$H^\infty + L_E$ IN SEVERAL VARIABLES

by

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1. INTRODUCTION.

Let D be a bounded strictly pseudoconvex domain in C^n with C^2 boundary and E a subset of ∂D. $A = A(D)$ is the algebra of the domain D (functions which are holomorphic on D and continuous on $\overline{D}$), $L = L(D)$ is the algebra of continuous and bounded functions on D and L_E is formed by those functions in L which extend continuously to E. $H^\infty = H^\infty(D)$ is the algebra of functions holomorphic and bounded on D and $H_E^\infty = H^\infty \cap L_E$, the algebra of functions in H^∞ which extend continuously to E.

In this paper we prove that $H^\infty + L_E$ is a closed subalgebra of L, with respect to the sup norm on D. The proof follows the pattern of the one variable case but at several points different techniques are needed.

2. AN APPROXIMATION THEOREM.

A crucial point in proving that $H^\infty + L_E$ is closed, is the possibility to approximate a function in H^∞ by functions in H_E^∞ uniformly and boundedly over the sets of D which are at a positive distance of E. The following therorem was proved in [9]; we give here a different proof that does not need the solution of $\overline{\partial}$ with estimates.

Theorem 1. Let D be a bounded strictly pseudoconvex domain in C^n with C^2 boundary, $E \subset \partial D$ a closed set and $f \in H^\infty(D)$. Then there exists a sequence F_n of functions in $H^\infty(D)$ which are analytic across E, such that $\|F_n\| \leqslant C \cdot \|f\|$ and $F_n \to f$ uniformly on the sets $S \subset D$ which are at a positive distance of E. (The constant C depends only on D).

$$\text{Let } H(z,\xi) = \frac{K(z,\xi)}{\varphi^n(z,\xi)} \text{ for } \xi \in \partial D,\ z \in \bar{D}\setminus\{\xi\}$$

be the Henkin's kernel of the domain D (following [6]). The function

$$\xi \to |\xi - z| \cdot |H(z,\xi)|$$

is integrable on ∂D with respect to the surface measure $d\sigma$, for $z \in \bar{D}$. The proof of Theorem 1 is based on the fact that this function is also in $L^p(\partial D, d\sigma)$ for some $p > 1$.

Lemma 1. With the precedent notation, we have:

$$\sup_{z \in \bar{D}} \int_{\partial D} |\xi - z| \cdot |H(z,\xi)|^{1+\epsilon}\, d\sigma(\xi) < +\infty, \text{ for } 0 < \epsilon < \frac{1}{2n}.$$

Proof of the lemma. It is known ([6]) that there exist positive numbers δ_1, δ_2 such that for $z \in (\partial D)_{\delta_1}$ (neighborhood of ∂D of radius δ_1) and $\xi \in B(z, \delta_2)$ (ball of center $\underline{z}$ and radius δ_2), the Henkin's change of variables is possible.

For $z \notin (\partial D)_{\delta_1}$ there is no problem, since $\varphi(z,\xi)$ is bounded by bellow far of ∂D. Also, when $\xi \notin B(z, \delta_2)$ the integral is uniformly bounded with respect to $\underline{z}$. So we have to bound, independently of $\underline{z}$, the integral

$$I(z) = \int_{\partial D \cap B(z,\delta_2)} \frac{|\xi - z|}{|\varphi^2(z,\xi)|^{n/2+\epsilon'}}\, d\sigma(\xi) \quad \text{where} \quad \epsilon' = n \cdot \epsilon/2,\ z \in (\partial D)_{\delta_1}.$$

We make now the announced replacement of the complex coordinates $\xi_1, \xi_2, \ldots, \xi_n$ by real coordinates $t_1, t_2, \ldots, t_{2n}$ so that:

$$t_i(z) = 0;\ t_1 = \rho(\xi) - \rho(z) \text{ and } t_2 = \operatorname{Im} \phi(z,\xi),$$

ρ being a defining function for D, strictly pluri-subharmonic in a neighborhood of $\bar{D}$. We have the estimates:

$$d\sigma \leqslant K\, dt_2 \ldots dt_{2n};\ |\xi - z| \leqslant |t_1| + (t_2^2 + \ldots + t_{2n}^2)^{1/2} \quad \text{and also ([6]):}$$

$$|\varphi|^2 \geqslant \mathrm{Re}^2 \varphi + \mathrm{Im}^2 \varphi \geqslant (|\rho(z)| + t_2^2 + \ldots + t_{2n}^2)^2 + t_2^2.$$

Consequently we obtain: $I(z) \leqslant I_1(z) + I_2(z)$, where I_1, I_2 are given by the following integrals:

$$I_1(z) = \int_{\partial D \cap B(z_1 \delta_2)} \frac{|\rho(z)|\, d t_2 \ldots d t_{2n}}{[(|\rho(z)| + t_2^2 + \ldots + t_{2n}^2)^2 + t_2^2]^{n/2+\epsilon'}}$$

$$I_2(z) = \int_{\partial D \cap B(z_1 \delta_2)} \frac{(t_2^2 + \ldots + t_{2n}^2)^{1/2}\, d t_2 \ldots d t_{2n}}{[(|\rho(z)| + t_2^2 + \ldots + t_{2n}^2)^2 + t_2^2]^{n/2+\epsilon'}}$$

Let $r = (t_2^2 + \ldots + r_{2n}^2)$, $t_2 = r \cdot \cos\alpha$ and let ω be the surface area of the (2 n 3)-dimensional unit sphere. Then in spherical coordinates we obtain:

$$I_2(z) \leqslant \omega \cdot \int_0^{\delta_2} dr \int_0^{n} \frac{r r^{2n-2} (\sin\alpha)^{2n-3}}{(r^4 + r^2 \cdot \cos^2\alpha)^{n/2+\epsilon'}} d\alpha,$$

and putting $s = \cos\alpha$.

$$I_2(z) \leqslant \omega \cdot \int_0^{\delta_2} dr \int_{-1}^{1} \frac{r r^{n-2} (1 \quad s^2)^{n-2}}{r^{2\epsilon'} (r^2 + s^2)^{n/2+\epsilon'}} \cdot ds \leqslant$$

$$\leqslant \omega \cdot \int_0^{\delta_2} \frac{r \cdot dr}{r^{2\epsilon'}} \int_{-1}^{1} \frac{ds}{(r^2 + s^2)^{1+\epsilon'}}$$

If we use that $$\int_{-\infty}^{\infty} \frac{ds}{(r^2 + s^2)^\lambda} = \frac{K(\lambda)}{(2r)^{2\lambda-1}} \quad \text{for } \lambda > 1/2 \quad (*)$$

(see [5] p. 159) we arrive to:

$$I_2(z) \leqslant K \int_0^{\delta_2} \frac{r^{1-2\epsilon'}}{r^{1+2\epsilon'}}\, dr = \frac{K}{1-4\epsilon'}\, \delta_2^{1-4\epsilon'}$$

which is bounded when $1 - 4\epsilon' > 0$ i.e. when $\epsilon < 1/2\,n$.

For the first integral we have:

$$I_1(z) \leqslant \omega \cdot |\rho(z)| \int_0^{\delta_2} dr \int_{-1}^{1} \frac{r^{n-2}\, ds}{r^{2\epsilon'} \left[\left(\frac{|\rho(z)|}{r} + r\right)^2 + s^2\right]^{n/2+\epsilon'}} \leqslant$$

$$\leqslant \omega \cdot |\rho(z)| \int_0^{\delta_2} \frac{dr}{r^{2\epsilon'}} \int_{-1}^{1} \frac{ds}{\left[\left(\frac{|\rho(z)|}{r} + r\right)^2 + s^2\right]^{1+\epsilon'}} \leqslant$$

$$\leqslant K \cdot \omega \cdot |\rho(z)| \int_0^{\delta_2} \frac{dr}{r^{2\epsilon'} \left(\frac{|\rho(z)|}{r} + r\right)^{1+2\epsilon'}}, \text{ according to } (*).$$

From this inequality we obtain:

$$I_1(z) \leqslant K \cdot \omega \cdot |\rho(z)| \int_0^{\delta_2} \frac{r \cdot dr}{(|\rho(z)| + r^2)^{1+2\epsilon'}} \leqslant$$

$$\leqslant K \cdot \omega \cdot |\rho(z)| \int_0^{\delta_2} \frac{r \cdot dr}{(|\rho(z)| + r^2)^{1+½}} \leqslant$$

$$\leqslant K \cdot \omega \cdot |\rho(z)| \int_0^{\infty} \frac{r \cdot dr}{(|\rho(z)| + r^2)^{1+½}}$$

since $(|\rho(z)| + r^2)^{1+2\epsilon'} > (|\rho(z)| + r^2)^{1+½}$ when $2\epsilon' < 1/2$ i.e. $\epsilon < 1/2\,n$.

Finally we use the equality

$$\int_0^{\infty} \frac{x^k\,dx}{(a\,x^2+c)^{k+\frac{1}{2}}}=\frac{\sqrt{\pi}\,\Gamma(k)}{\sqrt{a}\cdot(2\sqrt{a\,c})^k\,\Gamma(k+\frac{1}{2})}$$

(a, c > 0), ([5] p. 185) to obtain:

$$I_1(z)\leqslant K'\,|\rho(z)|\,\frac{1}{|\rho(z)|^{\frac{1}{2}}}=K'\,|\rho(z)|^{\frac{1}{2}}$$ which is bounded.

Proof of theorem 1. We fix $f\in H^{\infty}$, $S\subset D$ at a positive distance of E and $\epsilon>0$. We have to find a function $F\in H^{\infty}$, holomorphic through E such that

$$\|F-f\|_S<\epsilon \quad\text{and}\quad \|F\|\leqslant C\cdot\|f\|.$$

We begin with a sequence $f_n\in H(\bar{D})$ (functions holomorphic in a neighborhood of D) such that $f_n(z)\to f(z)$ for all $z\in D$ and $\|f_n\|\leqslant M\cdot\|f\|$, where M depends only on D (same proof that approximation theorem of [6] but applied to a function $f\in H^{\infty}$). Since (f_n) is a bounded set in $L^{\infty}(\partial D, d\sigma)$ we may take a parcial sequence, let be still (f_n), such that $f_n\to g$, $g\in L^{\infty}$ in the weak topology (with respect to L^1). It follows that $f_n\to g$ weakly in L^2 and we may take some convex combinations (g_n) of the f_n's such that $g_n\to g$ in L^2 and we have also $\|g_n\|\leqslant M\,\|f\|$. By taking a parcial sequence, let be still (g_n), we have $g_n\to g$ a.e. in ∂D. By means of the Poisson integral we see that $P[g_n](z)\to P[g](z)$ for every $z\in D$; since we have that $P[f_n](z)=f_n(z)\to f(z)$ for $z\in D$ and the g_n's are convex combinations of the f_n's we obtain $P[g_n](z)\to f(z)$ i.e.

$$P[g](z)=f(z)$$

from what it follows $g(\xi)=f(\xi)$ a.e. in ∂D. So we have obtained $g_n\in H(\bar{D})$ with $\|g_n\|\leqslant M\cdot\|f\|$ and $g_n(\xi)\to f(\xi)$ a.e. in ∂D; in particular, $g_n\to f$ in $L^p(\partial D, d\sigma)$.

Now we take a C^{∞} function χ on C^n with value 0 in a neighborhood of E, value 1 in a neighborhood of S and $0\leqslant\chi\leqslant 1$, and define the sequence:

$$h_n(z)=T_{\chi}(f-g_n)=\int_{\partial D}\chi(\xi)\,(f(\xi)-g_n(\xi))\,H(z,\xi)\,d\sigma(\xi)=$$

$$\chi(z)\,(f(z)-g_n(n))+\int_{\partial D}(\chi(\xi)-\chi(z))\,(f(\xi)-g_n(\xi))\,H(z,\xi)\,d\sigma(\xi)$$

The functions h_n are in H^∞ (D) and extend analytically across E. Moreover, if $|\chi(\xi) - \chi(z)| \leq K_\chi \cdot |\xi - z|$ and we use Schwarz's inequality for $p = 1 + \epsilon'$ with $0 < \epsilon' < 1/2\,n$ and $p^{-1} + q^{-1} = 1$, we obtain:

$$\|h_n - \chi(f - g_n)\|_D \leq \sup_{z \in \bar{D}} \int_{\partial D} |\chi(\xi) - \chi(z)| \cdot |f(\xi) - g_n(\xi)| \cdot |H(z,\xi)| \, d\sigma(\xi) \leq$$

$$\leq K_\chi \cdot \sup_{z \in \bar{D}} \left| \int_{\partial D} |\xi - z|^p |H(z,\xi)|^p \, d\sigma(\xi) \right|^{\frac{1}{p}} \cdot \left| \int_{\partial D} |f(\xi) - g_n(\xi)|^q \, d\sigma(\xi) \right|^{\frac{1}{q}}$$

Now, by Lemma 1, there is some constant $\underline{K}$ such that:

$$\|h_n - \chi(f - g_n)\|_D \leq K_\chi \, K \, \|f - g_n\|_q$$

By the coice of (g_n) there is some positive integer n_0 such that

$$\|h_n - \chi(f - g_n)\|_D < \epsilon \qquad \text{for } n > n_0.$$

So the function $F = h_n + g_n$ (for some $n > n_0$) is in H^∞, is holomorphic through E and verifies:

$$\|F - f\|_S = \sup_{z \in S} |h_n + g_n - f| = \sup_{z \in S} |h_n + \chi(f - g_n)| < \epsilon$$

Moreover:

$$\|F\|_D \leq \|h_n\|_D + \|g_n\|_D \leq \|h_n - \chi(f - g_n)\|_D + \|\chi(f - g_n)\|_D + M\,\|f\| \leq$$

$$\leq \epsilon + \|f\| + 2\,M\,\|f\| = \epsilon + (2\,M + 1) \cdot \|f\| \leq C \cdot \|f\|.$$

3. A LOCALIZATION PROPERTY.

Let Y be the spectrum of the algebra H^∞ (D), where D is a bounded strictly pseudoconvex domain in C^n. Since $A \subset H^\infty$ we have a continuous projection $\pi\colon Y \to \bar{D} = \mathrm{Spec}\,(A)$, given by $\pi(\varphi) = \lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_i = \phi(z_i)$ if $\varphi \in Y$.

For every $\lambda \in D$ let $Y_\lambda = \pi^{-1}(\lambda)$ be the fiber of Y over $\underline{\lambda}$. If $\lambda \in D$ this fiber reduces to the point $\underline{\lambda}$, as it is easily seen writing a function $f \in H^\infty$ such that $f(\lambda) = 0$ as:

$$f(z) = (z_1 \quad \lambda_1) g_1(z) + \ldots + (z_n - \lambda_n) g_n(z) \text{ with } g_i \in H^\infty.$$

The fact that this decomposition is possible may be proved in the same manner that in case $f \in A$ ([8]).

Now we want to show that if $f \in H^\infty$ and $\phi \in Y_\lambda$ with $\lambda \in \partial D$, the action $\phi(f)$ of ϕ over f depends only on the values of f near λ.

We begin by observing that for a function $f \in H^\infty$ which extends analytically through $\lambda \in \partial D$ and for every $\phi \in Y_\lambda$ we have $\phi(f) = f(\lambda)$. To see this is true, it suffices to take a strictly pseudoconvex domain $\widetilde{D}$ containing $D \cup \lambda$ suh that $f \in H^\infty(\widetilde{D})$ ([8]) and to apply now the preceding decomposition, in case $f(\lambda) = 0$.

The fact that $\phi(f) = f(\lambda)$ for every $\phi \in Y_\lambda$ is also true for a function $f \in H^\infty$ that extends continuously to λ. This is a consequence of an approximation property which is a particulr case of a theorem of [9]. We give here a direct proof using Henkin's kernel.

Theorem 2. Let D be a bounded strictly pseudoconvex domain in C^n with C^2 boundary and $\lambda \in \partial D$. Then every function $f \in H^\infty(D)$ wich extends continuously to λ can be approximated uniformly on D by functions in $H^\infty(D)$ which extend analytically across λ.

Proof. Let us suppose $f(\lambda) = 0$ and, given $\epsilon > 0$, we shall find $F \in H^\infty$ analytic at λ with $\|f \quad F\|_D < \epsilon$.

Let W be a neighborhood of $\underline{\lambda}$ such that $|f(z)| < \epsilon/2$ when $z \in W$ and let us take a C^∞ function χ, $0 \leqslant \chi \leqslant 1$ with value 1 in a neighborhood of λ and whose support be contained in W. Consider now the function defined by

$$h(z) = T_\chi f(z) = \int_{\partial D} \chi(\xi) f(\xi) H(z, \xi) d\sigma(\xi),$$

so that $h \in H^\infty(D)$ and $\underline{h}$ is holomorphic in a domain $\widetilde{D}$ that contains D and $\partial D \setminus \operatorname{supp}(\chi)$. We take also supp (χ) small enough so that little translations of $\widetilde{D}$ in the direction of ν, the outward normal to ∂D at λ, will conver $\overline{D}$.

Since

$$f(z) - h(z) = \int_{\partial D} (1 - \chi(\xi)) f(\xi) H(z, \xi) d\sigma(\xi)$$

the function $f - h \in H^\infty$ and is holomorphic at λ. If we find a function $k \in H(\bar{D})$ such that $\|h - k\|_D < \epsilon$ then $F = f - h + k$ is a solution of our problem. To find k remember that we have, for $z \in D$:

$$h(z) = p(z) + \chi(z) f(z) \qquad \text{where p is continuous on } \bar{D}.$$

The function χf extends by zero to $\tilde{D}$ and so the function p, equal to h off supp (χ), extends to $\tilde{D}$. For $\delta > 0$ small, define

$$h^\delta(z) = h(z - \delta\nu) \qquad \text{if } z \in \bar{D} \text{ and, in the same way, } p^\delta \text{ and } (\chi f)^\delta.$$

Now we have:

$$\|h^\delta(z) - h(z)\|_D \leqslant \|p^\delta(z) - p(z)\|_D + \|(\chi f)^\delta(z) - (\chi f)(z)\|_D \leqslant$$

$$\leqslant \|p^\delta(z) - p(z)\| + \|(\chi f)^\delta(z)\| + \|(\chi f)(z)\|.$$

The term $\|p^\delta(z) - p(z)\|$ tends to zero when $\delta \to 0$ since p is uniformly continuous on $\tilde{D}$. Also $\|(\chi f)^\delta\|$ and $\|\chi f\|$ are small by the choice of W. So it suffices to take $k = h^\delta$ for some small δ.

Corollary. For D as in Theorem 2, if $f \in H^\infty$ extends continuously to $\lambda \in \partial D$ and $\phi \in Y_\lambda$, we have $\phi(f) = f(\lambda)$.

In order to prove our local property we need to extend a character of $H^\infty(D)$ to a character of $H^\infty(U \cap D)$, where U is some neighborhood of λ. This extension is possible using the following.

Theorem 3. Let D be a bounded strictly pseudoconvex domain in C^n with C^2 boundary, $\lambda \in \partial D$ and U an open set containing λ. If $f \in H^\infty(D \cap U)$, there exists a function $F \in H^\infty(D)$ such that $F - f$ is analytic through λ.

Proof. Let us take a strictly pseudoconvex domain $\tilde{D} \supset D \cup \{\lambda\}$ with $\tilde{D} \setminus D \subset U$, and $0 \leqslant \chi \leqslant 1$ a C^∞ function with supp $(\chi) \subset U$ and χ equals 1 in a neighborhood of $\widetilde{D} \setminus D$. Then, $\omega = -f \bar{\partial} \chi$ is a (0, 1)-form, $\bar{\partial}$-closed and bounded on $\widetilde{D}$. According to [7] there is a bounded function u on $\widetilde{D}$ such that $\bar{\partial} u = \omega$. The function $F = u + f\chi$ is bounded in D and $\bar{\partial} F = 0$. Also $F - f$ is holomorphic in a neighborhood of λ since we have $F \quad f = u$ and $\bar{\partial}(F \quad f) = \bar{\partial} u = - f \bar{\partial} \chi = 0$ near λ.

Theorem 4. Let D, $\lambda \in \partial D$ and U be as in Theroem 3 and let $\phi \in Y_\lambda(D)$ be a character of $H^\infty(D)$ over λ. Then, there exists a character $\widetilde{\phi}$ of the algebra $H^\infty(D \cap U)$ such that $\phi(F) = \widetilde{\phi}(F_{|U})$ for every $F \in H^\infty(D)$.

Proof. For every $f \in H^\infty(D \cap U)$ we can take, according to Theorem 3, a function $F \in H^\infty(D)$ such that $F - f$ is continuous at λ and $(F - f)(\lambda) = 0$. Define

$$\widetilde{\phi}(f) = \phi(F) \quad \text{for every } f \in H^\infty(D \cap U).$$

This definition is independent of F because if we take also $\widetilde{F} \in H^\infty(D)$ with $(\widetilde{F} - f)(\lambda) = 0$, we have that $\widetilde{F} - F = (\widetilde{F} - f) + (f - F)$ is continuous and vanishes at λ. By corollary to Theorem 2, $\phi(\widetilde{F}) = \phi(F)$.

Also it is easily seen that $\widetilde{\phi}$ is multiplicative on $H^\infty(D \cap U)$, since taking $F, G \in H^\infty(D)$ with $(F - f)(\lambda) = (G - g)(\lambda) = 0$ for $f, g \in H^\infty(D \cap U)$, we see that the function $FG - fg = F(G - g) + g(F - f)$ is continuous at λ and vanishes at this point.

Corollary 1. For $f \in H^\infty(D)$, $\lambda \in \partial D$ and $\phi \in Y_\lambda$ there is a sequence of points $\lambda_n \in D$ such that $\lambda_n \to \lambda$ and $f(\lambda_n) \to \phi(f)$.

Proof. Let us suppose $\phi(f) = 0$ and we prove the existence of $\lambda_n \to \lambda$ with $f(\lambda_n) \to 0$. In other case, we would have $|f(z)| \geqslant \delta > 0$ in some neighborhood U of λ. So $f^{-1} \in H^\infty(D \cap U)$ and by Theorem 4, $\phi(f) \neq 0$ for every $\phi \in Y_\lambda$.

Corollary 2. Let $f \in H^\infty(D)$, $\lambda \in \partial D$ and let $\hat{f}$ be the Gelfand transform of f. Then we have:

$$\|\hat{f}\|_{Y_\lambda} = \limsup_{\substack{z \to \lambda \\ z \in D}} |f(z)|$$

Proof. Corollary 1 gives $\|\hat{f}\|_Y \leqslant \limsup |f(z)|$. The other inequality follows from the fact that if $\lambda_n \to \lambda$ and $f(\lambda_n) \to \rho$, there is a $\phi \in Y_\lambda$ such that $\phi(f) = \rho$.

4. $H^\infty + L_E$ IS A CLOSED SUBSPACE OF L.

Theorem 5. If D is a bounded strictly pseudoconvex domain in C^n with C^2 boundary and $E \subset \partial D$ is a closed set, the sum $H^\infty + L_E$ is a closed subspace of L.

By a well known result of Banach spaces, it is enough to prove the equality:

$$d(h, H_E^\infty) = d(h, H^\infty) \qquad \text{for all } h \in L_E$$

where $\underline{d}$ is the distance corresponding to sup norm in L.

The inequality $d(h, H_E^\infty) \geqslant d(h, H^\infty)$ is trivial and the reverse inequality is contained in the following lemma, in case $\eta = d(h, H^\infty) + \epsilon$ if $f \in H^\infty$ and $d(f, h) < \eta$:

Lemma 2. Given D as in Theorem 5, $E \subset \partial D$, closed, $h \in L_E$ and $\eta > 0$, let V be the ball in L with center h and radius η. Then, given $f \in V \cap H^\infty$ there is a sequence F_n of functions in $H_E^\infty \cap V$ such that $F_n \to f$ uniformly on sets of D which are at a positive distance of E.

Proof. We give a sketch of the proof; details are the same that in Lemma 5.3 of [3].

Let M_E be the Stone-Cech compactification of $D \cup E$, so that functions in L_E are now continuous functions on M_E and uniform convergence on sets of D at a positive distance of E is equivalent to uniform convergence on compact subsets of $M_E \setminus E$. If $f \in V \cap H^\infty$, in order to see that f lies in the closure of $H_E^\infty \cap V$ with respect to this convergence we need to prove that for any measure μ with compact support in $M_E \setminus E$ and for any real number a with

$$a > \sup \left\{ \operatorname{Re} \int g \, d\mu : g \in H_E^\infty \cap V \right\}$$

we have also $a \geqslant \operatorname{Re} \int f \, d\mu$.

We take $\eta' < \eta$ such that $f \in V'$ (ball of center h and radius η') a define a continuous real-linear functional χ on H_E^∞ by:

$$g \xrightarrow{\ \chi\ } \operatorname{Re} \int g \, d\mu .$$

Using Lemma 4.1 of [3] χ can be extended to a functional ϕ on L_E, obtaining a measure ν on M_E such that:

$$\mathrm{Re} \int g \, d\nu \leq a \text{ for every } g \in L_E \cap V'$$

and $\mathrm{Re} \int g \, d\mu = \mathrm{Re} \int g \, d\nu$ if $g \in H_E^\infty$ i.e. $\mu - \nu$ is orthogonal to H_E^∞.

Let us now consider the space $\mathcal{F}$ of functions $\tilde{f}$ which are Borel and bounded on M_E, continuous on $M_E \setminus E$ and such that there is a sequence $f_n \in H_E^\infty$ with $\sup \|f_n\| < \infty$, $f_n \to \tilde{f}$ uniformly on sets of D at a positive distance of E and $\tilde{f}_n \to f$ weakly in $L^\infty(\nu)$. Functions in $\mathcal{F}$ are bounded and holomorphic on D and $\mu - \nu$ is orthogonal to $\mathcal{F}$, since

$$\int f_n \, d(\mu - \nu) \to \int \tilde{f} \, d(\mu - \nu).$$

If $\rho\colon \mathcal{F} \to H^\infty$ is the restriction map to D, given by $\rho(\tilde{f}) = f = \tilde{f}_{|D}$, we see that ρ is surjective: in fact, by Theorem 1 we can take $f_n \in H_E^\infty$ uniformly bonded and converging uniformly on sets at a positive distance of E to any $f \in H^\infty$ and passing to a parcial sequence we can get also $f_n \to f$ weakly in $L^\infty(d\nu)$.

We have also:

Lemma 3. The restriction map $\rho\colon \mathcal{F} \to H^\infty$ is one-to-one.

Proof of Lemma 3. Let be $f_n \in H_E^\infty$ with $f_n \to \tilde{f}$ uniformly on compact sets of $M_E \setminus E$, $\sup \|f_n\| < \infty$ and $f_n \to \tilde{f}$ weakly in $L^\infty(\nu)$; let us suppose $\rho(\tilde{f}) = 0$ i.e. $f_n(z) \to 0$ for every $z \in D$ and we shall see that $\tilde{f} = 0$ ν-a.e. or $\tilde{f}_{|E} = 0$ ν-a.e.; since μ is zero on E it is sufficient to see that $\tilde{f}_{|E} = 0$ $(\mu - \nu)$-a.e. We take $\varphi \in C(E)$ and show that

$$\int \tilde{f} \varphi \, d(\mu - \nu) = 0$$

and, in fact, we may take $\varphi \in C^\infty(\bar{D})$.

Consider the (0,1)-forms $\omega_n = f_n \cdot \bar{\partial} \varphi$; they are $\bar{\partial}$-closed, uniformly bounded and $\omega_n \to 0$ pointwise on D and so, weakly in $L^\infty(D, dm)$. By Theorem 1.1 of [2] there are functions $u_n \in C(\bar{D})$ such that $\bar{\partial} u_n = \omega_n$ and $u_n \to 0$ uniformly on $\bar{D}$. If we take $F_n = f_n \cdot \varphi - u_n$ we have that $F_n \in H_E^\infty$ and

$$\int \tilde{f} \varphi \, d(\mu - \nu) = \lim_n \int f_n \varphi \, d(\mu - \nu) = \lim_n \int F_n \, d(\mu \quad \nu) + \lim_n \int u_n \, d(\mu - \nu) = 0.$$

Now we take again the proof of Lemma 2: first of all we see that, for any $g \in \mathcal{F}$ with $\rho(g) = f$, the fact that $\|f - h\| < \eta'$, implies $\|h - g\|L^\infty(\nu|_E) < \eta'$. It is enough to prove the inequality

$$|\phi(g) - h(\lambda)| \leqslant \|f - h\| < \eta'$$

for every character ϕ of $L^\infty(\nu)$ which belongs to the fiber over a point $\lambda \in E$. Now, the restriction of ϕ to $\rho^{-1}(H^\infty)$ gives a character of H^∞; since the cluster values of the function $f - h(\lambda)$ at λ are bounded by $\|f - h\|$ we obtain, by Corollary 2 to Theorem 4: $|\phi(f) - h(\lambda)| \leqslant \|f - h\|$. Since $\rho(g) = f$ we have $\phi(f) = \phi(g)$ and so $|\phi(g) - h(\lambda)| \leqslant \|f - h\|$.

The rest of the proof of Lemma 2 is a consequence of this estimate and details are as in case of one variable ([3]).

Theorem 5 may be formulated in a different context: if $E \subset \partial D$ is closed, we define L_E^∞ to be the algebra of cunctions in $L^\infty(\partial D)$ which are essentially continuous at every point of E. If we identify H^∞ with a closed subalgebra of $L^\infty(\partial D)$ and put $H_E^\infty = H^\infty \cap L_E^\infty$ we obtain:

Theorem 5'. With same hypothesis that in Theorem 5 we have

$$d(h, H^\infty) = d(h, H_E^\infty) \quad \text{for every} \quad h \in L_E^\infty$$

and, consequently, $H^\infty + L_E^\infty$ is a closed subspace of $L^\infty(\partial D)$.

Proof. We need only to observe that $f \to \tilde{f}$ being the transformation given by Poisson integral, $\tilde{f} = P[f]$ for $f \in L^\infty(\partial D)$, we have $\tilde{f} \in L_E$ if $f \in L^\infty$ and $f \to \tilde{f}$ is an isometry.

Remark. The results of Theorema 5 and 5' are also true for $E \subset \partial D$ not necessarilly closed, but this generalizaton offers no new dificulty in several variables.

5. THE ALGEBRA $H^\infty + L_E$.

We prove, here, that the sum $H^\infty + L_E$ is also a closed subalgebra of L. For simplicity, we suppose that $E \subset \partial D$ is closed.

Theorem 6. If $E \subset \partial D$ is closed, where D is a bounded strictly pseudoconvex domain in C^n with C^2 boundary, then the sum $H^{\infty} + L_E$ is a closed subalgebra of L.

Proof. Let M be the Stone-Cech compactification of D and M_λ the fiber of M over $\lambda \in \partial D$. We shall prove that

$$H^{\infty} + L_E = \left\{ g \in L : g|_{M_\lambda} \in H^{\infty}_{|M_\lambda} \quad \text{for all} \quad \lambda \in E \right\},$$

one inclusion relation being evident; this will prove that $H^{\infty} + L_E$ is an algebra.

Let $\pi: M \to \bar{D}$ be the natural projection and put $E^{-1} = \pi^{-1}(E)$. If B denotes the restriction of $H^{\infty} + L_E$ to E^{-1}, since L_E contains all functions in $C(\bar{D})$ which vanishes on E one sees that B is a closed subspace of $C(E^{-1})$ and $H^{\infty} + L_E$ is formed by functions in B extended continuously to M in any way. We regard C(E) as a subalgebra of $C(E^{-1})$; if we prove that B is a C (E)-module, a theorem of Bishop ([4]) ensures that $f \in B$ when $f_{|M_\lambda} \in B_{|M_\lambda}$ for every $\lambda \in E$, since M_λ are the level sets of C (E).

That B is a C (E)-module is a direct consequence of the fact that $H^{\infty} + C(\bar{D})$ is a algebra, when D is a strictly pseudoconvex domain ([1]): if $f = f_1 + f_2 \in H^{\infty} + L_E$ with $f_1 \in H^{\infty}$ and $f_2 \in L_E$ we obtain for $g \in C(E)$, or $g \in C(D)$, that $f_1 \cdot g \in H^{\infty} + L_E$ and $f_2 \cdot g \in L_E$.

Theorem 6'. With the same hypothesis that in Theorem 6, $H^{\infty} + L_E^{\infty}$ is a closed subalgebra of $L^{\infty}(\partial D)$.

Proof. We may do the same reasoning that in proof ot Theorem 6, substituing M by $X = \mathrm{Spec}\,(L^{\infty}(\partial D))$ and identifying L^{∞} with C (X).

Finally we determine the spectrum of the algebra $H^{\infty} + L^{\infty} \subset L^{\infty}(\partial D)$:

Theorem 7. With the same hypothesis that in Theorem 6, the fiber of the spectrum of the algebra $H^{\infty} + L_E^{\infty}$ over a point $\lambda \in \partial D$ is the fiber of Spec (H^{∞}) if $\lambda \in E$ and the fiber of Spec (L^{∞}) if $\lambda \notin E$.

Proof. The second affirmation is a consequence of the fact that every function in $L^{\infty}(\partial D)$ equals, in a neighborhood of λ, a function in L_E^{∞}. To prove the first one we need only to show that if $\lambda \in E$ and $\varphi \in Y_\lambda$ ($Y = \mathrm{Spec}\,(H^{\infty})$), then ϕ extends to a character of $H^{\infty} + L_E^{\infty}$.

Let $f = g + h \in H^\infty + L_E^\infty$ with $g \in H^\infty$, $h \in L_E^\infty$ and define a continuous function $\hat{f}$ on $Y \setminus D$ by $\hat{f}(\phi) = \hat{g}(\varphi) + h(\lambda)$. We have to see that

$$\hat{g}(\phi) \cdot \hat{h}(\phi) = \widehat{gh}(\phi) \quad \text{for } g \in H_E^\infty \text{ and } h \in L_E^\infty .$$

First of all, the definition of $\hat{f}$ does not depend on the decomposition of f as a sum, as it is easily seen. Now we consider $X = \mathrm{Spec}(L^\infty(\partial D))$; the inclusion $H^\infty \subset L^\infty$ gives us a continuous map $\tau : X \to Y$, the restriction of characters, which is not one-to-one. If $\phi \in Y_\lambda$, taking a Hanh-Banach extension of ϕ to L^∞, we obtain a probability measure m_φ on X such that

$$\hat{f}(\phi) = \phi(f) = \int_X \tilde{f}\, dm_\varphi \quad \text{for every} \quad f \in H^\infty,$$

noting by $\tilde{h}$ the function in C (X) which corresponds to $h \in L^\infty(\partial D)$, by means of the Gelfand transform.

The measure m_φ is supported by the fiber X_λ, of X over λ: this is a consequence of the fact that every $\lambda \in \partial D$ is a pick point for the algebra A (D), D being strictly pseudoconvex. So if $f \in H^\infty + L_E^\infty$ we have also the equality

$$\hat{f}(\varphi) = \int_X \hat{f}\, dm_\phi$$

because putting $f = g + h$ with $g \in H^\infty$, $h \in L_E^\infty$, we obtain:

$$\hat{f}(\phi) = \hat{g}(\phi) + h(\lambda) = \int_X \tilde{g}\, dm_\varphi + \int_X \tilde{h}\, dm_\varphi = \int_X \tilde{f} dm_\phi.$$

Finally, if $g \in H^\infty$, $h \in L_E^\infty$ we have:

$$\hat{g}(\phi) \cdot \hat{h}(\phi) = \int_X \tilde{g}\, dm_\varphi \cdot h(\lambda) = \int_{X_\lambda} \tilde{h} \cdot \tilde{g}\, dm_\varphi = \int_X \tilde{h} \cdot \tilde{g}\, dm_\phi =$$

$$= \int_X \widetilde{h \cdot g}\, dm_\varphi = \widehat{h \cdot g}(\phi),$$

that is all to be proven.

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