EMBEDDINGS AND IMMERSIONS OF BRANCHED COVERING SPACES

by

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1. INTRODUCTION.

In [4], Hilden proved that all 3-fold dihedral branched coverings of S^n, $n = 3$ or 4 with orientable branch set can be embedded in $S^3 \times S^2$ in such a way that the branched covering space map can be factored through the embedding. The objective of this paper is to generalize this result to those types of branched coverings of S^n, $n = 3$ or 4 which have representations onto either the rotation group of a tetrahedron, an icosahedron, or a polygon with an odd number of sides. The central idea is that these branched coverings of S^n all have representations factor through their corresponding binary polyhedral subgroups of continuous unit quaternions.

The main results of this paper are:

Theorem 6. *If* $p : M^n \to S^n$, $n = 3$ *or* 4 *is a tetrahedral branched covering space with branch set a polyhedral knot or link if* $n = 3$ *or a locally flat link of two-spheres if* $n = 4$. *Then there is a locally flat embedding* $e : M^n \to S^n \times S^2$ *so that diagram* (1) *is commutative.*

$$\begin{array}{ccc} & & S^n \times S^2 \\ & \overset{e}{\nearrow} & \downarrow \\ M^n & \xrightarrow{\ p\ } & S^n \end{array} \qquad (1)$$

Theorem 7. *If* $M^n \to S^n$, $n = 3$ *or* 4 *is an icosahedral branched covering space or an* r*-fold* (r *odd, and* $r \geqslant 3$) *dihedral branched covering space branched over a* *polyhedral knot or link if* $n = 3$, *or a locally flat knot or link of two-spheres if* $n = 4$. *Then there is a locally flat immersion* $e : M^n \to S^n \times S^2$ *so that diagram* (1) *is commutative.*

2. DEFINITIONS AND ASSUMPTIONS.

All work will be done in the piecewise linear category. For the standard definitions and notations of branched covering space theory, we refer to [4] or [6].

A branched covering space $p : M^n \to S^n$, $n = 3$ or 4 with orientable branch set K is said to be a tetrahedral branched covering space if it has an associated representation $\rho : \pi_1 (S^n - K) \to \Sigma_4$ which maps the fundamental group onto the alternating group A_4 by sending each meridian generator to a 3-cycle. Since A_4 is isometric to the rotation goup of a regular tetrahedron, this representation may be thought of as sending meridians to 120° rotations of a regular retrahedron about an axis through a vertex and the midpoint of the vertex's opposite face.

A branched covering space of S^n, $n = 3$ or 4 is called an icosahedral branched covering space if it has a representation $\rho : \pi_1 (S^n - K) \to \Sigma_{12}$ which maps the fundamental group onto the alternating group A_5 or Σ_{12} by sending meridians to permutations which are the product of certain disjoint 5-cycles. There is a natural isomorphism between this subgroup of Σ_{12} and the rotation group of a regular icosahedron. Under the isomorphism the permutations which are the product of two disjoint 5-cycles are mapped to 72° clockwise rotations about an axis through two diametrically opposite vertices.

If r is an odd positive integer larger than two, a branched covering of S^n, $n = 3$ or 4, is called an r-fold dihedral branched covering space whenever its associated representation $\rho : \pi_1 (S^n - K) \to \Sigma_r$ maps the fundamental group onto a dihedral subgroup D_r of order 2 r by sending meridians to permutations which are products of $(r - 1)/2$ disjoint transpositions. This representation has a geometrical representation as a map onto the rotation group of a regular r-gon in which meridians are sent to 180° rotations about a vertex and the midpoint of its opposite edge.

The proofs of the main results depend on a factorization of the above representations through certain subgroups of continuous unit quaternions. Let Q denote the binary tetrahedral group, let I be the binary icosahedral group and let H_r denote the dicyclic group of order 4 r, r odd. Each of these groups are subgroups of the topological group of continuous quaternions, [1, p. 68]. For each of these groups there is a natural two to one homomorphism, η, of the binary polyhedral group onto its corresponding group of rotations. In each case the kernel of the homomorphisme is -1, the only element of order two in the topological group of all unit quaternions.

3. LINK GROUPS AND BINARY POLYHEDRAL GROUPS.

In this sections we show that the tetrahedral, icosahedral, and dihedral representations of knot and link groups factor through their corresponding subgroups of unit quaternions.

Lemma 1. Let μ_1 *denote the fundamental group of the complement of a polyhedral knot or link in* S^3 *or the fundamental group of the complement of a link of locally flat twospheres in* S^4. *If there is a tetrahedral representation* $\rho : \pi_1 \to \Sigma_4$ *then there is a homomorphism* $\theta : \pi_1 \to Q$ *so that* $\eta\, \theta = \rho$.

Proof. Every polyhedral knot or link in S^3 has a Wirtinger presentation [5, pg. 57] as does every link of locally flat two-spheres in S^4 [3, pg. 132]. The Wirtinger presentation of π_1 has the form $\langle x_1, \ldots, x_n; r_1, \ldots, r_n \rangle$ where each x_i is a meridian and the relator r_i either has the form

$$x_k\, x_i\, x_k^{-1} = x_{i+1} \quad \text{or} \quad x_k\, x_{i+1}\, x_k^{-1} = x_i.$$

Since ρ maps each generator x to a 3-cycle, the set $\eta^{-1}\, \rho\,(x)$ has exactly two elements; one element has order three, the other is of order six. Let $\theta\,(x)$ be the element of order six in $\eta^{-1}\, \rho\,(x)$. Since the two elements of $\eta^{-1}\, \rho\,(x)$ are negatives of each other: $\theta\,(x_k\, x_i\, x_k^{-1}) = \pm\, \theta\,(x_{i+1})$. Suppose that

$$\theta\,(x_k\, x_i\, x_x^{-1}) = -\,\theta\,(x_{i+1}).$$

Then

$$[\theta\,(x_k\, x_i\, k_k^{-1})]^3 = \theta\,(x_k)\, \theta\,(x_i)^3\, \theta\,(x_k)^{-1} = -\,\theta\,(x_{i+1})^3.$$

Since $\theta\,(x_i)^3$ and $\theta\,(x_{i+1})^3$ have order two they both must equal -1. This is a contradiction.

An analogous argument for the relations of the form $x_k\, x_{i+1}\, x_k^{-1} = x_i$ shows that θ preserves all the relators in the Wirtinger presentation of π_1. The construction of θ makes it clear that $\eta\, \theta = \rho$.

Lemma 2. If π_1 *is a knot group as in Lemma 1 and* $\rho : \pi_1 \to \Sigma_{12}$ *is an icosahedral representation, then there is a map* $\theta : \pi_1 \to I$ *so that* $\eta\, \theta = \rho$.

Proof. This argument is essentially the same as the one given in Lemma 1. The difference lies in the order of the elements in the set $\eta^{-1}\ \rho\,(x)$: One of these has order five and the other order ten. Define $\theta\,(x)$ to be the element of order ten.

Lemma 3. *For odd integers* r *greater than two, link group* π_1, *and an* r*-fold dihedral representation* $\rho : \pi_1 \to \Sigma_r$, *there is a map* $\theta : \pi_1 \to H_r$ *so that* $\eta\ \theta = \rho$.

Proof. One presentation for the dicyclic group H_r is $\langle A, B : A^r = B^2 = (AB)^2 \rangle$. The element A^2 of H_r generates a normal cyclic subgroup of order 2 r [1, pg. 75]. The quotient of H_r by this cyclic group is isomorphic to Z_4. Let μ denote the quotient map $H_r \to Z_4$ whose kernel is generated by A^2. Let G_r denote the pullback of Z_4 and D_r. over Z_2. We now have the following commutative

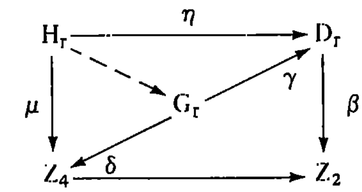
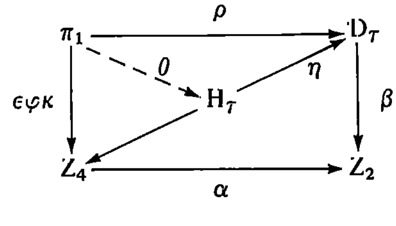


diagram. The maps δ and γ are projections while the maps α and β are the conical maps. Since both η and μ are surjective and both G_r and H_r have 4 r elements, the two groups are isomorphic.

Now let $\epsilon : Z \to Z_4$ and $\kappa : \pi_1 \to H_1$ be natural maps. Let $\varphi : H_1 \to Z$ be defined by sending meridian generator of H_1 to either + 1 or - 1. As Hilden points out [4], making a choice is equivalent to choosing an orientation for S^n and each component of the link.

The diagram below is commutative since ρ sends each meridian generator to the product of (r - 1/2 disjoint transpositions.



Since H_T is the pullback of Z_4 and D_T there must exist a map $\theta : \pi_1 \to H_T$ so that $\eta\,\theta = \rho$.

4. THE BASIC CONSTRUCTION.

Let R ($\bar{R}$) be unordered (ordered) quadruples of points that lie at the vertices of a regular tetrahedron in the standard 2-sphere. The space $\hat{R}$ is homeomorphic to the orthogonal group O(3). Let $\hat{R}$ be the component of O(3) which contains the identity. This component is the special orthogonal group SO (3). We will consider R to be a subspace of R. Notice that there is a natural map $\hat{R} \to R$ of ordered quadruples to unordered ones which is a 12-fold regular covering space map.

Lemma 4. The first homotopy group π_1 (R) *is isomorphic to* Q. *The second homotopy group* π_2 (R) *is the trivial group.*

Proof. Consider S^3 to be the topological group of all unit quaterions. There is a two-fold covering space map $S^3 \to \hat{R} = SO(3)$. For each q in S^3 let T_q denote the rotation in SO (3) to which q is sent by the covering space map. Now let V be the vertices of a fixed regular tetrahedron in the unit sphere, and define $f : S^3 \to R$ by the formula $f(q) = T_q(V)$. The subgroup K of S^3 is defined by $f^{-1}(V)$.

There is a natural map $g : S^3/K \to R$ defined by $g(x) = f(p^{-1}(x))$, where p is the projection $S^3 \to S^3/K$. Since f is a quotient space map, g must be a homeomorphism.

Since there is a 2 : 1 homomorphism from K to the group of rotations of the regular tetrahedron with vertices V, K must be conjugate to Q in S^3. Hence K must be isomorphic to Q. Thus S^3/Q is homeomorphic to S^3/K which in turn is homeomorphic to R.

Since Q is a finite group acting without fixed points on S^3, $\pi_1(S^3/Q)$ is isomorphic to Q. The homotopy group π_2 (R) is trivial as R is covered by $\hat{R}$ which in turn is covered by S^3, and $\pi_2(S^3) = 0$.

In a similar way, if S denotes the space of unordered 12-tuples of points that lie at the vertices of a regular icosahedron in the standard 2-sphere, and T denotes the space of unordered r-tuples of points that lie at the vertices of a regular polygon with an odd number of sides, then we have:

Lemma 5. The group π_1 (S) *is isomorphic to the binary icosahedral group and* π_1 (T) *is isomorphic to the dicyclic group of order* 4 r. *Both the groups* π_2 (S) *and* π_2 (T) *are trivial.*

With the above lemmas the main theorems can now be proved. The technique is analogous to the one given in [4].

Sketch of proof of Theorem 6. Let N be a tube neighborhood of the branch set. Using Lemma 4, the first step is to construct a function f : 2-skeleton $(S^n - \text{Int}\, N) \cup \partial N \to R$ so that the induced map f_* equals the map

$$\theta : \pi_1 (S^n - \text{Int}\, N) \to \pi_1 (R).$$

The map f can then be extended to all of $S^n - \text{Int}\, N$. Using the extension of f, a map $h : M^n - p^{-1} (\text{Int}\, N) \to S^2$ is constructed by "lifting paths" to the regular covering $\hat{R}$ of R. The map h is then extended to p^{-1} (Int N) by treating this set as a linear disc bundle and extending h each disc simultaneously by coning.

Sketch of proof of Therorem 7. The proof is exactly the same as the one above except that the discs in S^2 which h is extended by the cone procedure overlap. So the map $(p, h) : M^n \to S^n \times S^2$ is an immersion.

5. Remarks.

José Montesinos has pointed out to me that the space of unordered tuples of points on the 2-sphere which lie at the vertices of a regular polyhedron is a Seifert fiber space with threee exceptional fibers. Each exceptional fiber corresponds to an axis of rotation through the polyhedron. The space of unordered quadruples for the positions of a regular tetrahedron in the 2-sphere is

$$(O\ o\ 0 \mid -1, (3, 1), (3, 1), (2,1)).$$

For the icosahedron the space of unordered 12-tuples is

$$(O\ o\ 0 \mid -1, (5, 1), (2, 1), (3, 1)).$$

and for the regular polyhedron with an odd number of edges the fiber space is $(O\ o\ 0 \mid -1, (r, 1), (2, 1), (2, 1))$, (compare [5]).

Noticeably absent from the list of rotation groups in this paper is Σ_4, the rotation group of an octahedron. There does not appear to be a method of constructing a homomorphism from the link group to the binary octahedral subgroup of continous unit quaternions which projects to a representation of the link group which sends meridians to permutations with the same cycle structure. This type of construction was used in lemmas one and two.

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