\beta(X_1^*, X) | _{X_\gamma^*})$ equicontinuity of $\left\{ \sum_{j=1}^{n} Jx_j : n \geqslant 1 \right\}$; for in this case $(X_\gamma^*, \beta(X_1^*, X) | _{X_\gamma^*})$, being complete, is a barrelled space (cf. [9], p 368). Consequently, $\left\{ \sum_{j=1}^{n} Jx_j : n \geqslant 1 \right\}$ is $\beta(X_{\gamma 1}^{**}, X_\gamma^*)$-bounded. This fact yields the construction of a VVSS equipped with two locally convex topologies in the following.

Definition 4.3: In addition to our earlier restrictions, let us also assume that $(X_\gamma^*; \beta(X_1^*, X) | _{X_\gamma^*})$ is infrabarrelled and set

$$\widetilde{X} = \left\{ \{Jx_j\} : \sum_{j\geqslant 1} Jx_j \text{ converges in } \sigma(X_{\gamma 1}^{**}, X_\gamma^*) \right\}.$$

Then $\widetilde{X}$ is a vector space with resepct to usual pointwise addition and scalar multiplication. Further, it can be equipped with two locally convex topologies τ and τ^* defined respectively bu the families of seminorms

$$D_\tau = \left\{ q_{B^*} : B^* \text{ varies over } \beta(X_1^*, X) | _{X_\gamma^*} \text{ bounded subsets of } X_\gamma^* \text{ and} \right.$$

$$\left. q_{B^*}(\{Jx_j\}) = \sup_n p_{B^*}(\sum_{j=1}^{n} Jx_j) \text{ for } \{Jx_j\} \in \widetilde{X} \right\}; \text{and}$$

$$D_{\tau^*} = \left\{ \hat{q}_{B^*} : B^* \text{ varies over } \beta(X_1^*, X) \mid_{X_\gamma^*} \text{ bounded subsets of } X_\gamma^* \text{ and} \right.$$

$$\left. \hat{q}_{B^*}(\{Jx_j\}) = \sum_{j \geq 1} \frac{p_{B^*}(Jx_j)}{2^j} \text{ for } \{Jx_j\} \in \widetilde{X} \right\}.$$

One can easily check that $\tau^* \subset \tau$ and so the triplet $(\widetilde{X}, \tau, \tau^*)$. is a bi-l.c. TVS denoted by $\widetilde{X}_b$, that is $\widetilde{X}_b \equiv (\widetilde{X}, \tau, \tau^*)$. Concerning its basic structural properties we have

Proposition 4.4: The VVSS $(\widetilde{X}, \tau)$ and $(\widetilde{X}, \tau^*)$ are GK-spaces.

Proof: Since $\tau^* \subset \tau$, it is enough to show that $(\widetilde{X}, \tau^*)$ is a GK-space. So consider a net $\{F_\delta : \delta \in \Lambda\}$ and a point F in $\widetilde{X}$ with $F_\delta = \{Jx_j^\delta\}$, $\delta \in \Lambda$ and $F = \{Jx_j\}$ such that $F_\delta \longrightarrow F$ in τ^*. Then for $\beta(X_1^*, X) \mid_{X_\gamma^*}$ bounded subset B^* of X_γ^*,

$$\hat{q}_{B^*}(F_\delta - F) = \sum_{j \geq 1} \frac{p_{B^*}(Jx_j^\delta - Jx_j)}{2^j}$$

$$\overset{\delta}{\longrightarrow} 0$$

$$\Rightarrow p_{B^*}(Jx_j^\delta - Jx_j) \overset{\delta}{\longrightarrow} 0 \text{ for each } j \geq 1.$$

Thus $(\widetilde{X}, \tau^*)$ is a GK-space.

Note: If $N_i = \{\delta_i^{Jx} : x \in X\}$, $i \geq 1$, $\{N_i\}$ is clearly an S.D. for $(\widetilde{X}, \tau^*)$. Also it is an S.D. for $(\widetilde{X}, \sigma(\widetilde{X}, \phi(X_\gamma^*)))$. However it would be interesting to investigate the form of the generalized Köthe dual of $\widetilde{X}$ and establish realtrionship with its topological duals (cf. [3]) so as to have an insight into the various structural properties of the space $\widetilde{X}$ and the role played by $\{N_i\}$ in view of the results of [4].

Proposition 4.5: $\widetilde{X}_b = (\widetilde{X}, \tau, \tau^*)$ is a normal bi-l.c. TVS.

Proof: Let us consider a net $\{F_\delta : \delta \in \Lambda\}$ and an F in $\widetilde{X}_b$ with $F_\delta = \{Jx_j^\delta\}$, $\delta \in \Lambda$ and $F = \{Jx_j\}$ such that $F_\delta \to F$ in $(\widetilde{X},\tau^*)$. Then by Proposition 4.4, $p_{B^*}(Jx_j^\delta) \to p_{B^*}(Jx_j)$ for each $\beta(X_1^*,X)|_{X_\gamma^*}$ -bounded subset B^* of X_γ^* and $j \geqslant 1$. We now fix a $\beta(X_1^*,X)|_{X_\gamma^*}$ -bounded set B^* in X_γ^*. Then for $f \in B^*$ and $n \in \mathbb{N}$,

$$|\sum_{j=1}^{n}(Jx_j)(f)| = \underline{\lim}_{\delta}|\sum_{j=1}^{n}J(x_j^\delta)(f)|$$

$$\leqslant \underline{\lim}_{\delta}\ p_{B^*}(\sum_{j=1}^{n}J(x_j^\delta))$$

$$\leqslant \underline{\lim}_{\delta}\ q_{B^*}(F_\delta)$$

Since the right hand side is independent of $f \in B^*$ and $n \in \mathbb{N}$, we get

$$q_{B^*}(F) \leqslant \underline{\lim}_{\delta}\ q_{B^*}(F_\delta).$$

Hence $\widetilde{X}_b$ is a normal bi-l.c. TVS.

Preposition 4.6: If (X,T) is complete, $\widetilde{X}_b$ is a γ-complete bi-l.c. TVS.

Proof: Let us consider a γ-Cauchy net $\{F_\delta : \delta \in \Lambda\}$ in $\widetilde{X}_b$ where $F_\delta = \{Jx_j^\delta\}$, $\delta \in \Lambda$. Then for a given $\beta(X_1^*,X)|_{X_\gamma^*}$ bounded set B^* and $\epsilon > 0$ there exists a positive constant M depending on B^* and a $\delta_0 \in \Lambda$ such that

$$(*)\qquad q_{B^*}(F_\delta) \leqslant M,\ \forall \delta \in \Lambda;$$

$$(**)\qquad q_{B^*}(F_\delta - F_\eta) < \epsilon,\ \forall \delta, \eta \geqslant \delta_0.$$

Consequently, from (**), $\{Jx_j^\delta : \delta \in \Lambda\}$ is a Cauchy net in JM_j for each $j \geqslant 1$. Now, by the completene of each $(M_j, T|_{M_j})$ and Theorem 3.3 the completeness of $(JM_j, \beta(X^{**}_{\gamma 1}, X^*_\gamma)|_{JM_j})$ follows. Hence there exist $x_j \in M_j, j \geqslant 1$ such that

$$\{Jx_j^\delta\} \to Jx_j \text{ in } \beta(X^{**}_{\gamma 1}, X^*_\gamma)|_{JM_j} \text{ and } j \geqslant 1.$$

In order to dispose of the proof completely, we need to show that

(a) $F = \{Jx_j\} \in \widetilde{X}$; and

(b) $F_\delta \xrightarrow{\delta} F$ in $\widetilde{X}_b$.

For proving (a) let us consider a $\beta(X_1^*, X)|_{X^*_\gamma}$ bounded subset B^* of X^*_γ. Then for $n \in \mathbb{N}$ and $f \in B^*$,

$$\left|\sum_{j=1}^{n} (Jx_j)(f)\right| \leqslant \overline{\lim_{\delta}}\; q_{B^*}(F_\delta) \leqslant M$$

$$\Rightarrow p_{B^*}\Big(\sum_{j=1}^{n} Jx_j\Big) \leqslant M, \ \forall n \geqslant 1$$

or,

$$q_{B^*}(F) \leqslant M.$$

Hence (a) follows.

For proving (b), it is sufficient to show that $F_\delta \to F$ in $(\widetilde{X}, \tau^*)$ in view of (*) and (a). Therefore consider a $\beta(X_1^*, X)$-bounded subset B^* of X^*_γ. Then by (*) and (a),

$$q_{B^*}(F_\delta - F) \leqslant 2M, \ \forall \delta \in \Lambda$$

and so

$$p_{B^*}(Jx_j^\delta - Jx_j) \leqslant 4M, \ \forall \delta \in \Lambda, j \geqslant 1.$$

Now for given $\epsilon > 0$ there exist $N \in \mathbb{N}$ and a δ_0 in Λ depending on ϵ and N such that

$$\sum_{j>N} \frac{1}{2^j} < \frac{\epsilon}{8M}$$

and

$$p_{B*}(Jx_j^\delta \quad Jx_j) < \frac{\epsilon}{2N}, \quad \forall \delta \geqslant \delta_0 \text{ and } 1 \leqslant j \leqslant N.$$

Consequently,

$$\hat{q}_{B*}(F_\delta \quad F) \leqslant \sum_{j=1}^{n} \frac{p_{B*}(Jx_j^\delta \quad Jx_j)}{2^j} + 4M \sum_{j>N} \frac{1}{2^j}$$

$$< \epsilon, \ \forall \delta > \delta_0$$

and hence (b) holds. This completes the proof.

Finally, we have

Proposition 4.7: Let (X,T) be also sequentially complete. Then the space $\widetilde{X}_b \equiv (\widetilde{X},\tau,\tau^*)$ is the canonical γ-completion of $X_b \equiv (X,T,T^*)$.

Proof: For $x \in X$, let us note that the series $\sum_{j \geqslant 1} Jx_j$ where $x = \sum_{j \geqslant 1} x_j$ in (X,T). $\sigma(X_{\gamma_1}^{**}, X_\gamma^*)$ converges by Theorem 3.3. Thus we can define a map $E: X \to \widetilde{X}$ by

$$E(x) = \{Jx_j\}$$

where $x \in X$ with $x = \sum_{j \geqslant 1} x_j$, $x_j \in M_j$, $j \geqslant 1$. Clearly E is linear and one-to-one.

In order to prove the result, we need show

(i) $\widetilde{X}_b \equiv (\widetilde{X},\tau,\tau^*)$ is normal and γ-complete;

(ii) $E: (X,T) \to (\widetilde{X},\tau)$ and $E: (X,T^*) \to (\widetilde{X},\tau^*)$ are topological isomorphisms into; and

(iii) E(X) is γ-dense in $\widetilde{X}$.

We have already proved (i) in Propositions 4.5 and 4.6.

For proving (ii), let us first show the T-τ continuity of E. Therefore, consider a seminorm q_{B*} of τ and x in X with $x = \sum_{j \geqslant 1} x_j$, $x_j \in M_j$, $j \geqslant 1$. Since J is T-$\beta(X_{\gamma_1}^{**}, X_\gamma^*) \mid_{JX}$ continuous by Theorem 3.3, there exist a constant $M > 0$ and $p \in D_T$ such that

$$(*) \qquad p_{B*}(Jx) \leqslant Mp(x), \ \forall x \in X.$$

Consequently,

$$q_{B^*}(Ex) = \sup_n p_{B^*}\left(\sum_{j=1}^{n} Jx_j\right)$$

$$\leqslant M \sup_n p\left(\sum_{j=1}^{n} x_j\right) = M\, p\,(x).$$

Since the topologies T and T̃ are equivalent by Proposition 2.4, the T-τ continuity of E follows.

For showing the τ-T continuity of E^{-1}, consider a seminorm $p \in D_T$. By Theorem 3.3, there exists a seminorm $p_{B_1^*}$ of $\beta(X_{\gamma 1}^{**}, X_\gamma^*)$ such that

$$(**) \qquad p(x) \leqslant p_{B_1^*}(Jx), \quad \forall\, x \in X$$

$$\Rightarrow p(x) \leqslant \sup_n p_{B_1^*}\left(\sum_{j=1}^{n} Jx_j\right) = q_{B_1^*}(Ex)$$

for $x = \sum_{j\geqslant 1} x_j$, $x_j \in M_j$, $j \geqslant 1$. Hence E is a T-τ topological isomorphism from X into $\widetilde{X}$.

For T*-τ* continuity of E, note that for each $\hat{q}_{B^*}$,

$$\hat{q}_{B^*}(Ex) \geqslant M\, p^*(x), \quad \forall\, x \in X$$

for some $p \in D_T$ and some $M > 0$ by (*); and for the τ*-T* continuity of E^{-1} we have from (**)

$$p^*(x) \leqslant \hat{q}_{B_1^*}(Ex), \; x \in X.$$

Hence E is a T*-τ* topological isomorphism.

To prove (iii), consider an $F = \{Jx_j\} \in \widetilde{X}$. For $n \geqslant 1$, define

$$F_n = \{Jx_1, \cdots, Jx_n, 0, 0, \cdots\}.$$

Then $F_n \in E(X)$, for each $n \geqslant 1$ and the sequence $\{F_n\}$ is τ-bounded, for

$$q_{B^*}(F_n) \geqslant q_{B^*}(F), \ \forall n \geqslant 1.$$

Also, for an arbitrary $\beta(X_1^*, X) \mid X_\gamma^*$ bounded subset B^* and given $\epsilon > 0$, there exists n_0 depending on B^* and ϵ such that

$$\sum_{j \geqslant n+1} \frac{p_{B^*}(Jx_j)}{2^j} < \epsilon, \ \forall n \geqslant n_0.$$

Consequently,

$$\hat{q}_{B^*}(F_n - F) < \epsilon, \ \forall n \geqslant n_0.$$

Thus $E(X)$ is γ-sequentially dense in $\widetilde{X}_b$ and the result is completely proved.

5. Relationship Between k-Reflexivity and γ-Reflexivity

In this section we prove results which lead us to establish the relationship between k- and γ-reflesivities of a canonical bi-l.c. TVS and deduce a known result in the basis theory. As mentioned earlier, (X,T) is a Mazur infrabarrelled space with e-S.D. $\{M_n; P_n\}$, and the γ-dual $(X_\gamma^*, \beta(X_1^*, X) \mid X_\gamma^*)$ of the corresponding canonical bi-l.c. TVS is a Mazur infrabarrelled space. Let us also recall the map E introduced in Proposition 4.7, namely, $E: X \longrightarrow \widetilde{X}$, with $E(x) = \{Jx_j\}$, $x = \sum_{j \geqslant 1} x_j$, $x_j \in M_j$, $j \geqslant 1$. Define $e: \widetilde{X} \longrightarrow X_{\gamma 1}^{**}$ by

$$e(\{Jx_j\}) = \sum_{j \geqslant 1} Jx_j, \ \forall \{Jx_j\} \in \widetilde{X},$$

the convergence of the series being relative to $\sigma(X_{\gamma 1}^{**}, X_\gamma^*)$.

In terms of the map e, we characterize k-reflexivity in

Proposition 5.1: For a sequentially complete space (X,T), the canonical bi-l.c. TVS $X_b \equiv (X,T,T^*)$ is k-reflexive if and only if c is onto.

Proof: Let X_b be k-reflexive. Then by Proposition 3.8 each M_j is reflexive. For showing that c is onto, let us consider an F in $X^{**}_{\gamma 1}$. Then

$$F = \sum_{j \geqslant 1} P_j^{**}(F) \text{ in } \sigma(X^{**}_{\gamma 1}, X^*_\gamma).$$

Since $JM_j = M_j^{**}$ by reflexivity of M_j and the map φ_j in Proposition 3.7 is identity on JM_j it follows that $J(M_j) = P_j^{**}(X^{**}_{\gamma 1})$. Hence there exist $x_j \in M_j$, $j \geqslant 1$ such that

$$P_j^{**}(F) = Jx_j, \; j \geqslant 1.$$

Therefore

$$F = \sum_{j \geqslant 1} Jx_j \text{ in } \sigma(X^{**}_{\gamma 1}, X^*_\gamma)$$

$$\Rightarrow F = c(\{Jx_j\}), \text{ that is, c is onto.}$$

Conversely, let c be onto, Then for $F \in X^{**}_{\gamma\gamma}$,

$$F = \sum_{j \geqslant 1} Jx_j \text{ in } \sigma(X^{**}_{\gamma 1}, X^*_\gamma).$$

Also by Proposition 2.13,

$$F = \sum_{j \geqslant 1} P_j^{**}(F) \text{ in } \beta(X^{**}_{\gamma 1}, X^*_\gamma).$$

Hence

$$Jx_j = P_j^{**}(F), \; \forall j \geqslant 1.$$

Thus J is an onto map from M_j to $P_j^{**}(X_{\gamma\gamma}^{**})$. Now invoking the proof of Theorem 3.8, J maps X onto $X_{\gamma\gamma}^{**}$, that is X_b is k-reflexite.

As an immediate consequence of the preceding result, we have

Proposition 5.2: If the S.D. of a sequentially complete space (X,T) is shrinking, then e: $\widetilde{X} \longrightarrow X_{11}^{**}$ is onto if and only if each M_j is reflexive.

Proof: Since the S.D. is shrinking, $X_\gamma^* = X_1^*$ by Proposition 2.13 and therefore $X_{\gamma 1}^{**} = X_{11}^{**}$. Now the result is immediate from the preceding proposition and Theorem 3.8.

For our next result concerning the characterization of boundedly complete S.D., we need

Lemma 5.3: If (X,T) is sequentially complete, then

$$E(X) = \left\{ \{ Jx_j \} : x_j \in M_j, j \geqslant 1 \text{ and } \sum_{j\geqslant 1} Jx_j \text{ converges in } \beta(X_{\gamma 1}^{**}, X_\gamma^*) \right\}.$$

Proof: In view of Proposition 3.3, it is sufficient to show the existence of a point x in X for each sequence $\{ x_j \}$ with $x_j \in M_j$, $j \geqslant 1$ and $\sum_{j\geqslant 1} Jx_j$ converges in $\beta(X_{\gamma 1}^{**}, X_\gamma^*)$ such that $\sum_{j=1}^{n} x_j \longrightarrow x$ in (X,T). Therefore, consider such a sequence $\{ Jx_j \}$. Then $\left\{ \sum_{j=1}^{n} Jx_j \colon n \geqslant 1 \right\}$ is a β $(X_{\gamma 1}^{**}, X_\gamma^*)$-Cauchy sequence in $X_{\gamma 1}^{**}$. Hence $\left\{ \sum_{j=1}^{n} x_j \colon n \geqslant 1 \right\}$ is a T-Cauchy sequence in (X,T) by the inequality (3.4) in the proof of Theorem 3.3. Hence there exists an x in X such that $x = \sum_{j\geqslant 1} x_j$. This completes the proof.

Theorem 5.4: The S.D. $\{ M_n ; P_n \}$ for a sequentially complete space (X,T) is boundedly complete if and only if E is onto.

Proof: Let $\{ M_n \}$ be boundedly complete. Then it follows from Theorem 3.3 that $\{ JM_n \}$ is also boundedly complete for $(JX, \beta(X_{\gamma 1}^{**}, X_\gamma^*) \mid_{JX})$. Now to prove the equality $\widetilde{X} = E(X)$, we need show that $\widetilde{X} \subset E(X)$ as the other inclusion is trivially true. Therefore, consider $Jx_j \in \widetilde{X}$. Then $\left\{ \sum_{j=1}^{n} Jx_j : n \geqslant 1 \right\}$ is $\beta(X^{**}_{\gamma 1}, X^{*}_{\gamma})$-bounded by the note following Proposition 4.2. Thus $\sum_{j \geqslant 1} Jx_j$ converges in $\beta(X^{**}_{\gamma 1}, X^{*}_{\gamma})$ by the above arguments and hence $\{ Jx_j \} \in E(X)$.

For the converse, let us consider a T-bounded sequence $\left\{ \sum_{j=1}^{n} x_j : n \geqslant 1 \right\}$, $x_j \in M_j$, $j \geqslant 1$. Then applying Theorem 3.3, we get $\{ Jx_j \} \in \widetilde{X} = E(X)$. Hence there exists an x in X such that $x = \sum_{j \geqslant 1} x_j$. This establishes the result completely.

An immediate consequence of this result is

Corollary 5.5: For a sequentially complete l.c. TVS (X,T), X_b is γ-complete if and only if $\widetilde{X} = E(X)$.

Proof: Follows immediately from Proposition 2.14 and 5.4.

We are now prepared to prove the main result of this section, namely,

Theorem 5.6: If (X,T) is sequentially complete, then X_b is γ-reflexive if and only if X_b is k-reflexive and γ-complete.

Proof: Let X_b be γ-reflexive. Then $J: X \longrightarrow X^{**}_{\gamma 1}$ is onto. Consequently, $J: X \longrightarrow X^{**}_{\gamma\gamma}$ is onto. Hence X_b is k-reflexive.

For showing the γ-completeness of X_b, it is enough to show that $\widetilde{X} \subset E(X)$ in view of Corollary 5.5. So, consider $\{ Jx_j \} \in \widetilde{X}$. Then $\sum_{j \geqslant 1} Jx_j$ convdrges in $\sigma(X^{**}_{\gamma 1}, X^{*}_{\gamma})$ to some element, say F, in $X^{**}_{\gamma 1}$. Hence by the γ-reflexivity of X_b there exists a y in X such that

$$(*) \qquad F = Jy.$$

Since $y = \sum_{j \geqslant 1} y_j$, $y_j \in M_j$, $j \geqslant 1$ in (X,T), applying Theorem 3.3, we have

$$Jy = \sum_{j \geqslant 1} Jy_j$$

the convergence of the series being relative to $\beta(X_{\gamma 1}^{**}, X_\gamma^*)$ and hence also relative to $\sigma(X_{\gamma 1}^{**}, X_\gamma^*)$. Thus

$$\sum_{j\geq 1} Jx_j = \sum_{j\geq 1} Jy_j.$$

Now operating both sides on $P_j^*(f)$, $f \in X_j^*$, we obtain

$$x_j = y_j, \ \forall j \geq 1.$$

Consequently, $\sum_{j\geq 1} Jx_j$ converges in $\beta(X_{\gamma 1}^{**}, X_\gamma^*)$. Hence by Lemma 5.3. $\{Jx_j\} \in E(X)$ and $\widetilde{X} \subset E(X)$.

Conversely, let X_b be k-reflexive and γ-complete. For showing γ-reflexivity of X_b, consider an F in $X_{\gamma 1}^{**}$. Then applying Proposition 5.1,

$$F = e(\{Jx_j\}) = \sum_{j\geq 1} Jx_j$$

for some $\{Jx_j\} \in \widetilde{X}$. Also by Theorem 5.4. $\widetilde{X} = E(X)$ and so there exists an $x \in X$ such that

$$Jx = \sum_{j\geq 1} Jx_j.$$

Thus $F = Jx$ for some $x \in X$ and hence X_b is γ-reflexive.

As an immediate consequence, we have

Corollary 5.6: Let (X,T) be a sequentially complete infrabarrelled space with a Schauder basis $\{x_i; f_i\}$ such that $(X_\gamma^*, \beta(X_1^*, X) \mid_{X_\gamma^*})$ is a barrelled space. Then X_b is γ-reflexive if and only if X_b is γ-complete.

Proof: Since a sequentially complete infrabarrelled space is barrelled, (X,T) is a Mazur space by Proposition 2.2. Now the result follows immediately from Theorem 5.6 as each M_j, being one dimensional space, is reflexive.

Finally, we deduce the following result due to Retherford (14, Theorem 2.3, p. 281) by using the techniques of bi-locally convex spaces as follows:

Theorem 5.7: If (X,T) is a barrelled, semireflexive (and hence reflexive) complete space with a Schauder basis $\{x_i; f_i\}$, then $\{x_i; f_i\}$ is both shrinking and boundedly complete.

Proof: The space X_b is γ-reflexive and saturated by Proposition 2.10 and hence $(X_\gamma^*, \beta(X_1^*, X)|_{X_\gamma^*})$ is barrelled (6, p. 228). Consequently, X_b is γ-complete by Corollary 5.6. The result now follows from Propositions 2.13 and 2.14.

The authors gratefully acknowledge their thanks to Professor P. K. Kamthan for his useful suggestions and comments during the preparation of this paper.

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