

AN ESTIMATE OF THE RATE OF CONVERGENCE
OF THE TRIANGULAR MATRIX MEANS OF THE FOURIER
SERIES OF FUNCTIONS OF BOUNDED VARIATION*

by

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In this paper a theorem concerning degree of approximation of a class of functions of bounded variation has been established. Our result includes, as a special case, a recent result of Bojanic and Mazhar (1980).

1. Let $\Lambda = (\lambda_{n,k})$, $n, k \geq 0$ be an infinite lower triangular matrix. Given a sequence $\{s_n\}$ we define its Λ -means by

$$\sigma_n = \sum_{k=0}^n \Delta \lambda_{n,k} s_k,$$

where $\Delta \lambda_{n,k} = \lambda_{n,k} - \lambda_{n,k+1}$. Let $\lambda_{n,0} = 1$ and we write $\alpha_n = \lambda_{n,n}$. Let $T_n(x)$ denote the Λ -means of the Fourier series of a 2π -periodic function f of bounded variation on $[-\pi, \pi]$, where $\{s_n\}(x)$ are the partial sums of the Fourier series. We prove the following theorem:

Theorem: *Let $\{\Delta \lambda_{n,k}\}$ be positive and non-decreasing with respect to k and let $\{\alpha_n\}$ be decreasing, then for every f of bounded variation on $[-\pi, \pi]$, we have*

$$\left| T_n(x) - \frac{f(x+0) + f(x-0)}{2} \right| \leq \frac{9}{2} \frac{\alpha_n}{1-\alpha_1} \sum_{k=0}^{n-1} V_a^b(f)(\varphi_x) \left(\frac{1}{\alpha_{k+1}} - \frac{1}{\alpha_k} \right),$$

where $\varphi_x(t) = f(x+t) + f(x-t) - f(x+0) - f(x-0)$ and $V_a^b(f)$ denotes the total variation of f on $[a, b]$.

Taking $\Delta \lambda_{n,k} = \frac{p_{n-k}}{P_n}$, $P_n = p_0 + p_1 + \dots + p_n$,

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with $\{p_k\}$ positive and non-increasing such that $p_n \rightarrow \infty$, as $n \rightarrow \infty$, we get the following result of Bojanic and Mazhar (1980):

$$\left| N_n(x) - \frac{f(x+0) + f(x-0)}{2} \right| \leq \frac{9}{2} - \frac{p_1}{p_n} \sum_{k=0}^n p_k V_0^{\frac{\pi p_0}{p_k}} (\varphi_x),$$

where $N_n(x)$ is the (N, p_n) mean of the partial sums of the Fourier series of f .

2. To prove our theorem we need the following lemmas:

$$\text{Lemma 1. Let } K_n(t) = \sum_{k=0}^n \Delta \lambda_{n,k} D_k(t)$$

$$\text{and } \lambda_n(y) = \int_0^y K_n(t) dt, \text{ where } D_k(t) = \frac{\sin((k+\frac{1}{2})t)}{2 \sin t/2}, \Lambda = (\lambda_{n,k}), n, k \geq 0$$

is an infinite lower triangular matrix with $\lambda_{n,0} = 1$ and $\{\Delta \lambda_{n,k}\}$ being non-negative. Then

$$|\lambda_n(y)| \leq 2\pi, \quad 0 \leq y \leq \pi.$$

$$\begin{aligned} \text{Proof. } |\lambda_n(y)| &\leq \sum_{k=0}^n \Delta \lambda_{n,k} \left| \int_0^y D_k(t) dt \right| \\ &\leq \Delta \lambda_{n,0} \frac{\pi}{2} + \lambda_{n,1} \frac{\pi}{2} + \sum_{k=1}^n \Delta \lambda_{n,k} \left| \sum_{v=1}^k \frac{\sin(v)y}{v} \right| \\ &\leq \frac{\pi}{2} + 2\sqrt{\pi} \leq 2\pi, \end{aligned}$$

by using the well known estimate:

$$\left| \sum_{k=1}^n \frac{\sin(k)y}{k} \right| \leq 2\sqrt{\pi} \text{ for every } y \text{ and } n, \text{ Natanson (1964).}$$

Lemma 2. Let $\{\Delta \lambda_{n,k}\}$ be non-negative and non-decreasing with respect to k and let

$$\gamma_n(y) = \int_y^\pi K_n(t) dt.$$

$$\text{Then } |\gamma_n(y)| \leq \frac{\pi^2 \alpha_n}{2y}, \quad 0 < y \leq \pi.$$

Proof of Lemma 2. We have

$$\begin{aligned} K_n(t) &= \sum_{k=0}^n \Delta \lambda_{n,n-k} D_{n-k}(t) \\ &= \frac{1}{2 \sin t/2} \operatorname{Im} \left\{ e^{i(n+1/2)t} \sum_{k=0}^n \Delta \lambda_{n,n-k} e^{-ikt} \right\}, \end{aligned}$$

so that

$$\begin{aligned} |K_n(t)| &\leq \frac{1}{2 \sin t/2} \left| \sum_{k=0}^n \Delta \lambda_{n,n-k} e^{-ikt} \right| \\ &\leq \frac{\alpha_n}{2 \sin t/2} \max_{0 \leq k \leq n} \left| \sum_{r=0}^k e^{-irt} \right| \\ &\leq \frac{\alpha_n}{2 (\sin t/2)^2} \leq \frac{\pi^2}{2 t^2} \alpha_n. \end{aligned}$$

$$\begin{aligned} \text{Thus } |\gamma_n(y)| &\leq \int_y^\pi |K_n(t)| dt \leq \frac{\pi^2}{2} \cdot \alpha_n \int_y^\pi \frac{dt}{t^2} \\ &\leq \frac{\pi^2}{2} \cdot \frac{\alpha_n}{y}. \end{aligned}$$

3. Proof of the theorem. Writing

$$T_n(x) = \sum_{k=0}^n \Delta \lambda_{n,k} s_k(x), \quad \text{where } s_k(x) \text{ is the } k-th$$

partial sum of Fourier series, we have in view of $\lambda_{n,0} = 1$,

$$\begin{aligned}
 B(n) &\equiv T_n(x) - \frac{f(x+0) + f(x-0)}{2} = \sum_{k=0}^n \Delta \lambda_{n,k} \left(s_k(x) - \frac{f(x+0) + f(x-0)}{2} \right) \\
 &= \sum_{k=0}^n \Delta \lambda_{n,k} \cdot \frac{1}{\pi} \int_0^\pi \varphi_x(t) D_k(t) dt \\
 &= \frac{1}{\pi} \int_0^\pi \varphi_x(t) \sum_{k=0}^n \Delta \lambda_{n,k} D_k(t) dt \\
 &= \frac{1}{\pi} \int_0^\pi \varphi_x(t) K_n(t) dt = \frac{1}{\pi} \left(\int_0^\delta + \int_\delta^\pi \right) \\
 &= B_1(n) + B_2(n), \text{ say.}
 \end{aligned}$$

Applying Lemma 1,

$$\begin{aligned}
 B_1(n) &= \frac{1}{\pi} \int_0^\delta \varphi_x(t) K_n(t) dt \\
 &= \frac{1}{\pi} [\varphi_x(t) \lambda_n(t)]_0^\delta - \frac{1}{\pi} \int_0^\delta \lambda_n(t) d\varphi_x(t) \\
 &= \frac{1}{\pi} \varphi_x(\delta) \lambda_n(\delta) - \frac{1}{\pi} \int_0^\delta \lambda_n(t) d\varphi_x(t) \\
 &\leq 2 |\varphi_x(\delta)| + 2 \int_0^\delta |d\varphi_x(t)| \leq 4 V_0^\delta(\varphi_x).
 \end{aligned}$$

Next applying Lemma 2, we have

$$B_2(n) = \frac{1}{\pi} \int_\delta^\pi K_n(t) \varphi_x(t) dt$$

$$\begin{aligned}
&= -\frac{1}{\pi} [\gamma_n(t) \varphi_x(t)]_{-\delta}^{\pi} + \frac{1}{\pi} \int_{-\delta}^{\pi} \gamma_n(t) d\varphi_x(t) \\
&= \frac{1}{\pi} \gamma_n(\delta) \varphi_x(\delta) + \frac{1}{\pi} \int_{-\delta}^{\pi} \gamma_n(t) d\varphi_x(t) \\
&\leq \frac{\pi}{2} \cdot \frac{\alpha_n}{\delta} V_0^\delta(\varphi_x) + \frac{\alpha_n \pi}{2} \int_{-\delta}^{\pi} \frac{|d\varphi_x(t)|}{t} \\
&= \frac{\pi \alpha_n}{2 \delta} V_0^\delta(\varphi_x) + \frac{\pi \alpha_n}{2} \int_{-\delta}^{\pi} \frac{dV_0^t(\varphi_x)}{t} \\
&= \frac{\alpha_n}{2} V_0^\pi(\varphi_x) + \frac{\pi}{2} \alpha_n \int_{-\delta}^{\pi} \frac{V_0^t(\varphi_x)}{t^2} dt.
\end{aligned}$$

Thus

$$|B(n)| \leq 4 V_0^\delta(\varphi_x) + \frac{\alpha_n}{2} V_0^\pi(\varphi_x) + \frac{\alpha_n}{2} \int_1^{\pi/\delta} V_0^{\pi/t}(\varphi_x) dt.$$

Choosing $\delta = \pi \alpha_n$ and using the hypothesis that $\{\alpha_n\}$ is decreasing we have for $n \geq 1$

$$\begin{aligned}
\int_1^{\frac{1}{\alpha_n}} V_0^t(\varphi_x) dt &= \sum_{k=0}^{n-1} \int_1^{\frac{1}{\alpha_{k+1}}} V_0^t(\varphi_x) dt \\
&\leq \sum_{k=0}^{n-1} V_0^{\pi \alpha_k}(\varphi_x) \left(\frac{1}{\alpha_{k+1}} - \frac{1}{\alpha_k} \right) \equiv d_{n,\varphi}, \text{ say.}
\end{aligned}$$

Also

$$\begin{aligned}
\alpha_n d_{n,\varphi} &\geq \alpha_n V_0^{\pi \alpha_n}(\varphi_x) \left(\frac{1}{\alpha_n} - 1 \right) \\
&= V_0^{\pi \alpha_n}(\varphi_x) (1 - \alpha_n) > (1 - \alpha_1) V_0^{\pi \alpha_n}(\varphi_x)
\end{aligned}$$

and

$$\begin{aligned} & \frac{\alpha_n}{2} (1/\alpha_1 - 1) V_0^\pi(\varphi_x) \\ & \leq \frac{\alpha_n}{2} \sum_{k=0}^{n-1} V_0^{\pi\alpha_k}(\varphi_x) \left(\frac{1}{\alpha_{k+1}} - \frac{1}{\alpha_k} \right) = \frac{\alpha_n}{2} d_{n,\varphi}. \end{aligned}$$

Thus

$$\frac{\alpha_n}{2} V_0^\pi(\varphi_x) \leq \frac{\alpha_1}{1-\alpha_1} \frac{\alpha_n}{2} d_{n,\varphi},$$

and $4 V_0^\delta(\varphi_x) \leq \frac{4}{1-\alpha_1} \alpha_n \cdot d_{n,\varphi}$. Combining these estimates, we have

$$\begin{aligned} |B(n)| & \leq \frac{4}{1-\alpha_1} \alpha_n d_{n,\varphi} + \frac{\alpha_1}{1-\alpha_1} \frac{\alpha_n}{2} d_{n,\varphi} + \frac{\alpha_n}{2} d_{n,\varphi} \\ & = \frac{9}{2} \frac{\alpha_n}{1-\alpha_1} d_{n,\varphi}. \end{aligned}$$

This completes the proof of the theorem.

Corollary. Let $\Delta \lambda_{n,k} = \frac{q_k}{Q_n}$ where $Q_n = \sum_{k=0}^n q_k$, $\{q_k\}$ is positive and

non-decreasing and $\left\{ \frac{q_k}{Q_k} \right\}$ is decreasing, then

$$\begin{aligned} \left| \bar{N}_n(x) - \frac{f(x+0) + f(x-0)}{2} \right| & \leq \frac{9}{2} \frac{q_n}{Q_n} \frac{Q_1}{q_0} \sum_{k=0}^{n-1} V_0^{\frac{\pi q_k}{Q_k}}(\varphi_x) \\ & \quad \times \left(\frac{Q_{k+1}}{q_{k+1}} - \frac{Q_k}{q_k} \right), \quad \text{where} \end{aligned}$$

$\bar{N}_n(x)$ is the (\bar{N}, q_k) mean of the Fourier series of $f(x)$.

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