NONORIENTABLE, INCOMPRESSIBLE SURFACES OF GENUS 3 IN $M_\varphi\left(\frac{\lambda}{\mu}\right)$ MANIFOLDS

by

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In this paper we investigate nonorientable, incompressible surfaces of genus 3 in $M_\varphi\left(\frac{\lambda}{\mu}\right)$ manifolds, that is manifolds which arise from punctured-torus bundles over S^1 by Dehn surgery (see Part 3). We are interested especially in $M_\varphi\left(\frac{\lambda}{\mu}\right)$ manifolds which are irreducible and non-Haken. J. H. Rubinstein analyzed the following problem [Ru]: Let M be an orientable, closed, irreducible, non-Haken 3-manifold. How many pairwise non-isotopic, nonorientable, incompressible, genus 3 surfaces which represent the same element of H_2 (M, Z_2) can be embedded in M ? It is proved in [Ru] that 3 is an upper bound, and that for Seifert fibered spaces 1 is an upper bound. We generalise the last result for manifolds $M_\varphi\left(\frac{\lambda}{\mu}\right)$, and we prove that, with the exception of

$$M_\varphi\left(\frac{1}{1}\right), \varphi = -\bar{\alpha}^2 \beta\bar{\alpha}^2 \beta\bar{\alpha}^2 \beta$$

(see 1.1 and Part 3), 1 is an upper bound. We examine closer the case of $M_\varphi\left(\frac{1}{1}\right)$ and we prove that $M_\varphi\left(\frac{1}{1}\right)$ is not a Seifert fibered space. We give strong evidence that $M_\varphi\left(\frac{1}{1}\right)$ contains three, pairwise non-isotopic, nonorientable incompressible, genus 3 surfaces but we do not prove it and it is still an open problem.

0. INTRODUCTION.

We work in the PL (or equivalently smooth) category. We refer to [H] or [J] for basic terminology.

In the first part of the paper we state the results contained in [F – H] and [P – 2] concerning the classification of incompressible, ∂-incompressible (or closed) surfaces (orientable or not) in M_φ-punctured torus bundles over S^1. In the second part we insert the tables containing all nonorientable, incompressible, ∂-incompressible surfaces of genus $\leqslant 3$ embedded in M_φ. In the third part we find all nonorientable, incompressible surfaces of genus 3 in irreducible, non-Haken manifolds obtained from M_φ by Dehn surgery. In particular we prove the following theorem:

Theorem 3.2. Let $M_\varphi(\frac{\lambda}{\mu})$ be an irreducible, non-Haken, orientable 3-manifold obtained by Dehn surgery of type $\frac{\lambda}{\mu}$ from a punctured torus bundle over S^1 with a monodromy map φ (see Part 3). Let K_1 and K_2 be nonorientable, incompressible, genus 3 surfaces in $M_\varphi(\frac{\lambda}{\mu})$ such that the classes of K_1 and K_2 are equal in $H_2(M_\varphi(\frac{\lambda}{\mu}), Z_2)$. Then K_1 is isotopic to K_2 with the possible exception of the case of $M_\varphi(\frac{1}{1})$ with $\varphi = -\bar{\alpha}^2\beta\bar{\alpha}^2\beta\bar{\alpha}^2\beta$ (see 1.1). We identify φ with its equivalence class defined by conjugation and taking the inverse.

In the fourth part of the paper we examine closer the case of genus 3 surfaces in

$$M_\varphi\left(\frac{1}{1}\right) \text{ for } \varphi = -\bar{\alpha}^2\beta\bar{\alpha}^2\beta\bar{\alpha}^2\beta.$$

We give some evidence that the three constructed surfaces are not isotopic. We prove that the manifold $M_\varphi(\frac{1}{1})$ is not Seifert fibered. We divide $M_\varphi(\frac{1}{1})$ by a free Z_3-action to get the manifold $M_\psi(\frac{2}{3})$ with $\psi = -\bar{\alpha}^2\beta$. $M_\psi(\frac{2}{3})$ is an irreducible, non-Haken, non-Seifert fibered manifold. On the other hand, the

Jørgensen's decomposition does not produce a hyperbolic structure on $M_\psi\ (\frac{2}{3})$ [B – P – Ż] however Thurston conjectures that this manifold posses a hyperbolic structure [T; Conjecture 1.1].

Now we recall the definition of an incompressible and ∂-incompressible surface (for S^2 and D^2 slightly different then that of [H] and [J]).

Definition 0.1.

a) Let M be a 3-manifold and F a surface which is either properly embedded in M or contained in ∂ M. We say that F is *compressible in M* if one of the following conditions is satisfied:

(i) F is a 2-sphere which bounds a 3-cell in M, or

(ii) F is a 2-cell and either $F \subset \partial M$ or there is a 3-cell $X \subset M$ with $\partial X \subset F \cup \partial M$, or

(iii) there is a 2-cell $D \subset M$ with $D \cap F = \partial D$ and with ∂D not contractible in F.

We say that F is *incompressible* if it is not compressible.

b) Let F be a submanifold of a manifold M. We say that F is *π_1-injective in M* if the inclusion-induced homomorphism from $\pi_1(F)$ to $\pi_1(M)$ is an injection.

c) Let F be a surface properly embedded in a compact 3-manifold M. We say that F is *∂-incompressible in M* if there is no a 2-disk $D \subset M$ such that $D \cap F = \alpha$ is an arc in ∂ D, $D \cap \partial M = \beta$ is an arc in ∂ D, with $\alpha \cap \beta = \partial \alpha = \partial \beta$ and $\alpha \cup \beta = \partial D$, and α is not parallel to ∂ F in F.

1. INCOMPRESSIBLE SURFACES IN PUNCTURED TORUS BUNDLES OVER S^1.

The classification of orientable, incompressible, ∂-incompressible surfaces in punctured torus bundles over S^1 is done in [F–H] and [C–J–R] and the nonorientable case is done in [P–2].

Let M_φ be a once-punctured torus bundle over S^1 with a hyperbolic monodromy map $\varphi \in SL(2, Z)$: that is $M_\varphi = F \times R/_\sim (x, t) \sim (\varphi(x), t + 1)$ where F denotes a punctured torus. For convenience we shall usually not distinguish the open manifold M_φ from its natural compactification obtained by adding a boundary torus. Now we establish a coordinate system of $H_1(\partial M_\varphi)$.

Definition 1.1.

The second generator *(longitude),* of $H_1(\partial M_\varphi)$ is determined by the boundary of a fiber (with the clockwise orientation, see Fig. 1.1). To define the first generator *(meridian)* of H_1 (∂M_φ) we have to consider two cases:

a) tr $\varphi > 0$; so φ has two positive eigenvalues. Then the restriction of φ to the boundary of a fiber (∂ F is understood to be the set of angles) has four fixed points, say $\mp \alpha_1$ and $\mp \alpha_2$.

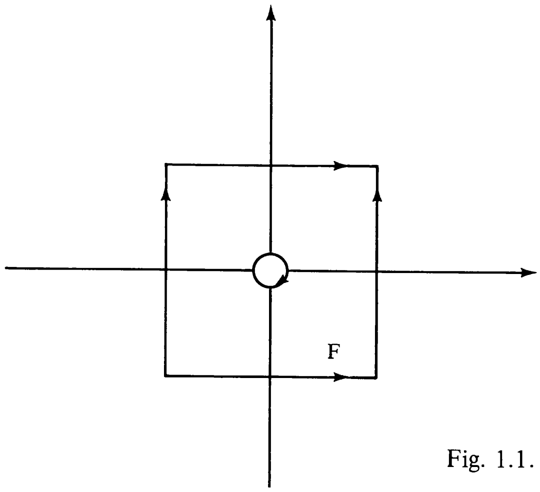


Fig. 1.1.

Now, the image under projection $F \times \mathbb{R} \to M_\varphi$ of the straight line in $\partial F \times \mathbb{R}$ which joins $(\alpha_1, 0)$ and $(\alpha_1, 1)$ is a circle which determines the first generator of H_1 (∂M_φ).

b) tr $\varphi < 0$; so φ has two negative eigenvalues. Then the restriction of $-\phi$ to ∂ F has four fixed points, say $\mp \alpha_1$ and $\mp \alpha_2$, so $\phi(\alpha_1) = -\alpha_1$. Let λ be the curve in $\partial F \times \mathbb{R}$ given by the equation $z = e^{\pi i t}$ where $z \in \partial F$ and $t \in \mathbb{R}$ (so λ joins $(\alpha_1, 0)$ and $(-\alpha_1, 1)$ with a negative half twist with respect to the chosen orientation of ∂ F). The image of λ under projection $F \times \mathbb{R} \to M_\phi$ determines the first generator of H_1 (∂M_ϕ). The slope of a curve on ∂M_ϕ is defined to be the quotient

$$\frac{\text{second coordinate of the curve}}{\text{first coordinate of the curve}}.$$

Let $\varphi' \in SL(2, Z)$ be a hyperbolic matrix, then φ' is conjugate to the matrix φ such that

$$\varphi = \pm \bar{\alpha}^{a_1} \beta^{a_2} \bar{\alpha}^{a_3} \beta^{a_4} \ldots \bar{\alpha}^{a_{2k-1}} \beta^{a_{2k}} \text{ where } k, a_i \geqslant 1 \qquad 1.1.$$

$$(i = 1, 2, \ldots, k) \text{ and } \bar{\alpha} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \beta = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

From now on, we assume that each hyperbolic φ is of the form 1.1.

Definition 1.2.

a) By the diagram of PSL (2, Z) we will understand the following graph, W,′ placed in a circle. The set of vertices of W′ is: $W = Q \cup (\infty)$. Two vertices

$$\frac{p_1}{q_1}, \frac{p_2}{q_2} \in W \text{ are joined by an edge iff } \det\begin{bmatrix} p_1 & p_2 \\ q_1 & q_2 \end{bmatrix} = \mp 1$$

(Fig. 1.2 a)) ; see [H – T] or [F – H].

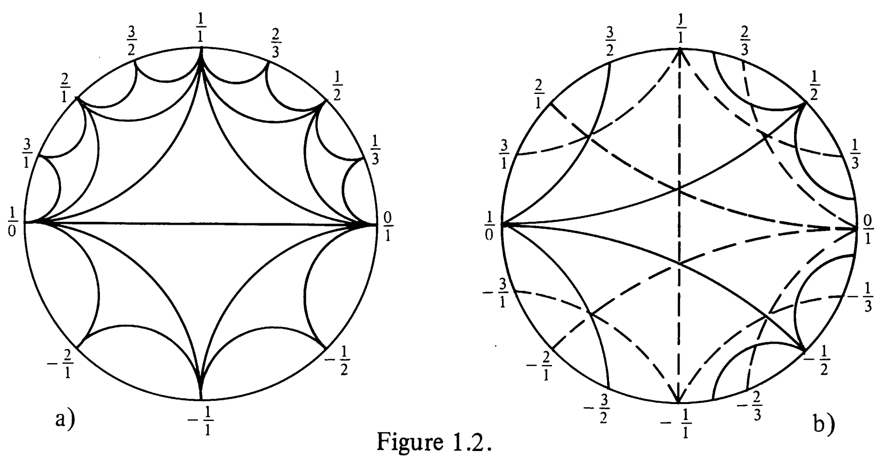


Figure 1.2.

b) Let $W_0 = \left\{\frac{p}{q} \in W \ : \ p \text{ and } q \text{ are odd}\right\}$,

$$W_1 = \left\{\frac{p}{q} \in W \ : \ q \text{ is even}\right\}, \quad W_2 = \left\{\frac{p}{q} \in W \ : \ p \text{ is even}\right\}.$$

We define $\overline{W}$ (respectively $\overline{W}_0$, $\overline{W}_1$ and $\overline{W}_2$) to be the graph with vertices W (resp. W_0, W_1 and W_2) and such that two vertices

$$\frac{p_1}{q_1}, \frac{p_2}{q_2} \in W$$

(resp. W_0, W_1 and W_2) are joined by an edge iff

$$\det\begin{bmatrix} p_1 & p_2 \\ q_1 & q_2 \end{bmatrix} = \mp 2$$

(Fig. 1.2b)). An edge-path

$$\ldots \frac{p_{i-1}}{q_{i-1}}, \frac{p_i}{q_i}, \frac{p_{i+1}}{q_{i+1}} \ldots$$

in $\overline{W}$ is said to be minimal if for each i

$$\frac{p_{i+1}}{q_{i+1}} \neq \frac{p_{i-1}}{q_{i-1}}$$

(i.e. we do not go back and forth). An edge path in the diagram of PSL (2, Z) is said to be minimal if no two succesive edges lie in the same triangle of the diagram. If $\varphi \in$ SL (2, Z) then the described graphs possess naturally defined action of φ, i.e. if

$$\varphi = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{then } \varphi\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} ay + bx \\ cy + dx \end{pmatrix}$$

$\left(\text{in particular the slope } \frac{1}{0} \text{ goes to } \frac{a}{c} \text{ and } \frac{0}{1} \text{ to } \frac{b}{d}\right)$.

For each φ-invariant edge-path γ in $\overline{W}$ we can uniquely associate a closed surface, say S_γ^c (see Construction 2.9 [P – 2]). For each φ-invariant edge-path γ in the diagram of PSL (2, Z) we can uniquely assign the surface S_γ (orientable or not). Let $\overline{S}_\gamma$ be the boundary of a regular neighborhood of S_γ for γ of odd length (so S_γ nonorientable); see [P–2].

Definition 1.3.

Consider a symbol $\gamma\,[\epsilon_1, \ldots, \epsilon_k]$ where γ is a minimal, φ-invariant edge-path in $\overline{W}$ of length k and $(\epsilon_1, \ldots, \epsilon_k) \in (Z_2)^k$. For such a symbol we have uniquely associated surface, say $S_\gamma\,(\epsilon_1, \ldots, \epsilon_k)$, in M_φ ([P – 2; Construction 2.10]). Furthermore this symbol uniquely determines an invariant (minimal or not) edge-path γ' in the diagram of PSL (2, Z):

If γ is defined by a sequence

$$\ldots \frac{a_{-2}}{b_{-2}}, \frac{a_0}{b_0}, \frac{a_2}{b_2}, \ldots, \frac{a_{2k}}{b_{2k}}, \ldots$$

where

$$\varphi\left(\frac{a_i}{b_i}\right)=\frac{a_{i+2k}}{b_{i+2k}},$$

then γ' is defined by the sequence of vertices

$$\dots\frac{a_{-1}}{b_{-1}},\frac{a_0}{b_0},\dots,\frac{a_{2k-1}}{b_{2k-1}},\frac{a_{2k}}{b_{2k}},\dots$$

where

$$\frac{a_{2i+1}}{b_{2i+1}}=\begin{cases}\dfrac{\frac{1}{2}(a_{2i+2}-a_{2i})}{\frac{1}{2}(b_{2i+2}-b_{2i})} & \text{if } \epsilon_{i+1}=0,\\[3ex] \dfrac{\frac{1}{2}(a_{2i+2}+a_{2i})}{\frac{1}{2}(b_{2i+2}+b_{2i})} & \text{if } \epsilon_{i+1}=1.\end{cases}$$

This formula is valid for

$$\frac{a_{2i}}{b_{2i}},\frac{a_{2i+2}}{b_{2i+2}}\geqslant 0.$$

We use the assumption that γ' is ϕ-invariant, to get all vertices of γ' (ϕ is of the form 1.1). In fact γ' is associated with the boundary of a regular neighborhood of $S_\gamma(\epsilon_1,\dots,\epsilon_k)$ in M_ϕ.

In considerations below we consider a period of γ with vertices $\frac{a_i}{b_i}\geqslant 0$.

Let $\sigma_i=\begin{cases}1 & \text{if } \frac{a_{2i+2}}{b_{2i+2}}>\frac{a_{2i}}{b_{2i}} \quad \text{(i.e. the edge } \frac{a_{2i}}{b_{2i}},\frac{a_{2i+2}}{b_{2i+2}} \text{ goes left on the diagram of PSL}(2,Z))\\ -1 & \text{if } \frac{a_{2i+2}}{b_{2i+2}}<\frac{a_{2i}}{b_{2i}} \quad \text{(i.e. the edge } \frac{a_{2i}}{b_{2i}},\frac{a_{2i+2}}{b_{2i+2}} \text{ goes right).}\end{cases}$

We introduce an equivalence relation among symbols $\gamma\,[\epsilon_1, \ldots, \epsilon_k]$ by elementary equivalences:

$$\gamma\,[\epsilon_1, \ldots, \epsilon_i, \epsilon_{i+1}, \ldots, \epsilon_k] \sim \gamma\,[\epsilon_1, \ldots, 1-\epsilon_i, 1-\epsilon_{i+1}, \ldots, \epsilon_k]$$

if

$$\frac{a_{2i+2}}{b_{2i+2}} = \frac{a_{2i-2} \pm 2\,a_{2i}}{b_{2i-2} \pm 2\,b_{2i}}$$

and either (i) $\epsilon_i = \epsilon_{i+1} = 0$ and $\sigma_{i-1} = -\sigma_i$ or (ii) $\epsilon_i = 1 - \epsilon_{i+1}$ and $\sigma_{i-1} = \sigma_i$.

Definition 1.4.

We define, here, a new graph, which we call the special graph. The set of vertices of the special graph consists of ordered pairs of slopes

$\left(\frac{a}{b}, \frac{c}{d}\right)$ which satisfy: $\det\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \mp 1$. Two vertices $\left(\frac{a_1}{b_1}, \frac{c_1}{d_1}\right)$ and $\left(\frac{a_2}{b_2}, \frac{c_2}{d_2}\right)$

are joined by an edge iff either

(i) $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ and $\det\begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix} = \mp 2$ or

(ii) $\frac{c_1}{d_1} = \frac{c_2}{d_2}$ and $\det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = \mp 2.$

An edge-path γ in the special graph defines two edge-paths in teh graph $\overline{W}$. Namely if

$$\gamma = \ldots\left(\frac{a_1}{b_1}, \frac{c_1}{d_1}\right), \left(\frac{a_2}{b_2}, \frac{c_2}{d_2}\right)\ldots \quad \text{then}$$

$$\gamma_1 = \ldots \frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3} \ldots$$

$$\gamma_2 = \ldots \frac{c_1}{d_1}, \frac{c_2}{d_2}, \frac{c_3}{d_3} \ldots.$$

(we allow here, for simplicity, repetitions of consecutive slopes)

We say that an edge-path γ in the special graph is minimal if the associated edge-paths γ_1 and γ_2 are minimal in $\overline{W}$. We say that γ is ϕ-invariant if $\phi(\gamma)=\gamma$ or $-\gamma$ ($-\gamma$ is obtained from γ by changing the order of slopes in each vertex of γ). For each ϕ-invariant, edge-path γ in the special graph, we have uniquely associated surface S_γ^{sp} in M_ϕ ([P–2; Construction 2.12]). If $\phi(\gamma)=\gamma$ then S_γ^{sp} consists of two components and if $\phi(\gamma)=-\gamma$ then S_γ^{sp} is connected.

Now we are ready to formulate the classification theorems.

Theorem 1.5 [P – 2].

Let M_ϕ be a punctured-tous bundle over S^1 with a hyperbolic monodromy map ϕ. Then:

a) Each closed, connected, incompressible surface in M_ϕ is either

(i) a torus parallel to the boundary, or

(ii) isotopic to one of a finite number (0,1 or 3) nonorientable surfaces S_γ^c, where γ is a minimal, ϕ-invariant edge-path in $\overline{W}$.

b) Each connected, incompressible, ∂-incompressible surface, S, in M_ϕ with ∂ S parallel to the boundary of a fiber is isotopic to a fiber.

c) Each connected, incompressible, ∂-incompressible surface, S, in M_ϕ with ∂S ($\neq \phi$) transverse to each fiber is either

(i) isotopic to one of the surfaces S_γ indexed by a minimal, ϕ-invariant edge-path γ in the diagram of PSL (2, Z) or to $\overline{S}_\gamma$ where the length of γ is odd and γ is minimal and ϕ-invariant in the diagram of PSL (2, Z), or

(ii) isotopic to one of the surfaces $S_\gamma(\epsilon_1, \ldots, \epsilon_k)$, where γ is a minimal, ϕ-invariant edge-path in $\overline{W}$, or

(iii) isotopic to a surface S_γ^{sp} associated to a minimal, ϕ-invariant edge-path γ in the special graph with $\phi(\gamma)=-\gamma$.

Theorem 1.6 [P – 2].

Let γ be a minimal, ϕ-invariant edge-path in the diagram of PSL (2, Z), $\overline{W}$ or in the special graph. Then, surfaces S_γ, $\overline{S}_\gamma$, S_γ^c, $S_\gamma(\epsilon_1, \ldots, \epsilon_k)$ and S_γ^{sp} are incompressible, ∂-incompressible (or closed), if defined. Two surfaces from the above are isotopic iff the following conditions are satisfied:

(i) the surfaces are associated with the same γ (up to sign, in the case of S_γ^{sp}).

(ii) they are in the same class (S_γ, $\overline{S}_\gamma$, S_γ^c, $S_\gamma(\epsilon_1, \ldots, \epsilon_k)$ or S_γ^{sp}) and

(iii) $[\epsilon_1', \ldots, \epsilon_k'] \sim [\epsilon_1'', \ldots, \epsilon_k'']$ if we deal with surfaces of type $S_\gamma(\epsilon_1, \ldots, \epsilon_k)$ (see Definition 1.3).

***Proposition 1.7** [P – 2].*

Let g (S) denote the genus of a surface S, b (S) the number of boundary components of S and sl (S) – the slope of ∂ S on ∂ M_ϕ. The following table establishes dependences among γ (of length k), b (S), g (S), sl (S) . . . (compare [F – H; Table 1]):

Table 1.1.

S	k	tr ϕ	g (S)	b (S)	sl (S)	orientation
S_γ	odd	positive	k + 1	1	$\frac{L-R}{4}$	nonorientable
		negative			$\frac{L-R+2}{4}$	
	even	positive	$\frac{k}{2}-\frac{b(S)}{2}+1$	h.c.f. (L – R, 4)	$\frac{L-R}{4}$	orientable
		negative		h.c.f. (L – R + 2, 4)	$\frac{L-R+2}{4}$	
$\bar{S}_\gamma$	odd	positive	k	2	$\frac{L-R}{4}$	orientable
		negative		2	$\frac{L-R+2}{4}$	
S^c_γ	either	either	2 + k	0	–	nonorientable
S^∂_γ	either	either	2 + k	1	$\frac{1}{0}$	
$S_\gamma(\epsilon_1,\dots,\epsilon_k)$	either	positive	k + 2 – b (S)	h.c.f. $((\Sigma\,\sigma_i\epsilon_i), 2)$	$\frac{\sum_{i=1}^{k}\sigma_i\epsilon_i}{2}$	
		negative		h.c.f. $((\Sigma\,\sigma_i\epsilon_i)+1, 2)$	$\frac{\sum_{i=1}^{k}\sigma_i\epsilon_i+1}{2}$	
S^{sp}_γ $\phi(\gamma) = -\gamma$	either	positive	k + 2 – b (S)	$\frac{1}{2}$ h.c.f. $(L_{\gamma_1'} - R_{\gamma_1'}, 4)$	$\frac{L_{\gamma_1'}-R_{\gamma_1'}}{8}$	nonorientable
		negative			$\frac{L_{\gamma_1'}-R_{\gamma_1'}+4}{8}$	

2. NONORIENTABLE, INCOMPRESSIBLE SURFACES OF GENUS ⩽ 3 IN M_ϕ.

The following lemma is the key lemma in the proof of Theorem 3.2.

Lemma 2.1.

The following Tables describe all nonorientable, incompressible, ∂-incompressible surfaces of genus ⩽ 3 in manifolds M_ϕ (ϕ-hyperbolic). If S_1 and S_2 are surfaces in M_ϕ associated to different edge-paths and if there exists a homeomorphism f: $M_\phi \to M_\phi$ such that $f(S_1) = S_2$.

a) Table 2.1; nonorientable, incompressible, ∂-incompressible surfaces of genus ⩽ 3 in M_ϕ, associated with edge-paths γ of length k ⩽ 2. We use the terminology of previous sections, especially that of Proposition 1.7.

Table 1.2.

Type of S	k	g (S)	b (S)	ϕ	γ (one period)	sl (S)	S; exact description	Is S π_1 - injective?	Does an orientable, incompressible, ∂-incompressible surface of slope sl (S) exists? If yes what is its minimal genus.
S_γ	1	2	1	$\bar{\alpha}\beta^{a_2}$	$\frac{0}{1}, \frac{1}{1}$	$\frac{1}{4}$	S_γ	Yes	Yes (1)
				$-\bar{\alpha}\beta^{a_2}$		$\frac{3}{4}$			
$S_\gamma(\epsilon_1, \dots, \epsilon_k)$	1	1	2	$\bar{\alpha}^2\beta^{a_2}$	$\frac{0}{1}, \frac{2}{1}$	$\frac{0}{1}$	$S_\gamma(0)$	Yes if $a_2 > 1$ No if $a_2 = 1$	Yes (0) if $a_2 > 1$ No if $a_2 = 1$
		2	1			$\frac{1}{2}$	$S_\gamma(1)$	Yes	Yes (1)
		2	1	$-\bar{\alpha}^2\beta^{a}$		$\frac{1}{2}$	$S_\gamma(0)$	Yes if $a_2 > 1$ No if $a_2 = 1$	Yes (1) if $a_2 > 1$ No if $a_2 = 1$
		1	2			$\frac{1}{1}$	$S_\gamma(1)$	Yes	Yes (0)
	2	2	2	$\bar{\alpha}^2\beta^2$	$\frac{1}{1}, \frac{3}{1}, \frac{7}{3}$	$\frac{0}{1}$	$S_\gamma(1,1) = S_\gamma(0,0)$	No	Yes (0)
		3	1			$\frac{1}{2}$	$S_\gamma(1,0)$		Yes (1)
		3	1			$-\frac{1}{2}$	$S_\gamma(0,1)$		Yes (1)
		3	1	$-\bar{\alpha}^2\beta^2$		$\frac{1}{2}$	$S_\gamma(1,1) = S_\gamma(0,0)$		Yes (1)
		2	2			$\frac{1}{1}$	$S_\gamma(1,0)$		Yes (0)
		2	2			$\frac{0}{1}$	$S_\gamma(0,1)$		Yes (0)

Table 1.2. (Continuation)

Type of S	k	g(S)	b(S)	ϕ	γ (one period)	sl (S)	S; exact description	Is S π_1 – injective?	Does an orientable, incompressible, ∂-incompressible surface of slope sl (S) exists? If yes what is its minimal genus
$S_\gamma(\epsilon_1,\ldots,\epsilon_k)$	2	2	2	$\bar{\alpha}^4\beta^{a_2}$	$\frac{0}{1},\frac{2}{1},\frac{4}{1}$	$\frac{0}{1}$	$S_\gamma(0,0)$	No	Yes (0) if $a_2>1$ No if $a_2=1$
		3	1			$\frac{1}{2}$	$S_\gamma(1,0)=S_\gamma(0,1)$	No	No
		2	2			$\frac{1}{1}$	$S_\gamma(1,1)$	Yes	Yes (1)
		3	1	$-\bar{\alpha}^4\beta^{a_2}$		$\frac{1}{2}$	$S_\gamma(0,0)$	No	Yes (0) if $a_2>1$ No if $a_2=1$
		2	2			$\frac{1}{1}$	$S_\gamma(1,0)=S_\gamma(0,1)$	No	No
		3	1			$\frac{3}{2}$	$S_\gamma(1,1)$	Yes	Yes (2)
		2	2	$\bar{\alpha}^2\beta^{a_2}\bar{\alpha}^2\beta^{a_4}$	$\frac{0}{1},\frac{2}{1},\frac{4a_2+4}{2a_2+1}$	$\frac{0}{1}$	$S_\gamma(0,0)$	Yes if $a_2,a_4>1$ No if a_2 or $a_4=1$	Yes (1) if $a_2,a_4>1$ No if a_2 or $a_4=1$
		3	1			$\frac{1}{2}$	$S_\gamma(0,1)$	Yes	Yes (2)
						$\frac{1}{2}$	$S_\gamma(1,0)$	Yes	Yes (2)
		2	2			$\frac{1}{1}$	$S_\gamma(1,1)$	Yes	Yes (1)
		3	1	$-\bar{\alpha}^2\beta^{a_2}\bar{\alpha}^2\beta^{a_4}$		$\frac{1}{2}$	$S_\gamma(0,0)$	Yes if $a_2,a_4>1$ No if a_2 or $a_4=1$	Yes (2) if $a_2,a_4>1$ No if a_2 or $a_4=1$
		2	2			$\frac{1}{1}$	$S_\gamma(0,1)$	Yes	Yes (1)
						$\frac{1}{1}$	$S_\gamma(1,0)$	Yes	Yes (1)
		2	1			$\frac{3}{2}$	$S_\gamma(1,1)$	Yes	Yes (2)
		3	1	$\bar{\alpha}\beta^2\bar{\alpha}\beta^{a_4}$	$\frac{0}{1},\frac{2}{1},\frac{4}{3}$	$\frac{1}{2}$	$S_\gamma(1,0)$	No	Yes (1)
		2	2			$\frac{0}{1}$	$S_\gamma(0,0)=S_\gamma(1,1)$	No	No
		3	1			$\frac{1}{2}$	$S_\gamma(0,1)$	Yes if $a_4>1$ No if $a_4=1$	Yes (2) if $a_4>1$ No if $a_4=1$
		2	2	$-\bar{\alpha}\beta^2\bar{\alpha}\beta^{a_4}$		$\frac{1}{1}$	$S_\gamma(1,0)$	No	Yes (0)
		3	1			$\frac{1}{2}$	$S_\gamma(0,0)=S_\gamma(1,1)$	No	No
		2	2			$\frac{0}{1}$	$S_\gamma(0,1)$	Yes if $a_4>1$ No if $a_4=1$	Yes (1) if $a_4>1$ No if $a_4=1$

Table 1.2. (Continuation)

Type of S	k	g(S)	b(S)	ϕ	γ (one period)	sl(S)	S; exact description	Is S π_1 – injective?	Does an orientable, incompressible, ∂-incompressible surface of slope sl (S) exists? If yes what is its minimal genus
$S_\gamma(\epsilon_1,\dots,\epsilon_k)$	2	3	1	$\bar\alpha\beta\bar\alpha^{a_3}\beta\bar\alpha\beta^{a_6}$	$\frac{0}{1},\frac{2}{1},\frac{4a_3+4}{2a_3+3}$	$\frac{1}{2}$	$S_\gamma(1,0)$	Yes if $a_3>1$ No if $a_3=1$	Yes (2) if $a_3>1$ No if $a_3=1$
		2	2			$\frac{0}{1}$	$S_\gamma(0,0)$	Yes	Yes (1)
							$S_\gamma(1,1)$	Yes	Yes (1)
		3	1			$-\frac{1}{2}$	$S_\gamma(0,1)$	Yes if $a_6>1$ No if $a_6=1$	Yes (2) if $a_6>1$ No if $a_6=1$
		2	2	$-\bar\alpha\beta\bar\alpha^{a_3}\beta\bar\alpha\beta^{a_6}$		$\frac{1}{1}$	$S_\gamma(1,0)$	Yes if $a_3>1$ No if $a_3=1$	Yes (1) if $a_3>1$ No if $a_3=1$
		3	1			$\frac{1}{2}$	$S_\gamma(0,0)$	Yes	Yes (2)
							$S_\gamma(1,1)$	Yes	Yes (2)
		2	2			$\frac{0}{1}$	$S_\gamma(0,1)$	Yes if $a_6>1$ No if $a_6=1$	Yes (1) if $a_6>1$ No if $a_6=1$
S_γ^{sp} $\phi(\gamma)=-\gamma$	2	3	1	$\bar\alpha\beta^2$	$\left(\frac{1}{0},\frac{1}{1}\right),\left(\frac{1}{0},\frac{3}{1}\right),\left(\frac{5}{2},\frac{3}{1}\right)$	$\frac{1}{4}$	S_γ^{sp}	No	Yes (1)
				$-\bar\alpha\beta^2$		$\frac{3}{4}$			

b) Table 2.2. Nonorientable, incompressible, ∂-incompressible surfaces of genus 3 in M_ϕ, associated with edge-path γ of length $k\geqslant 3$. In fact, under these restrictions, the genus is always equal to 3, the edge-path always has length 3 and each surface S is of type $S_\gamma(\epsilon_1,\dots,\epsilon_k)$ with $b(S)=2$.

Table 1.3.

ϕ	γ	sl (S)	S; exact description	Is S π_1- injective?	Does an orientable, incompressible, ∂-incompressible surface of slope sl (S) exists? If yes what is its minimal genus
$\bar{\alpha}^2 \beta^4$	$\frac{1}{1}, \frac{3}{1}, \frac{7}{3}, \frac{11}{5}$	$\frac{0}{1}$	$S_\gamma(0,0,0) = S_\gamma(1,1,0) = S_\gamma(1,0,1)$	No	Yes (0)
		$-\frac{1}{1}$	$S_\gamma(0,1,1)$	No	Yes (1)
$-\bar{\alpha}^2 \beta^4$		$\frac{1}{1}$	$S_\gamma(1,0,0)$	No	Yes (0)
		$\frac{0}{1}$	$S_\gamma(0,1,0) = S_\gamma(1,1,1) = S_\gamma(0,0,1)$	No	No
$\bar{\alpha}^6 \beta^{a_2}$	$\frac{0}{1}, \frac{2}{1}, \frac{4}{1}, \frac{6}{1}$	$\frac{0}{1}$	$S_\gamma(0,0,0)$	No	Yes (0) if $a_2 > 1$ No if $a_2 = 1$
		$\frac{1}{1}$	$S_\gamma(1,1,0) = S_\gamma(1,0,1) = S_\gamma(0,1,1)$	No	No
$-\bar{\alpha}^6 \beta^{a_2}$		$\frac{1}{1}$	$S_\gamma(1,0,0) = S_\gamma(0,1,0) = S_\gamma(0,0,1)$	No	No
		$\frac{2}{1}$	$S_\gamma(1,1,1)$	Yes	Yes (2)
$\bar{\alpha}^4 \beta \bar{\alpha}^{a_3} \beta$	$\frac{1}{1}, \frac{3}{1}, \frac{5}{1}, \frac{10a_3+13}{2a_3+3}$	$\frac{0}{1}$	$S_\gamma(0,0,0) = S_\gamma(0,1,1) = S_\gamma(1,0,1)$	No	No
		$\frac{1}{1}$	$S_\gamma(1,1,0)$	Yes if $a_3 > 1$ No if $a_3 = 1$	Yes (2) if $a_3 > 1$ No if $a_3 = 1$
$-\bar{\alpha}^4 \beta \bar{\alpha}^{a_3} \beta$		$\frac{0}{1}$	$S_\gamma(0,0,1)$	No	Yes (0)
		$\frac{1}{1}$	$S_\gamma(1,1,1) = S_\gamma(1,0,0) = S_\gamma(0,1,0)$	No	Yes (1) if $a_3 = 2$ No if $a_3 \neq 2$

Table 1.3. (Continuation)

ϕ	γ	sl (S)	S; exact description	Is S π_1-injective?	Does an orientable, incompressible, ∂-incompressible surface of slope sl (S) exists? If yes what is its minimal genus
$\bar{\alpha}^2\beta^{a_2}\bar{\alpha}^4\beta^{a_4}$	$\frac{0}{1}, \frac{2}{1}, \frac{4a_2+4}{2a_2+1}, \frac{8a_2+6}{4a_2+1}$	$\frac{0}{1}$	$S_\gamma(0,0,0)$	No	Yes (1) if a_2 and $a_4 > 1$ No if a_2 or $a_4 = 1$
		$\frac{1}{1}$	$S_\gamma(1,1,0) = S_\gamma(1,0,1)$	No	Yes (2)
			$S_\gamma(0,1,1)$	Yes	Yes (2)
$-\bar{\alpha}^2\beta^{a_2}\bar{\alpha}^4\beta^{a_4}$		$\frac{1}{1}$	$S_\gamma(1,0,0)$	No	Yes (1)
			$S_\gamma(0,1,0) = S_\gamma(0,0,1)$	No	Yes (1)
		$\frac{2}{1}$	$S_\gamma(1,1,1)$	Yes	Yes (2)
$\bar{\alpha}\beta^2\bar{\alpha}\beta^{a_4}\bar{\alpha}^2\beta^{a_6}$	$\frac{0}{1}, \frac{2}{1}, \frac{4}{3}, \frac{8a_4+10}{6a_4+7}$	$\frac{0}{1}$	$S_\gamma(0,0,0) = S_\gamma(1,1,0)$ $S_\gamma(0,1,1)$	No Yes	Yes (1) if a_4 or $a_6 = 1$; (2) if a_4 and $a_6 \neq 1$
		$\frac{1}{1}$	$S_\gamma(1,0,1)$	No	Yes (1)
$-\bar{\alpha}\beta^2\bar{\alpha}\beta^{a_4}\bar{\alpha}^2\beta^{a_6}$		$\frac{1}{1}$	$S_\gamma(0,1,0)$	No	Yes (2) if either a_4 and $a_6 > 1$ or if a_4 or $a_6 = 3$ No in other cases
		$\frac{1}{1}$	$S_\gamma(1,1,1) = S_\gamma(0,0,1)$	No	Yes (1)
			$S_\gamma(1,0,0)$	No	Yes (1)
$\bar{\alpha}\beta^2\bar{\alpha}\beta^{a_4}$	$\frac{0}{1}, \frac{2}{1}, \frac{4}{1}, \frac{10}{4}$	$\frac{1}{1}$	$S_\gamma(1,1,0)$	No	Yes (1)
		$\frac{0}{1}$	$S_\gamma(1,0,1) = S_\gamma(0,1,1) = S_\gamma(0,0,0)$	No	No
$-\bar{\alpha}^3\beta^2\bar{\alpha}\beta^{a_4}$		$\frac{1}{1}$	$S_\gamma(1,1,1) = S_\gamma(1,0,0) = S_\gamma(0,1,0)$	No	No
		$\frac{0}{1}$	$S_\gamma(0,0,1)$	No.	Yes (1) if $a_4 > 1$ No if $a_4 = 1$

Table 1.3. (Continuation)

ϕ	γ	sl (S)	S; exact description	Is S π_1-injective?	Does an orientable, incompressible, ∂-incompressible surface of slope sl (S) exists? If yes what is its minimal genus
$\bar{\alpha}^3 \beta \bar{\alpha}^{a_3} \beta \bar{\alpha} \beta^{a_6}$	$\frac{0}{1}, \frac{2}{1}, \frac{4}{1}, \frac{8a_3+10}{2a_3+3}$	$\frac{1}{1}$	$S_\gamma(1,1,0)$	Yes if $a_3 > 1$ No if $a_3 = 1$	Yes (2) if $a_3 > 1$ No if $a_3 = 1$
		$\frac{0}{1}$	$S_\gamma(1,0,1) =$ $= S_\gamma(0,1,1)$	No	Yes (1)
			$S_\gamma(0,0,0)$	No	Yes (1)
$-\bar{\alpha}^3 \beta \bar{\alpha}^{a_3} \beta \bar{\alpha} \beta^{a_6}$		$\frac{1}{1}$	$S_\gamma(1,1,1)$	Yes	Yes (1) if $a_3 = 1$ (2) if $a_3 > 1$
			$S_\gamma(1,0,0) =$ $= S_\gamma(0,1,0)$	No	
		$\frac{0}{1}$	$S_\gamma(0,0,1)$	No	Yes (1) if $a_6 > 1$ No if $a_6 = 1$
$\bar{\alpha} \beta \bar{\alpha}^{a_3} \beta^2 \bar{\alpha}^{a_5} \beta \bar{\alpha} \beta^{a_8}$	$\frac{0}{1}, \frac{2}{1}, \frac{4a_3+4}{2a_3+3}, \phi\left(\frac{0}{1}\right)$	$\frac{0}{1}$	$S_\gamma(0,0,0)$	Yes if $a_5 > 1$ No if $a_5 = 1$	Yes (2)
			$S_\gamma(1,1,0)$	Yes	Yes (2)
			$S_\gamma(1,0,1)$	Yes if $a_3 > 1$ No if $a_3 = 1$	Yes (2)
		$-\frac{1}{1}$	$S_\gamma(1,0,1)$	Yes if $a_3 > 1$ No if $a_3 = 1$	Yes (2) if $a_8 > 1$ No if $a_8 = 1$
$-\bar{\alpha} \beta \bar{\alpha}^{a_3} \beta^2 \bar{\alpha}^{a_5} \beta \bar{\alpha} \beta^{a_8}$		$\frac{1}{1}$	$S_\gamma(1,0,0)$	Yes if a_3 and $a_5 > 1$. No if a_3 or $a_5 = 1$	Yes (2) if a_3 and $a_5 > 1$ or $a_i = 1$ and $a_j = 2$ Yes ($a_8 + 1$) if $a_i = 1$, $a_j = a_8 + 1$ Yes ($a_8 + 2$) if $a_i = 1$, $a_j = a_8 + 3$ ($\{i, j\} = \{3, 5\}$) No otherwise
		$\frac{0}{1}$	$S_\gamma(1,1,1)$	Yes	Yes (2)
			$S_\gamma(0,0,1)$	Yes if $a_8 > 1$ No if $a_8 = 1$	Yes (2)
			$S_\gamma(0,0,1)$	Yes if $a_5 > 1$ No if $a_5 = 1$	Yes (2)

Table 1.3. (Continuation)

ϕ	γ	sl (S)	S; exact description	Is S π_1 – injective?	Does an orientable, incompressible, ∂-incompressible surface of slope sl (S) exists? If yes what is its minimal genus
$\bar{\alpha}^2\beta^{a_2}\bar{\alpha}^2\beta^{a_4}\bar{\alpha}^2\beta^{a_6}$	$\frac{0}{1}, \frac{2}{1}, \frac{4a_2+4}{2a_2+1}, \phi\left(\frac{0}{1}\right)$	$\frac{1}{1}$	$S_\gamma(1, 1, 0)$	Yes	Yes (2)
			$S_\gamma(0, 1, 1)$	Yes	Yes (2)
			$S_\gamma(1, 0, 1)$	Yes	Yes (2)
		$\frac{0}{1}$	$S_\gamma(0, 0, 0)$	Yes if a_2, a_4 and $a_6 > 1$; No if a_2, a_4 or $a_6 = 1$	Yes (2) if either a_2, a_4 and $a_6 > 1$ or a_2, a_4 or $a_6 = 2$ No otherwise
$-\bar{\alpha}^2\beta^{a_2}\bar{\alpha}^2\beta^{a_4}\bar{\alpha}^2\beta^{a_6}$		$\frac{2}{1}$	$S_\gamma(1, 1, 1)$	Yes	Yes (2)
		$\frac{1}{1}$	$S_\gamma(1, 0, 0)$	Yes if $a_4 > 1$ No if $a_4 = 1$	Yes (2) if either a_2, a_4 or $a_6 > 1$ No if $a_2 = a_4 = a_6 = 1$
			$S_\gamma(0, 1, 0)$	Yes if $a_6 > 1$ No if $a_6 = 1$	
			$S_\gamma(0, 0, 1)$	Yes if $a_2 > 1$ No if $a_2 = 1$	

Proof of Lemma 2.1. Remind, after [F – H] and [P – 2], that instead of analysing full diagram of PSL (2, Z) or $\overline{W}$ it is enough to consider the infinite strip associated with ϕ (given by 1.1) as on Fig. 2.1.

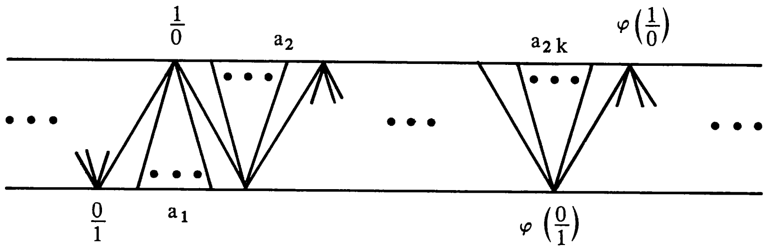


Fig. 2.1.

Now we use Theorems 1.5 and 1.6, and most importantly the table of Proposition 1.7. It follows from this table that we have the following possibilities:

1. S is of type S_γ, so the inequality $k + 1 \leqslant 3$ with k odd is satisfied, hence $k = 1$ and S is a Klein bottle with a hole and ϕ must be equal to $\pm\bar{\alpha}\beta^{a_2}$. Consider the diagram associated to ϕ:

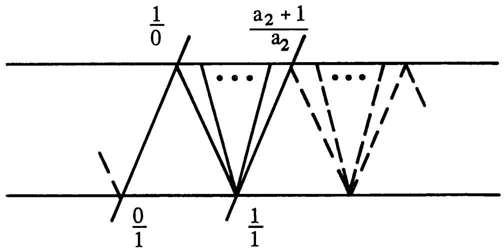


Fig. 2.2.

The analysis of this diagram yields the first part of Table 2.1.

2. S is of type S_γ $(\epsilon_1, \ldots, \epsilon_k)$ so the inequality $k + 2 - b(S) \leqslant 3$ is satisfied. We have several cases:

a) $k = 1$; ϕ must be equal to $\pm \bar{\alpha}^2 \beta^{a_2}$. Consider the diagram associated to ϕ:

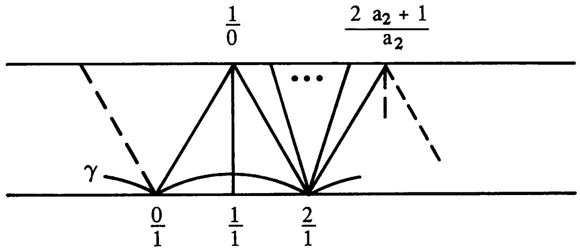


Fig. 2.3.

If $a_2 \neq 2$ then the only minimal invariant edge-path γ of length 1 is determined by vertices $\frac{0}{1}$, $\frac{2}{1}$. If $a_2 = 2$ we have the edge-path determined by vertices $\frac{1}{0}, \frac{5}{2}$ but we will not consider this edge-path (see the remark before Table 2.1). $\gamma = \ldots \frac{0}{1}, \frac{2}{1} \ldots$ leads us to four surfaces: (i) (if $\phi = \bar{\alpha}^2 \beta^{a_2}$) S_γ (0) with the slope equal to $\frac{0}{1}$ and S_γ (1) with the slope $\frac{1}{2}$ and (ii) (if $\phi = -\bar{\alpha}^2 \beta^{a_2}$) S_γ (0) with the slope $\frac{1}{2}$ and S_γ (1) with the slope $\frac{1}{1}$. In order to recognize whether S is π_1-injective we analyse the boundary of the tubular neighborhood of S (see Def. 1.3). To fill the last column in Table 2.1, we analyse Fig. 2.3 using the [F – H] theorem.

b) k = 2. We have several possibilities. Their analysis yields the remaining parts of Table 2.1.

(i) $\phi = \pm\overline{\alpha}\beta^2$ with $\gamma = \ldots \frac{1}{1}, \frac{3}{1}, \frac{7}{3} \ldots$ and surfaces S_γ (1, 1) = S_γ (0, 0)

S_γ (1, 0) and S_γ (0, 1).

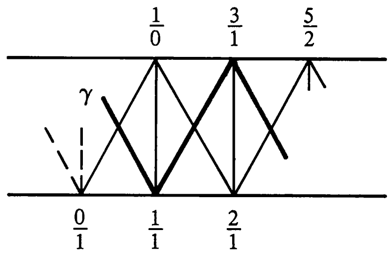


Fig. 2.4.

(ii) $\phi = \pm\overline{\alpha}^2\beta^{a_2}$ and $\gamma = \ldots \frac{0}{1}, \frac{2}{1}, \frac{4}{1} \ldots$

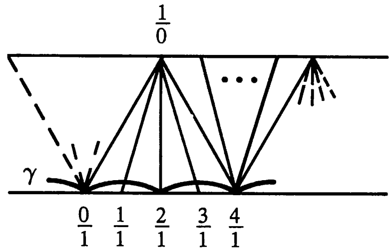


Fig. 2.5.

(iii) $\phi = \pm\overline{\alpha}^2\beta^{a_2}\overline{\alpha}^2\beta^{a_4}$ and $\gamma = \ldots \frac{0}{1}, \frac{2}{1}, \frac{4a_2+4}{2a_2+1} \ldots$

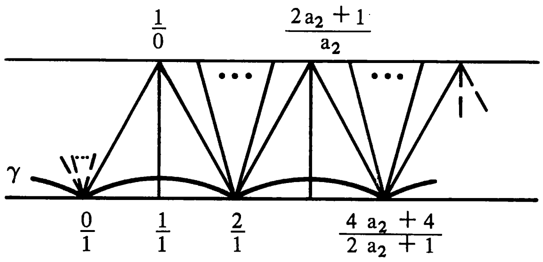


Fig. 2.6.

(iv) $= \pm\overline{\alpha}\beta^2\overline{\alpha}\beta^{a_4}$ and $\gamma = \ldots \frac{0}{1}, \frac{2}{1}, \frac{4}{3} \ldots$

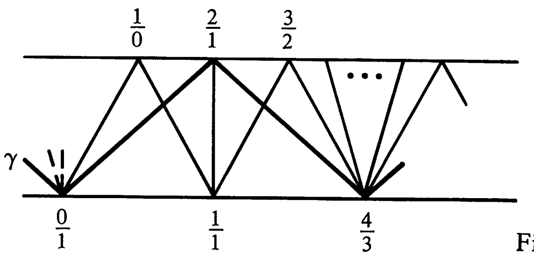


Fig. 2.7.

(v) $\phi = \pm\bar{\alpha}\,\beta\bar{\alpha}^{a_3}\,\beta\bar{\alpha}\beta^{a_6}$ and $\gamma = \dots \frac{0}{1}, \frac{2}{1}, \frac{4a_3+4}{2a_3+3} \dots$

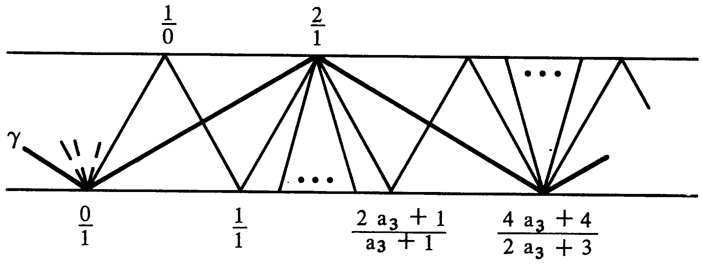


Fig. 2.8.

c) $k = 3$. We have several possibilities. Their analysis yields Table 2.2.

(i) $\phi = \pm\bar{\alpha}^2\,\beta^4$ and $\gamma = \dots \frac{1}{1}, \frac{3}{1}, \frac{7}{3}, \frac{11}{5} \dots$

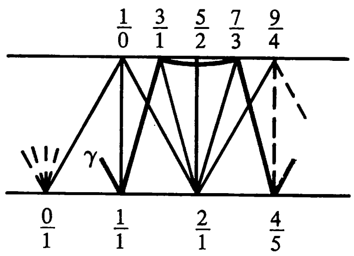


Fig. 2.9.

(ii) $\phi = \pm\bar{\alpha}^6\,\beta^{a_2}$ and $\gamma = \dots \frac{0}{1}, \frac{2}{1}, \frac{4}{1}, \frac{6}{1} \dots$

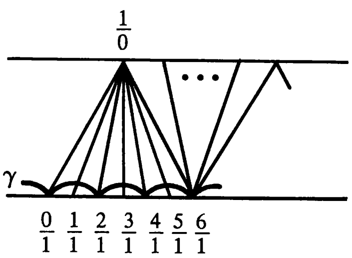


Fig. 2.10.

(iii) $\phi = \pm\bar{\alpha}^4 \beta \bar{\alpha}^{a_3} \beta$ and $\gamma = \ldots \frac{1}{1}, \frac{3}{1}, \frac{5}{1}, \frac{10\,a_3+13}{2\,a_3+3} \ldots$

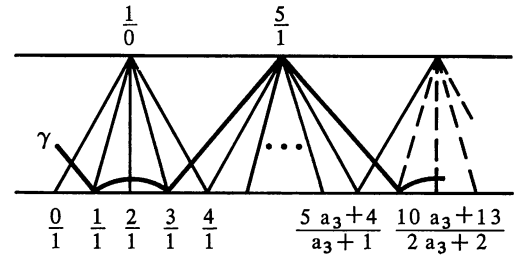


Fig. 2.11.

(iv) $\phi = \pm\bar{\alpha}^2 \beta^{a_2} \bar{\alpha}^4 \beta^{a_4}$ and $\gamma = \ldots \frac{0}{1}, \frac{2}{1}, \frac{4\,a_2+4}{2\,a_2+1}, \frac{8\,a_2+6}{4\,a_2+1} \ldots$

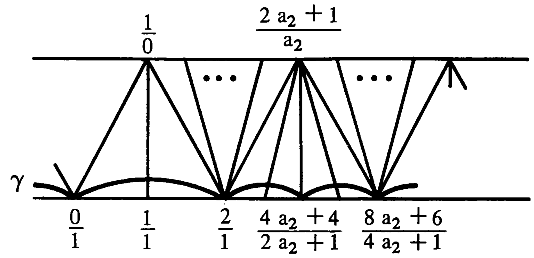


Fig. 2.12.

(v) $\phi = \pm\bar{\alpha}\beta^2 \bar{\alpha}\beta^{a_4} \bar{\alpha}^2 \beta^{a_6}$ and $\gamma = \ldots \frac{0}{1}, \frac{2}{1}, \frac{4}{3}, \frac{8\,a_4+10}{6\,a_4+7} \ldots$

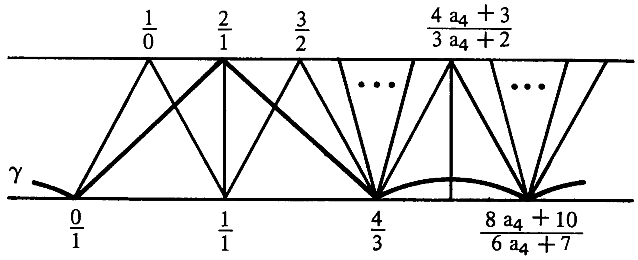


Fig. 2.13.

(vi) $\phi = \pm\bar{\alpha}^3 \beta^2 \bar{\alpha}\beta^{a_4}$ and $\gamma = \ldots \frac{0}{1}, \frac{2}{1}, \frac{4}{1}, \frac{10}{3} \ldots$

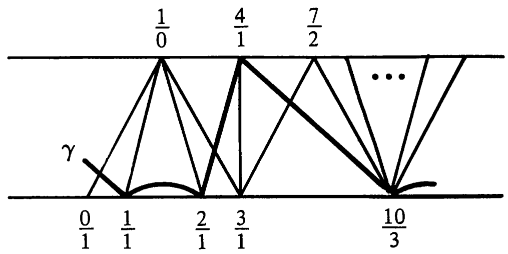


Fig. 2.14.

(vii) $\phi = \pm\, \overline{\alpha}^3 \beta \overline{\alpha}^{a_3} \beta \overline{\alpha} \beta^{a_6}$ and $\gamma = \dots \frac{0}{1}, \frac{2}{1}, \frac{4}{1}, \frac{8\,a_3 + 10}{2\,a_3 + 3} \dots$

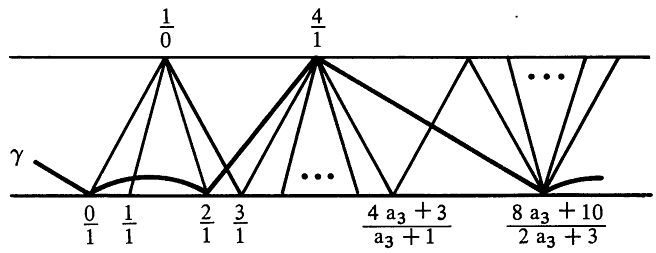


Fig. 2.15.

(viii) $\phi = \pm \overline{\alpha} \beta \overline{\alpha}^{a_3} \beta^2 \overline{\alpha}^{a_5} \beta \overline{\alpha} \beta^{a_8}$ and $\gamma = \dots \frac{0}{1}, \frac{2}{1}, \frac{4\,a_3 + 4}{2\,a_3 + 3}, \phi\left(\frac{0}{1}\right) \dots$

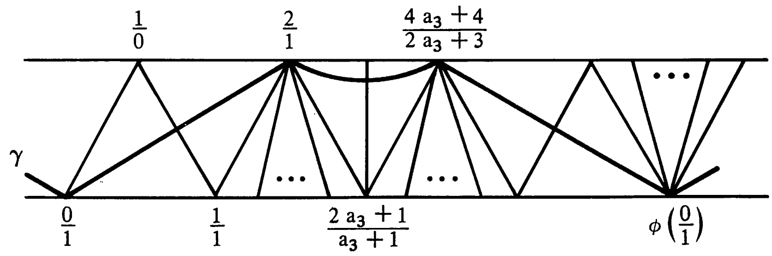


Fig. 2.16.

(ix) $\phi = \pm \overline{\alpha}^2 \beta^{a_2} \overline{\alpha}^2 \beta^{a_4} \overline{\alpha}^2 \beta^{a_6}$ and $\gamma = \dots \frac{0}{1}, \frac{2}{1}, \frac{4\,a_2 + 4}{2\,a_2 + 1}, \phi\left(\frac{0}{1}\right) \dots$

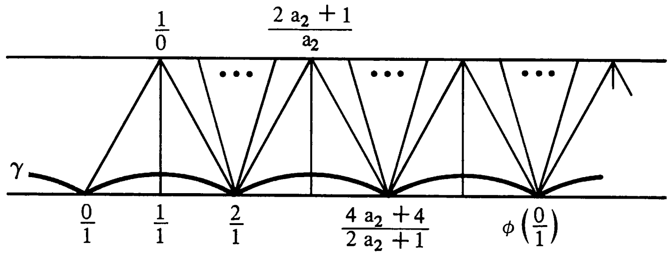


Fig. 2.17.

This ends the list of all ϕ for which there exists an invariant, minimal edge-path of length less or equal to 3 in $\overline{W}$.

3. S is of type S_γ^{sp} ($\phi(\gamma) = -\gamma$). Therefore the equality $k + 2 - b(S) \leqslant 3$ is satisfied (see Table 1.1). We have to consider three cases:

(a) $k = 1$; this case cannot happen because of hyperbolicity of ϕ.

(b) $k = 2$; ϕ must be equal to $\pm\overline{\alpha}^2\beta$ and $\gamma = \ldots\left(\frac{1}{0}, \frac{1}{1}\right), \left(\frac{1}{0}, \frac{3}{1}\right), \left(\frac{5}{2}, \frac{3}{1}\right), \ldots$

We can fill the last row of Table 2.1 analysing the diagram associated to ϕ (Fig. 2.18).

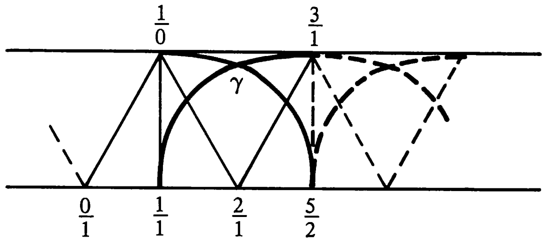


Fig. 2.18.

c) $k = 3$; the only possibilities for ϕ are $\pm\overline{\alpha}^2\beta\overline{\alpha}\beta$ and $\pm\overline{\alpha}^4\beta$ but in both cases S_γ^{sp} have the genus equal to 4.

This ends the proof of Lemma 2.1.

3. NONORIENTABLE, INCOMPRESSIBLE SURFACES OF GENUS 3 IN IRREDUCIBLE, NON-HAKEN MANIFOLDS $M_\phi\left(\frac{\lambda}{\mu}\right)$

In this part we consider manifolds $M_\phi\left(\frac{\lambda}{\mu}\right)$ obtained from M_ϕ by Dehn surgery along a curve on ∂M_ϕ of slope $\frac{\lambda}{\mu}$ (i.e. we glue together M_ϕ and $S^1 \times D^2$ along boundaries in such a way that the meridian of $S^1 \times D^2$ has the slope $\frac{\lambda}{\mu}$ on ∂M_ϕ).

Remark 3.1.

The manifold $M_\phi(\frac{\lambda}{\mu})$ is obtained from M_ϕ by the operation which is in fact only the second part of the original Dehn surgery (which consists of drilling and filling) and, perhaps, should be called Dehn filling.

Theorem 3.2.

Let $M_\phi(\frac{\lambda}{\mu})$ be an irreducible, non-Haken, orientable 3-manifold obtained by Dehn surgery of type $\frac{\lambda}{\mu}$ from a punctured torus bundle over S^1 with a monodromy map ϕ. Let K_1 and K_2 be nonorientable, incompressible, genus 3 surfaces in $M_\phi\,\frac{\lambda}{\mu}$ such that the classes of K_1 and K_2 are equal in

$$H_2\left(M_\phi\left(\frac{\lambda}{\mu}\right), Z_2\right).$$

Then K_1 is isotopic to K_2 with the possible exception of the case of $M_\phi\left(\frac{1}{1}\right)$ with $\phi = -\bar{\alpha}^2\beta\bar{\alpha}^2\beta\bar{\alpha}^2\beta$. We identify ϕ with its equivalence class defined by conjugation and taking the inverse.

Proof: If ϕ is periodic then $M_\phi\left(\frac{\lambda}{\mu}\right)$ is a Seifert fibered space and Theorem 3.2 follows from results of [Ru] and [P-3; Appendix II] in this case. If ϕ is parabolic then M_ϕ, and so $M_\phi\left(\frac{\lambda}{\mu}\right)$, contains either an embedded torus which does not disconnect the manifold or a Klein bottle. In this case we can conclude that $M_\phi\left(\frac{\lambda}{\mu}\right)$ is a Haken, Seifert or reducible manifold. For this let $M_\phi\left(\frac{\lambda}{\mu}\right)$ be a non-Haken, irreducible manifold. Then $M_\phi\left(\frac{\lambda}{\mu}\right)$ constains a Klein bottle which determines a one-sided Heegaard splitting of genus 2, but then $M_\phi\left(\frac{\lambda}{\mu}\right)$ is a Seifert fibered space. Therefore we can limit our investigation to the case of hyperbolic ϕ. Now, in order to prove Theorem 3.2, it is necessary to use the "composition" theorem which is very similar to Proposition 2.8 [P-2].

Proposition 3.3. (Composition theorem).

Let M_0 be a compact, irreducible 3-manifold with ∂M_0 equal to a collection of tori $T_1, T_2, \ldots, T_k$. Let $W_1, W_2, \ldots, W_k$ be a collection of solid tori. Let $\gamma_1, \gamma_2, \ldots, \gamma_k$ be a collection of nontrivial, simple cosed curves $\gamma_i \subset T_i$. Let M be a manifold obtained from M_0 by gluing $W_1, \ldots, W_k$ to M_0 along the boundaries (∂W_i is identified with T_i in such a way that γ_i becomes a meridian of W_i).

Then each incompressible surface S in M can be obtained by gluing togeher and incompressible, ∂-incompressible, non-parallel to the boundary surface S_0 in M_0, and incompressible surfaces $S_i \subset W_i$ (described in Lemma 3.4 below) such that $S_0 \cap T_i = \partial S_i$ (if $S_0 \cap T_i = \phi$ then $S_i = \phi$). Furthermore if $S_0 \cap T_i$ has more than one component then S_i is a collection of meridian disks in W_i.

Proof. It seems to be a known "folk" theorem, and can be derived from Proposition 2.8 [P-2] and Lemma 3.4.

Lemma 3.4. (See [P-2]).

Each incompressible, non-parallel to the boundary surface in solid torus $S^1 \times D^2$ is determined, up to isotopy, by slope

$$\frac{p}{q} \in W_1 \quad \left(\frac{p}{q} \neq \frac{1}{0}\right)$$

or is a collection of meridian disks. Genus of such a surface is equal to the length of the minimal edge-path in $\overline{W}_1$ from $\frac{1}{0}$ to $\frac{p}{q}$. If the length is > 0 then the surface is nonorientable and ∂-compressible.

In light of Proposition 3.3, in order to construct all nonorientable, incompressible surfaces of genus 3 in $M_\phi\left(\frac{\lambda}{\mu}\right)$, it is enough to find all incompressible surfaces in M_ϕ which are either

(i) orientable, ∂-incompressible, of genus 1, with one boundary component, or
(ii) closed, nonorientable of genus 3, or
(iii) nonorientable, ∂-incompressible of genus 3, or
(iv) nonorientable, ∂-incompressible, of genus $\leqslant 2$ with one boundary component.

Lemma 3.5.

Let S be an orientable, incompressible, ∂-incompressible surface of genus $g(S) \leqslant 1$ with one boundary component in M_ϕ. Then S is a puntured torus parallel to the fiber.

Proof. Lemma 3.5 follows from the Main Theorem of [F-H] (compare Theorem 1.5); Table 1.1 is usefull for the necessary computations.

Lemma 3.6.

Let S be a closed, nonorientable, incompressible surface of genus $\leqslant 3$ in M_ϕ. Then S is on genus 3 and $\phi = \pm \overline{\alpha}^2 \beta^{a_2}$ $(a_2 \geqslant 1)$. Furthermore S is unique, up to isotopy, in the class of $H_2(M_\phi, Z_2)$ determined by S.

Proof. Lemma 3.6 follows from Theorems 1.5, 1.6 and Proposition 1.7.

Now we can end the proof of Theorem 3.2. Each closed, nonorientable surface S in $M_\phi\left(\frac{\lambda}{\mu}\right)$ defines an element of $H^1\left(M_\phi\left(\frac{\lambda}{\mu}\right), Z_2\right)$. This element associates each simple closed curve in $M_\phi\left(\frac{\lambda}{\mu}\right)$ with intersection number of the curve and S. This allows us to make some easy but important observations:

1. If S_1 and $S_2 \subset M_\phi\left(\frac{\lambda}{\mu}\right)$ are constructed by using different minimal, invariant edge-paths in $\overline{W}$ then they represent different elements of

$$H^1\left(M_\phi\left(\frac{\lambda}{\mu}\right), Z_2\right)$$

so they are not isotopic.

2. If S was constructed from a surface with an odd (hence one) number of boundary components then it represents a different element of

$$H^1\left(M_\phi\left(\frac{\lambda}{\mu}\right), Z_2\right)$$

(so isotopy class) than one obtained from a surface with an even number (hence 0 or 2) of boundary components.

Now assume that $M_\phi\left(\frac{\lambda}{\mu}\right)$ is an irreducible, non-Haken manifold. It restricts drastically the number of possibilities and by using Lemmas 3.5 and 3.6 and Tables 2.1 and 2.2 we can verify that the only surfaces which yield the same element of $H^1\left(M_\phi\left(\frac{\lambda}{\mu}\right), Z_2\right)$ but are possibly not isotopic are surfaces which arise from $S_\gamma(1, 0, 0)$, $S_\gamma(0, 1, 0)$ and

$$S_\gamma(0, 0, 1) \text{ in } M_\phi\left(\frac{1}{1}\right) \quad \text{for } \phi = -\overline{\alpha}^2\beta\overline{\alpha}^2\beta\overline{\alpha}^2\beta \;,\; \gamma = \ldots \frac{0}{1}, \frac{2}{1}, \frac{8}{3}, \frac{30}{11} \ldots.$$

The verification is tedious but not difficult. In Table 3.1 we list all possible surfaces of genus 3 embedded in an irreducible, non-Haken $M_\phi(\lambda/_\mu)$. Furthermore we analyse closer threee special cases ("infinite classes"). To do this we introduce new notation; namely:

Let S be a surface in M_ϕ. Then $S^{(i)}$ denotes a surface in $M_\phi(\lambda/_\mu) = M_\phi \cup$ (a solid torus) where $S^{(i)}$ is obtained grom S by the construction of Proposition 3.3 by using S in M_ϕ and a surface of genus i in the solid torus.

"Infinite classes":

1. We start with a fiber F_0 in M_ϕ. The construction of Proposition 3.3 allows us to obtain the nonorientable surfaces $K = F_0^{(1)}$ of genus 3 in

$$M_\phi\left(\frac{2k+1}{2}\right)$$

where k is an integer. K is incompressible because $M_\phi\left(\frac{2k+1}{2}\right)$ contains no projective planes (see Table 2.1). One can verify whether a given manifold

$$M_\phi\left(\frac{2k+1}{2}\right)$$

is Haken by analysing the diagram of ϕ. K represents the element of

$$H^1\left(M_\phi\left(\frac{2k+1}{2}\right), Z_2\right)$$

in such a way that

$$[K](\ell) = \begin{cases} 1 & \text{if } \ell \text{ is a core of the attached solid torus;} \\ 0 & \text{if } \ell \text{ lies in the fiber of } M_\phi. \end{cases}$$

This distinguishes K from any other possible nonorientable surface of genus 3 in

$$M_\phi\left(\frac{2k+1}{2}\right),$$

with possible exception of these surfaces which come from S_γ (see case 3 b)). Then we refer to Theorem 7 of [Ru]. One can expect that if $M_\phi(\lambda/_\mu)$ in 1, is the same as that in 3 b) then $M_\phi(\lambda/_\mu)$ is a Seifert fibered space.

2. We start with a closed surface S_γ^c in M_ϕ. By Lemma 3.6 $\phi = \pm\bar{\alpha}^2\beta^{a_2}$. If $M_\phi(\lambda/\mu)$ does not contain any projective plane then S_γ^c is incompressible in $M_\phi(\lambda/\mu)$. If $M_\phi(\lambda/\mu)$ is irreducible and contains a projective plane then $\phi = \bar{\alpha}^2\beta$ and

$$\frac{\lambda}{\mu} = \frac{0}{1} \cdot M_\phi\left(\frac{0}{1}\right) = RP^3$$

hence S_γ^c is compressible in this case. From Fig. 2.3 we can conclude that $M_\phi(\lambda/\mu)$ is irreducible, non-Haken iff (see Corollary 2.2 [P-1]) either

a) $\phi = \bar{\alpha}^2\beta$ and $\frac{\lambda}{\mu} \neq \frac{1}{2}, -\frac{1}{4}, \frac{1}{0}$ or

b) $\phi = \bar{\alpha}^2\beta^{a_2}$ $(a_2 > 1)$ and $\frac{\lambda}{\mu} \neq \frac{1}{2}, \frac{0}{1}, \frac{-a_2}{4}, \frac{1}{0}$ or

c) $\phi = -\bar{\alpha}^2\beta$ and $\frac{\lambda}{\mu} \neq \frac{1}{1}, \frac{1}{4}, \frac{1}{0}$ or

d) $\phi = -\bar{\alpha}^2\beta^{a_2}$ $(a_2 > 1)$ and $\frac{\lambda}{\mu} \neq \frac{1}{1}, \frac{1}{2}, \frac{-a_2+2}{4}, \frac{1}{0}$.

S_γ^c can be distinguished from any other possible nonorientable surface of genus 3 in $M_\phi(\lambda/\mu)$ by considering its class in $H^1(M_\phi(\lambda/\mu), Z_2)$. Namely S_γ^c intersects a simple cosed curve of slope $\frac{1}{0}$ at one point and is disjoint from the curve of slope $\frac{0}{1}$ in the fiber, and from the core of the attached solid torus.

3. We start with a Klein bottle, K_b, with one boundary component in M_ϕ. Hence (see Table 2.1) either:

a) $\phi = \pm\bar{\alpha}^2\beta^{a_2}$ and $K = K_b^{(1)}$ is a nonorientable surface of genus 3 in

$$M_\phi\left(\frac{2c+1}{4c+2\pm 2}\right)$$

where c is an integer. Similarly as in the case 2 we see that K is incompressible.

By using Table 2.1 one can find out whether

$$M_\phi\left(\frac{2c+1}{4c+2\pm 2}\right)$$

is irreducible and non-Haken. K can be distinguished from any other possible monorientable surface of genus 3 in

$$M_\phi\left(\frac{2c+1}{4c+2\pm 2}\right)$$

by considering its class in

$$H^1\left(M_\phi\left(\frac{2c+2}{4c+2\pm 2}\right), Z_2\right).$$

b) $\phi = \pm\bar{\alpha}\beta^{a_2}$

(i) $\phi = \bar{\alpha}\beta^{a_2}$ and $K = K_b^{(1)}$ is a nonorientable surface of genus 3 in

$$M_\phi\left(\frac{2c+1}{8c+4\pm 2}\right)$$

where c is an integer.

(ii) $\phi = -\bar{\alpha}\beta^{a_2}$ and $K = K_b^{(1)}$ is a nonorientable surface of genus 3 in

$$M_\phi\left(\frac{3c+\frac{3\pm 1}{2}}{4c+2}\right)$$

where c is an integer such that $\left(3c+\frac{3\pm 1}{2}, 4c+2\right) = 1$.

K in (i) (and (ii)) is incompressible and $M_\phi(\lambda/_\mu)$ is irreducible, non-Haken iff:

In the case (i): $\dfrac{2c+1}{8c+4\pm 2} \neq \dfrac{-a_2}{4}$,

$$\text{In the case (ii):}\quad \frac{3c+\frac{3\pm 1}{2}}{4c+2} \neq \frac{-a_2+2}{4} \quad \text{(see Fig. 2.2).}$$

Table 3.1.

Nonorientable, incompressible surfaces of genus 3 in irreducible, non-Haken manifolds $M_{\phi}(\lambda/\mu)$. (The possibility of repetitions are not excluded).

Table 4.1.

No.	γ (one period)	ϕ	Exact description of the surface	$\frac{\lambda}{\mu}$	Remarks
1.	–	any	(fiber)[1]	$\frac{2k+1}{2}$ with finite number of exclusions for each ϕ. These exceptions can be read off the diagram of ϕ.	
2.	$\frac{0}{1}, \frac{2}{1}$	$\bar{\alpha}^2\beta^{a_2}$	S^c_γ	$\neq \frac{1}{2}, \frac{-a_2}{4}, \frac{1}{0}, \frac{0}{1}$	
		$-\bar{\alpha}\beta$	S^c_γ	$\neq \frac{1}{1}, \frac{1}{4}, \frac{1}{0}$	$M_{\phi}\left(\frac{1}{2}\right)$ is a Seifert fibered space
		$-\bar{\alpha}^2\beta^{a_2}(a_2>1)$	S^c_γ	$\neq \frac{1}{1}, \frac{-a_2+2}{4}, \frac{1}{0}, \frac{1}{2}$	
3. a)	$\frac{0}{1}, \frac{2}{1}$	$\bar{\alpha}^2\beta^{a_2}$	$S_\gamma(1)^{(1)}$	$\frac{2c+1}{4c+2\pm 2} \neq \frac{-a_2}{4}$	
		$-\bar{\alpha}^2\beta^{a_2}$	$S_\gamma(0)^{(1)}$	$\frac{2c+1}{4c+2\pm 2} \neq \frac{-a_2+2}{4}$	
b)	$\frac{0}{1}, \frac{1}{1}$	$\bar{\alpha}\beta^{a_2}$	$S_\gamma^{(1)}$	$\frac{2c+1}{8c+4\pm 2} \neq \frac{-a_2}{4}$	
		$-\alpha\beta^{a_2}$	$S_\gamma^{(1)}$	$\frac{3c+\frac{3\pm 1}{2}}{4c+2} \neq \frac{-a_2+2}{4}$ where $3c+\frac{3\pm 1}{2}$ is odd	
4.	$\frac{0}{1}, \frac{2}{1}, \frac{4}{1}$	$\bar{\alpha}^4\beta^{a_2}$	$S_\gamma(1,0)^{(0)}$	$\frac{1}{2}$	
		$-\bar{\alpha}^4\beta$	$S_\gamma(0,0)^{(0)}$	$\frac{1}{2}$	
5.	$\frac{0}{1}, \frac{2}{1}, \frac{4a+4}{2a+1}$	$-\bar{\alpha}^2\beta\bar{\alpha}^2\beta^{a_4}$	$S_\gamma(0,0)^{(0)}$	$\frac{1}{2}$	

Table 4.1.

No.	γ (one period)	ϕ	Exact description of the surface	$\frac{\lambda}{\mu}$	Remarks
6.	$\frac{0}{1}, \frac{2}{1}, \frac{4}{3}$	$\bar\alpha\beta^2\bar\alpha\beta$	$S_\gamma(0,1)^{(0)}$	$\frac{1}{2}$	
		$-\bar\alpha\beta^2\bar\alpha\beta^{a_4}$	$S_\gamma(0,0)^{(0)}$	$\frac{1}{2}$	
7.	$\frac{0}{1}, \frac{2}{1}, \frac{4a_3+4}{2a_3+3}$	$\bar\alpha\beta\bar\alpha\beta\bar\alpha\beta^{a_6}$	$S_\gamma(1,0)^{(0)}$	$\frac{1}{2}$	
		$\bar\alpha\beta\bar\alpha^{a_3}\beta\bar\alpha$	$S_\gamma(0,1)^{(0)}$	$-\frac{1}{2}$	
8.	$\frac{1}{1}, \frac{3}{1}, \frac{7}{3}, \frac{11}{5}$	$-\bar\alpha^2\beta^4$	$S_\gamma(0,1,0)^{(0)}$	$\frac{0}{1}$	
9.	$\frac{0}{1}, \frac{2}{1}, \frac{4}{1}, \frac{6}{1}$	$\bar\alpha^6\beta$	$S_\gamma(0,0,0)^{(0)}$	$\frac{0}{1}$	
		$\bar\alpha^6\beta^{a_2}$	$S_\gamma(1,1,0)^{(0)}$	$\frac{1}{1}$	
		$-\bar\alpha^6\beta^{a_2}$	$S_\gamma(1,0,0)^{(0)}$	$\frac{1}{1}$	
10.	$\frac{1}{1}, \frac{3}{1}, \frac{5}{1}, \frac{10a_3+13}{2a_3+3}$	$\bar\alpha^4\beta\bar\alpha^{a_3}\beta$	$S_\gamma(0,0,0)^{(0)}$	$\frac{0}{1}$	
		$\bar\alpha^4\beta\bar\alpha\beta$	$S_\gamma(1,1,0)^{(0)}$	$\frac{1}{1}$	
		$-\bar\alpha^4\beta\bar\alpha^{a_3}\beta; a_3 \neq 2$	$S_\gamma(1,1,1)^{(0)}$	$\frac{1}{1}$	
11.	$\frac{0}{1}, \frac{2}{1}, \frac{8}{3}, \frac{14}{5}$	$\bar\alpha^2\beta\bar\alpha^4\beta^{a_4}$	$S_\gamma(0,0,0)^{(0)}$	$\frac{0}{1}$	
12.	$\frac{0}{1}, \frac{2}{1}, \frac{4}{3}, \frac{18}{13}$	$-\bar\alpha\beta^2\bar\alpha\beta\bar\alpha^2\beta^{a_6}\ a_6 \neq 3$	$S_\gamma(0,1,0)^{(0)}$	$\frac{0}{1}$	
13.	$\frac{0}{1}, \frac{2}{1}, \frac{4}{1}, \frac{10}{3}$	$\bar\alpha^3\beta^2\bar\alpha\beta^{a_4}$	$S_\gamma(1,0,1)^{(0)}$	$\frac{0}{1}$	
		$-\bar\alpha^3\beta^2\bar\alpha\beta^{a_4}$	$S_\gamma(1,1,1)^{(0)}$	$\frac{1}{1}$	
		$-\bar\alpha^3\beta^2\bar\alpha\beta$	$S_\gamma(0,0,1)^{(0)}$	$\frac{0}{1}$	
14.	$\frac{0}{1}, \frac{2}{1}, \frac{4}{1}, \frac{18}{5}$	$\bar\alpha^3\beta\bar\alpha\beta\bar\alpha\beta^{a_6}$	$S_\gamma(1,1,0)^{(0)}$	$\frac{1}{1}$	
	$\frac{0}{1}, \frac{2}{1}, \frac{4}{1}, \frac{8a_3+10}{2a_3+3}$	$-\bar\alpha^3\beta\bar\alpha^{a_3}\beta\bar\alpha\beta$	$S_\gamma(0,0,1)^{(0)}$	$\frac{0}{1}$	

Table 4.1. (Continuation)

No.	γ (one period)	ϕ	Exact description of the surface	$\frac{\lambda}{\mu}$	Remarks
15.	$\frac{0}{1}, \frac{2}{1}, \frac{4a_3+4}{2a_3+3}, \frac{0}{1}$	$\bar{\alpha}\beta\bar{\alpha}^{a_3}\beta^2\bar{\alpha}^{a_5}\beta\bar{\alpha}\beta$	$S_\gamma(0,1,1)^{(0)}$	$\frac{1}{1}$	
	$\frac{0}{1}, \frac{2}{1}, \frac{8}{5}, \frac{16a_5+14}{10a_5+9}$	$-\bar{\alpha}\beta\bar{\alpha}\beta^2\bar{\alpha}^{a_5}\beta\bar{\alpha}\beta^{a_8}$ $a_5 \neq 2, a_8+1, a_8+3$	$S_\gamma(1,0,0)^{(0)}$	$\frac{1}{1}$	
16.	$\frac{0}{1}, \frac{2}{1}, \frac{8}{3}, \frac{16a_4+14}{6a_4+5}$	$\bar{\alpha}^2\beta\bar{\alpha}^2\beta^{a_4}\bar{\alpha}^2\beta^{a_6}$ $a_4, a_6 \neq 2$	$S_\gamma(0,0,0)^{(0)}$	$\frac{0}{1}$	
	$\frac{0}{1}, \frac{2}{1}, \frac{8}{3}, \frac{30}{11}$	$-\bar{\alpha}^2\beta\bar{\alpha}^2\beta\bar{\alpha}^2\beta$	$S_\gamma(1,0,0)^{(0)}$; $S_\gamma(0,1,0)^{(0)}$; $S_\gamma(0,0,1)^{(0)}$	$\frac{1}{1}$	$M_\phi\left(\frac{1}{1}\right)$ is not a Seifert fibered space (see Corollary 4.4)

4. *Manifold* $M_\phi\left(\frac{1}{1}\right)$ *for* $\phi = -\bar{\alpha}^2\beta\bar{\alpha}^2\beta\bar{\alpha}^2\beta$.

In this part we examine closer the case of surfaces

$K_0 = S_\gamma(0,0,1)^{(0)}$, $K_1 = S_\gamma(1,0,0)^{(0)}$ and $K_2 = S_\gamma(0,1,0)^{(0)}$ in $M_\phi\left(\frac{1}{1}\right)$ where $\phi = -\bar{\alpha}^2\beta\bar{\alpha}^2\beta\bar{\alpha}^2\beta$ and $\gamma = \cdots\frac{0}{1}, \frac{2}{1}, \frac{8}{3}, \frac{30}{11}\cdots$. We have the following:

Proposition 4.1.

Each incompressible, nonorientable surface of genus 3 in $M_\phi\left(\frac{1}{1}\right)$ is isotopic to K_0, K_1 or K_2. Furthermore there exists a free action of Z_3 on $M_\phi\left(\frac{1}{1}\right)$ with a generator T such that $T(K_i) = K_{i+1}$ (i is taken modulo 3) and

$$M_\phi\left(\frac{1}{1}\right)_{/Z_3} = M_\psi\left(\frac{2}{3}\right)$$

where $\psi = -\overline{\alpha}^2 \beta$.

Proof. The first part follows from Table 3.1; compare also Theorem 6 of [Ru]. The action of Z_3 is constructed as follows;

$$M_\phi = R \times F_{0/\sim} \quad (t, x) \sim (t+3, \phi(x)).$$

Let $T(t, x) = (t+1, \psi(x))$. T is well defined homeomorphism of order 3 and can be naturally extended to the action of Z_3 on $M_\phi\left(\frac{1}{1}\right)$ with the orbit space $M_\psi\left(\frac{2}{3}\right)$.

Conjecture 4.2.

K_0, K_1 and K_2 are not isotopic. If it is the case then we have an example showing that Rubinstein's Theorem 6 [Ru] can not be improved. On the other hand if Conjecture 4.2 is not true then we have an example of incompressible, ∂-incompressible surfaces which are not isotopic in M_ϕ, but isotopic after Dehn surgery i.e. in $M_\phi\left(\frac{1}{1}\right)\Big)$.

There is some evidence that K_0, K_1, K_2 are not isotopic. Namely:

Proposition 4.3.

K_i $(i = 0, 1, 2)$ can be isotoped in such a way that (i) K_i are pairwise transversal and $K_0 \cap K_1 = K_1 \cap K_2 = K_0 \cap K_2$ = a simple closed curve C, and (ii). If Y_0 = (the complement of the interior of a small regular neighborhoood of K_0 in $M_\phi\left(\frac{1}{1}\right)\Big)$, then the punctured tori $S_1 \cap Y_0$ and $S_2 \cap Y_0$ are not parallel (even isotopic modulo boundary) neither one to another nor to ∂Y_0.

Corollary 4.4.

$M_\phi\left(\frac{1}{1}\right)$ is not a Seifert fibered space.

Remark 4.5.

$M_\psi\left(\frac{2}{3}\right)$ is an irreducible, non-Haken, non-Seifert fibered manifold. Furthermore the Jϕrgensen's decomposition does not provide $M_\psi\left(\frac{2}{3}\right)$ with hyperbolic structure [B-P-Ż]. $M_\psi\left(\frac{2}{3}\right)$ seems to be a good candidate for testing Thurston Conjecture that the interior of every compact 3-manifold has a canonical decomposition into pieces which have geometric structures [T].

Proof. of Proposition 4.3.

Part (i) follows from Theorem 6 [Ru] because $M_\phi\left(\frac{1}{1}\right)$ is an irreducible, non-Haken manifold with $H^1\left(M_\phi\left(\frac{1}{1}\right), Z_2\right) = Z_2$. The proof of the part (ii) will be divided into several steps:

Step I.

Description of a one sided Heegaard splitting determined by K_0.

Let N (S_γ (0, 0, 1)) be a small regular neighborhood of S_γ (0, 0, 1) in M_ϕ. ∂ N (S_γ (0, 0, 1)) is an unknotted but compressible surface in M_φ with four boundary components. We will use the method of the proof of Corollary 2.2 [P-1]. ∂ N (S_γ (0, 0, 1)) is associated with the invariant edge-path (not minimal)

$$\ldots \frac{a_0}{b_0} = \frac{0}{1}, \frac{a_1}{b_1} = \frac{1}{0}, \frac{a_2}{b_2} = \frac{2}{1}, \frac{a_3}{b_3} = \frac{3}{1}, \frac{a_4}{b_4} = \frac{8}{3}, \frac{a_5}{b_5} = \frac{19}{7}, \frac{a_6}{b_6} = \frac{30}{11}, \frac{a_7}{b_7} = \frac{41}{15} \ldots,$$

see Definition 1.3 and Fig. 4.1.

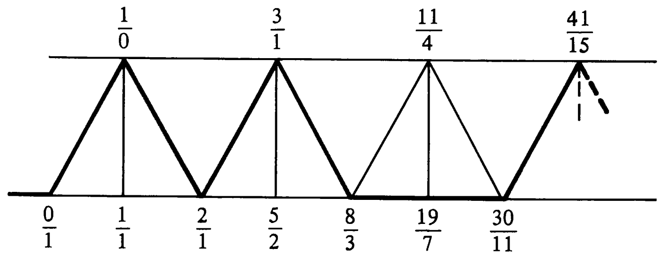


Fig. 4.1.

$\partial N(S_\gamma(0,0,1))$ cuts M_ϕ into two handlebodies H^0 and $H^e = N(S_\gamma(0,0,1))$ of genus 4 (see Fig. 4.2). It is important to recognize the "trace" of the boundary of $\partial N(S_\gamma(0,0,1))$ in H^0 and H^e (i.e. $\partial N(S_\gamma(0,0,1)) \cap \partial M_\phi \cap H^0$ and $\partial N(S_\gamma(0,0,1)) \cap \partial M_\phi \cap H^e$). In both cases it consists of two pairs of parallel curves γ_1, γ_1' and γ_2, γ_2' in H^e (resp. γ_1, γ_2' and γ_2, γ_1' in H^0). One curve from each pair is sketched in Fig. 4.2. The way the curves are identified in M_ϕ is shown in Fig. 4.2 too. Additionally we should remember that H^0 and H^e are glued together in M_ϕ by a level preserving map. To find the curves we use the ideas of Corollary 2.2 [P-1]. In particular we have to find the numbers c_i

as in Corollary 2.2 [P-1] $\frac{a_i}{b_i} = \frac{a_{i-2} + c_i a_{i-1}}{b_{i-2} + c_i b_{i-1}}$) :

$$\frac{2}{1} = \frac{0 + c_2 1}{1 + c_2 0} \qquad \text{so } c_2 = 2 ,$$

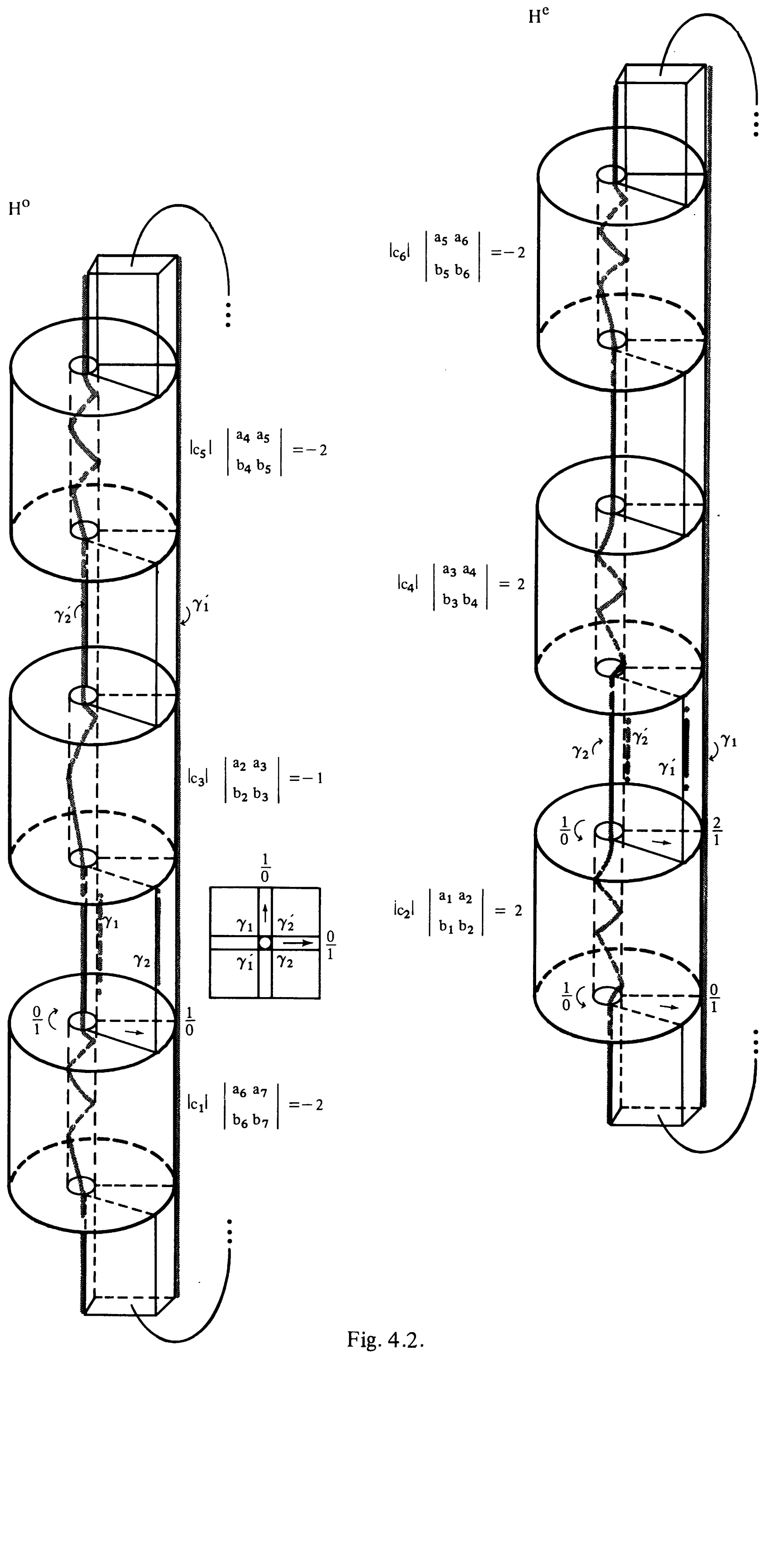
$$\frac{3}{1} = \frac{1 + c_3 2}{0 + c_3 1} \qquad \text{so } c_3 = 1 ,$$

$$\frac{8}{3} = \frac{2 + c_4 3}{1 + c_4 1} \qquad \text{so } c_4 = 2 ,$$

$$\frac{19}{7} = \frac{3 + c_5 8}{1 + c_5 3} \qquad \text{so } c_5 = 2 ,$$

$$\frac{30}{11} = \frac{8 + c_6 19}{3 + c_6 7} \qquad \text{so } c_6 = -2 ,$$

$$\frac{41}{15} = \frac{19 + c_7 30}{7 + c_7 11} \qquad \text{so } c_1 = c_7 = -2 .$$



H^o
H^e
|c6| | a5 a6 / b5 b6 | = −2
|c5| | a4 a5 / b4 b5 | = −2
|c4| | a3 a4 / b3 b4 | = 2
|c3| | a2 a3 / b2 b3 | = −1
|c2| | a1 a2 / b1 b2 | = 2
|c1| | a6 a7 / b6 b7 | = −2
γ1
γ2
γ1′
γ2′
1/0
0/1
2/1

Fig. 4.2.

The number $|c_i|$ defines the number of twists of the i-th segment of the respective curves. The direction is defined by the determinant

$$\begin{vmatrix} a_{i-1} & a_i \\ b_{i-1} & b_i \end{vmatrix}$$

(i.e. it is positive iff the i-th edge turns right on the diagram of PSL (2, Z) or equivalently starts on the top of the strip of Fig. 4.1.).

Recall that $C \subset K_0$ is a unique, up to isotopy, simple closed curve such that $K_0 - C$ is orientable. $N(K_0)$ (i.e. a regular neighborhood of K_0 in $M_\phi\left(\frac{1}{1}\right)$ can be identified with an I-bundle over K_0, so $\partial N(K_0)$ is an orientable double cover of S_0. Let C' denote the lifting of C to $\partial N(K_0)$. C' can be assumed to lie in ∂H^e and to be disjoint from the annuli defined by γ_1, γ_1' and γ_2, γ_2'. Hence C' lies in ∂H^0. The position of C' in ∂H^0 and ∂H^e is described in Fig. 4.4. Additionally Fig. 4.3 shows the position of C in $S_\gamma(0, 0, 1) = K_0 \cap H^e$:

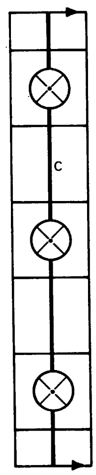


Fig. 4.3.

From Fig. 4.4 we can partially recover the structure of the one-sided Heegaard splitting of $M_\phi\left(\frac{1}{1}\right)$. Namely $H^0_{\{\gamma_1, \gamma_2\}} = H_2$ i.e. genus two handlebody (as in [P-1; Definition 1.2] $H^0_{\{\gamma_1, \gamma_2\}}$ means H^0 with two 2-handles glued to H^0 along γ_1 and γ_2), and Fig. 4.4 allows us to draw the picture of C' in ∂H_2. It is done in Fig. 4.5.

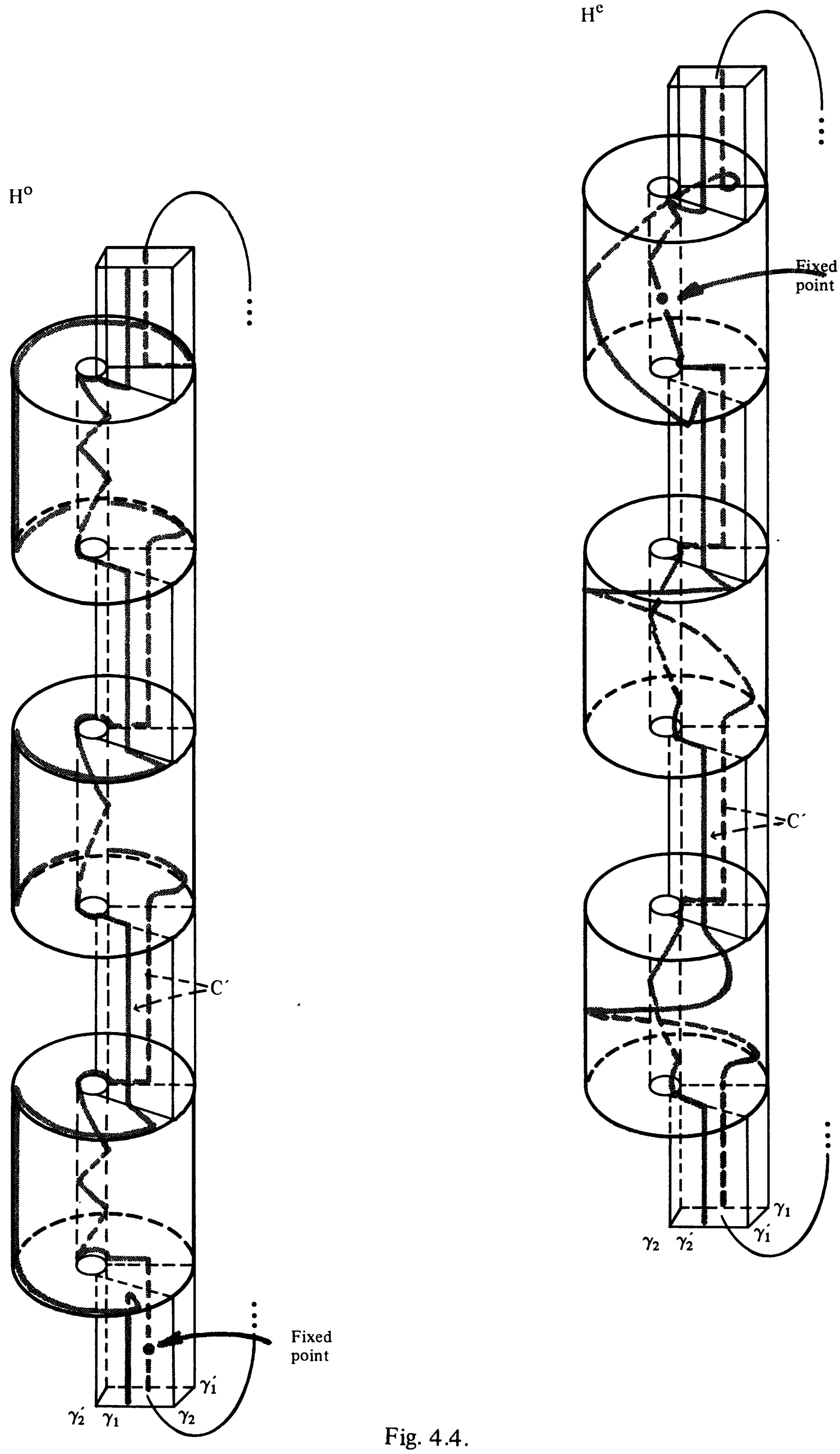


Fig. 4.4.

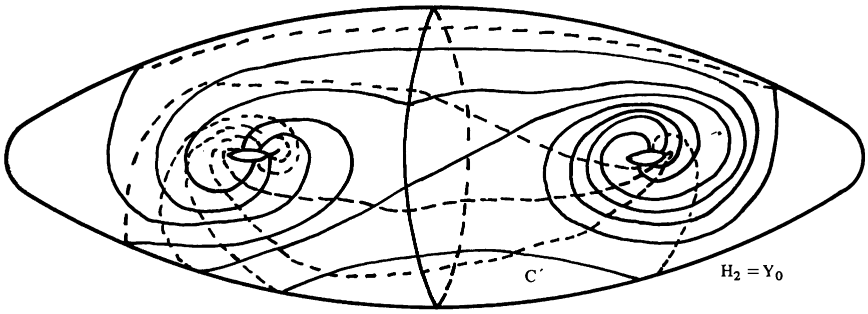


Fig. 4.5.

Algebraically C' $\subset H_2$ can be described as

$$[C'] = x_1^2\, x_2^2\, x_1^2\, x_2^2\, x_1^{-2}\, x_2^{-2}\, x_1^{-2}\, x_2^{-2} \in \pi_1\,(H_2).$$

To finish the description of $M_\phi\ \frac{1}{1}$ as a one sided Heegaard splitting one should know how the free involution on $\partial\, H_2$ looks like. We omit it because for our purposes it is enough to know that C' is invariant under this involution and that the sides of C' in $\partial\, H_2$ are interchanged.

Step II.

Incompressible, punctured tori in H_2 bounded by C′. This situation was studied in M. Kuperwaser's PhD-thesis [K].

Consider the decomposition of H_2 into two solid tori H_1' and H_1'' and the handlebody H_2' of genus 2 as follows:

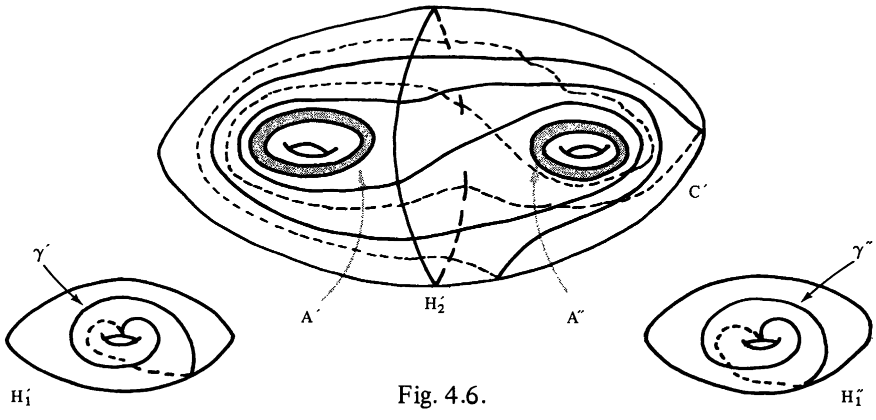


Fig. 4.6.

$H_2 = H_1' \cup H_2' \cup H_1''$; we glue H_2' with H_1' by identifying A' with the annulus which constituts a regular neighborhood of γ' in $\partial H_1'$. Similarly we glue H_1'' with H_2' (using γ'' and A''). Punctured tori bounded by C could be placed in $\partial H_2'$ as follows: $T_{1,1}'$ contains A' and is equal to a part of $\partial H_2'$ bounded by C'. $T_{1,1}''$ contains A'' and is equal to a part of $\partial H_2'$ bounded by C'. To prove that $T_{1,1}'$ (and $T_{1,1}''$) is not parallel to the boundary of H_2 we compute the length of a minimal word in $\pi_1(H_2)$ and in $\pi_1(H_2' \cup H_1')$ given by C' (see [L-S], [L] and [S]). With a natural choice of basis we have:

In $\pi_1(H_2)$: $[C'] = x_1^2 x_2^2 x_1^2 x_2^2 x_1^{-2} x_2^{-2} x_1^{-2} x_2^{-2}$, it is a minimal word in $\pi_1(H_2)$ so minimal length of $[C'] = 16$,

In $\pi_1(H_2' \cup H_1')$: $[C'] = x_2^1\, y\, x_1^2\, y\, x_1^{-2}\, y^{-1}\, x_1^{-1}\, y^{-1}$, substitute $z = x_1^2 y$ then $[C'] = z^2 x_1^{-2} z_1^{-2} x_1^2$, it is a minimal word in $\pi_1(H_2' \cup H_1')$ so minimal length of $[C'] = 8$,

In $\pi_1(H_2' \cup H_1'')$ the minimal length of $[C']$ is equal to 8 and in $\pi_1(H_2')$ the minimal length of $[C']$ is equal to 6. Additionally we can observe that if we isotope $T_{1,1}'$ (resp. $T_{1,1}''$) so as to be properly embedded in H_2 then the segment cut out by $T_{1,1}'$ (resp. $T_{1,1}''$) is a handlebody of genus 2. Minimal length of $[C']$ in the fundamental group of the segment is equal to 6. This information is enough to conclude that $T_{1,1}'$ and $T_{1,1}''$ are not parallel to the boundary of H_2 and $T_{1,1}'$ is not parallel (even isotopic) to $T_{1,1}''$.

Step III.

To end the proof of Proposition 4.3 we have to consider two possibilities.

1. K_0 is isotopic to K_1 (so to K_2) in $M_\phi\left(\frac{1}{1}\right)$. Then each incompressible, nonorientable surface of genus 3 in $M_\phi\left(\frac{1}{1}\right)$ is isotopic to K_0. Hence the surfaces in $M_\phi\left(\frac{1}{1}\right)$ which come grom $S_{1,1}'$ and $S_{1,1}''$ in H_2 are isotopic to K_0 so to K_1 and K_2.

2. K_0 is not isotopic to K_1 (so K_0 is not isotopic to K_2 and K_1 is not isotopic to K_2). We can assume (part (i) of Proposition 4.3) that

$$K_0 \cap K_1 = K_1 \cap K_2 = K_0 \cap K_2 = C.$$

Hence we conclude that $K_1 \cap Y_0$ is not isotopic neither to $K_2 \cap Y_0$ nor to the boundary of Y_0. On the other hand we know [Ru] or [K] that a given curve in ∂H_2 can bound at most two non-isotopic, non-isotopic ot the boundary, incompressible punctured tori. So $K_1 \cap Y_0$ and $K_2 \cap Y_0$ are isotopic to $S'_{1,1}$ and $S''_{1,1}$. This concludes the proof of Proposition 4.3.

Proof of Corollary 4.4. We need the classification of incompressible, nonorientable surfaces of genus 3 in non-Haken Seifert fiber spaces. It is almost done in [R]. The classification is completed in [P-3; Appendix II]. This classification implies that the pair (H_2, C') as on Fig. 4.5 will never occur in any Seifert fibered space.

Acknowledgements.

This paper is a part of the author's doctoral dissertation prepared at Columbia University under the supervision of Prof. J. Birman. I would like to express my gratitude to Prof. J. Birman whose assistance enabled me to accomplish this paper.

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