THE BEHAVIOUR OF TRANSFORMATIONS ON SEQUENCE SPACES

by

P. K. KAMTHAN

1. INTRODUCTION.

The study of different types of behaviour of a linear transformation T from a sequence space λ into another sequence space (for instance, the δ-nuclearity, δ-type, compactness, continuity, boundedness and so forth so on of T) may be considered as an outgrowth to the development of the overall theory of δ-nuclear spaces. In general, the problem of obtaining such a behaviour of T is too complicated to yield any satisfying solution unless we confine to special circumstances. Accordingly, one attempts to study this problem when either the space or the operator or both in question are restricted to a certain degree of generality and satisfaction, and that is what we are going to do precisely here. Indeed, we consider transformations (matrix or otherwise) from sequence spaces to sequence spaces and study their behaviour outlined above. Our basic results are Theorem 3.11 and 4.2 which respectively characterize the normal boundedness and δ-nuclearity of matrix transformations $[a_{ij}]$ in tems of analytic condicions comprizing the components a_{ij}'s.

2. PRELIMINARIES.

As usual we assume the reader to be familiar with the rudiments of locally convex spaces and for that reason it suffices to refer to some of the standard texts on the subject matter, e.g., [1] and [5], secondly, one's familiarity with the elements on sequence spaces and matrix transformations thereon will undoubtedly relieve him of any tension which he might expect to come ahead. So to be in tune with the discussion that follows, let us recall a few relevant facts from the theory of sequence spaces. (cf. [3] and [5] for several unexplained terms and results). A *sequence space* λ is a subspace of ω, the family of all sequences from the scalar field K such that λ contains the space generated by the unit vectors, that is, $\lambda \supset \varphi = \text{sp} \ \{ e^n : n \geq 1 \}$, where

$$e_i^n = \begin{cases} 1, & i = n; \\ 0, & i \neq n. \end{cases}$$

Let us recall the sequence spaces ℓ^1 and ℓ^∞ which are respectively the collections of absolutely summable and bounded sequences from ω; also let m_o denote the space generated by the set of sequences having coordinates zeros or ones. A sequence space λ is called *monotone* (resp. *normal*) if $m_o \lambda \subset \lambda$ (resp. $\ell^\infty \lambda \subset \lambda$). By λ^x we mean the *Köthe dual* of λ:

$$\lambda^x = \{ y \in \omega : p_y(x) \equiv \sum_{i \geq 1} |x_i y_i| < \infty, \text{for each x in } \lambda \}.$$

A sequence space λ is called *perfect* it $\lambda = \lambda^{xx} \equiv (\lambda^x)^x$. Clearly every perfect sequence space λ is normal and each normal sequence space λ is monotone; however, the reverse conclusions are not necessarily true (cf. [3]).

Unless specified otherwise, it should be understood unambiguously that whenever we talk of a topology on a given sequence space λ, this topology is nothing but the *normal topology* $\eta(\lambda, \lambda^x)$ generated by $\{ p_y : y \in \lambda^x, y > 0 \}$. It is known that $\eta(\lambda, \lambda^x)$ is compatible with the dual system $<\lambda, \lambda^x>$ and so for any locally convex topology $\mathcal{F}$ on λ compatible with $<\lambda, \lambda^x>$, bounded sets in λ are the same relative to either of the topologies $\sigma(\lambda, \lambda^x)$, $\eta(\lambda, \lambda^x)$, $\tau(\lambda, \lambda^x)$ or $\mathcal{F}$. Simple *(einfach)* bounded sets in $\lambda(P)$ and their properties were first considered by Köthe in [6], this notion was subsequently extended to an arbitrary sequence space in [2]. A bounded subset of a sequence space λ is called *simple* if it is contained in the normal hull of a point in λ, further, λ is called *simple*, if each bounded subset of it is simple. It is not difficult to see, for instance, one may consult [3], Exer. 2.5.12 that every normal simple sequence space λ is perfect.

If λ and μ are two sequence spaces, we write $[a_{ij}]$ for the matrix transformation $T \cdot \lambda \to \mu$, that is, for $x \in \lambda$

$$(Tx)_i = \sum_{j \geq 1} a_{ij} x_j, \qquad \forall\ i \geq 1.$$

The following result in this direction is essentially due to Köthe and Toeplitz [7] (cf. [3], Prop. 4.3.2); it will serve an useful observation for future reference.

Proposition 2.1. Let λ and μ be two sequence spaces with λ being monotone. Then a linear transformation $T : \lambda \to \mu$ is a matrix transformation if and only if T is $\sigma(\lambda, \lambda^x) - \sigma(\mu, \mu^x)$ continuous.

For our work we will also require ([2], [3], Th. 4.6.6).

Proposition 2.2. Let λ and μ be two normal sequence spaces with μ being simple. Let $T \equiv [a_{ij}]$. Then T is a matrix transformation, $T : \lambda \to \mu$ if and only if for each x in λ there exists y in μ, $y > 0$ such that

$$\sum_{j \geq 1} |a_{ij} x_j| \leq y_i, \qquad i \geq 1.$$

Finally, let us recall a few definitions concerning a linear operator $T : X \to Y$ where X and Y are locally convex spaces; T is said to be *bounded* (resp. *precompact, compact*) if for some neighbourhood U of zero in X, T(U) is bounded (resp. precompact, relatively compact) in Y.

3. CONTINUOUS AND BOUNDED OPERATORS.

Let λ and μ be two sequence spaces and let $T : \lambda \to \mu$ be an operator. The continuity (resp. boundedness) of T can be expressed comparatively in simpler analytical forms which we shall use later on. First we have.

Proposition 3.1. Let λ and μ be sequence spaces such that μ is perfect. Then a linear map $T : \lambda \to \mu$ is continuous if and only if

$$\forall\, v \in \mu^x, v > 0 \;\exists\; y \in \lambda^x, y > 0 \ni$$

$$\sup_n [p_v (Te^n)/p_y (e^n)] < +\infty. \tag{3.2}$$

Proof: Assuming the continuity of T first, we find

$$\forall\, v \in \mu^x, v > 0 \;\exists\; y \in \lambda^x, y > 0 \text{ and } M \neq M(x) > 0 \ni$$

$$p_v (Tx) \leq M\, p_y (x), \forall\, x \in \lambda$$

$$\Rightarrow p_v (Te^n) \leq M\, p_y (e^n), \qquad \forall\, n \geq 1.$$

Hence (3.2) follows.

Let now (3.2) be true. If $x \in \varphi$, we find

$$Tx = \sum_{i=1}^{n} x_i Te^i$$

$$\Rightarrow p_v (Tx) \leq M \sum_{i=1}^{n} |x_i|\, p_y (e^i) \leq M\, p_y (x).$$

Hence T is continuous from φ to μ. It is easily verified that μ is complete (cf. [4], p. 413); more over $\overline{\varphi} = \lambda$ (indeed, by Prop. 2.3.23 of [3], $\{e^n; e^n\}$ is a Schauder base for λ). Therefore T can be continuously extended to λ.

Next we have.

Proposition 3.3. Let λ and μ be two sequece spaces and $T : \lambda \to \mu$ a continuous operator. Then T is bounded if and only if

$$\exists \; y \in \lambda^x, y > 0 \ni \forall \; v \in \mu^x, v > 0$$

$$\sup_n [p_v(Te^n)/p_y(e^n)] < +\infty. \tag{3.4}$$

Proof: Assuming (3.4) to be true first, we find a constant $M_v > 0$ corresponding to each $v \in \mu^x$, $v > 0$ such that for some $y \in \lambda^x$, $y > 0$

$$p_v(Te^n) \leq M_v \, p_y(e^n), \qquad \forall \, n \geq 1.$$

Since T is continuous we find for each x in λ

$$p_v(Tx) \leq M_v \, p_y(x).$$

Hence

$$p_v(Tx) < M_v \quad \forall x \in U_y, \; U_y = \{z \in \lambda : p_y(z) \leq 1\}$$

giving thereby the boundedness of $T(U_y)$.

Let now T be bounded. Hence for some $y \in \lambda^x$, $y > 0$ and each $v \in \mu^x$, $v > 0$ there is an $M_v > 0$ such that

$$p_v(Tx) < M_v \quad \forall x \text{ in } U_y = \{z \in \lambda \cdot p_y(z) \leq 1\}.$$

Putting $x^n = e^n/p_y(e^n)$ we find that $\{x^n\} \subset U_y$. The result now easily follows.

Proposition 3.5. A family $\{f^\alpha : \alpha \in \Lambda\} \subset \lambda^x$ is equicontinuous on λ if and only if there exists y in λ^x, $y > 0$ such that

$$\sup \{|f_i^\alpha|/y_i : i \geq 1, \alpha \in \Lambda\} < +\infty.$$

Proof: This easily follows on the lines of proof of Proposition 3.1.

Concerning operators T on sequence spaces, there is another notion of boundedness contained in

Definition 3.6. An operator $T : \lambda \to \mu$ is said to be *simply* or *normally bounded* if there exists a zero neighbourhood U in λ such that T(U) is simple in μ.

It is clear that each simply bounded T is bounded. For the converse we require the range space to be simple. For instance, consider the space δ, the space of all entire sequences $\{ x_n \}$ with $x_n^{1/n} \to 0$ as $n \to \infty$. More generally speaking, all perfect nuclear sequence spaces are simple, where the nuclearity of the space λ is taken in the usual sense as discussed in [8], p. 70. To derive this result we recall the Grothendieck-Pietsch type criterion for the nuclearity of λ (cf. [3], Th. 4.7.9), namely.

Theorem 3.7. A sequence space λ is nuclear if and only if to each $u \in \lambda^x$, $u > 0$ there corresponds a $v \in \lambda^x$, $u < v$ such that $\{ u_n/v_n \}$ lies in ℓ^1.

The following result is essentially due to Köthe ([4], (5), p. 270).

Lemma 3.8. Each perfect nuclear sequence space is simple.

Using Lemma 3.8, we easily derive.

Proposition 3.9. Let λ and μ be sequence spaces with μ being nuclear and perfect. Then each operator $T : \lambda \to \mu$ is simply bounded if and only if T is bounded.

On the other hand there are bounded operators which are not simply bounded, for instance, consider.

Example 3.10. Let $I: \ell^1 \to \ell^1$ be the identity operator, ℓ^1 being equipped with $\eta(\ell^1, \ell^\infty)$ topology. I is bounded but not simply bounded (see also the remark below).

The foregoing example is a special case of the following more general result characterizing the simple boundedness of matrix transformations.

Theorem 3.11. Let λ and μ be sequence spaces where μ is normal. Let $T = [a_{ij}]: \lambda \to \mu$ be a matrix transformation. Then T is simply bounded if and only if there exists $y \in \lambda^x$, $y > 0$ such that

$$\left\{ (\sup_j [|a_{ij}|/y_j])_i \right\} \in \mu. \tag{3.12}$$

Prof: Let $T = [a_{ij}]$ be simply bounded. Hence there exist $y \in \lambda^x$, $y > 0$ and $u \in \mu$, $u > 0$ such that

$$|(Tz)_i| \leq u_i, \; i \geq 1, \text{ for } z \in U_y = \{ x \in \lambda : p_y(x) \leq 1 \}.$$

Therefore

$$\left| \sum_{j \geq 1} a_{ij} z_j \right| \leq u_i, \qquad \forall z \in U_y, i \geq 1.$$

Let $z^j = \{ 0, \ldots, 0, y_j^{-1}, 0, \ldots \}$. Then $z^j \in U_y$ for $j \geq 1$.

Thus

$$|a_{ij}| \, y_j^{-1} \leq u_i, \qquad \forall i, j \geq 1$$

and this proves (3.12).

Let now (3.12) be true. Put

$$u_i = \sup_j [|a_{ij}|/y_j], \text{ for some } y \text{ in } \lambda^x.$$

Consider U_y as before and let $z \in U_y$. Tnen

$$|(Tz)_i| \leq \sum_{j \geq 1} |a_{ij}| \, |z_j| \leq \sum_{j \geq 1} u_i \, y_j \, |z_j|$$

$$\Rightarrow |(Tz)_i| \leq u_i, \quad \forall i \geq 1.$$

Hence T is simply bounded.

Remark: Let $\lambda = \mu = \ell^1$ and $T = [a_{ij}]$ where $a_{ii} = 1$, $i \geq 1$ and $a_{ij} = 0$ for $i \neq j$. Let $y \in \ell^\infty$ be such that $\{ y_i^{-1} \} \in \ell^1$. Hence $1/y_i \to 0$ as $i \to \infty$; however, this statement means that $y \notin \ell^\infty$. Consequently T is not simply bounded.

4. δ-NUCLEAR OPERATORS

Throughout this section, δ stands for an arbitrary sequence space. First of all let us recall.

Definition 4.1. Let X and Y be two locally convex spaces. A linear map $T : X \to Y$ is said to be *δ-nuclear* provided there exist sequences $\{\alpha_n\}$, $\{f_n\}$ and $\{y_n\}$ with $\{\alpha_n\} \in \delta$;

$\{f_n\} \subset X^*$, f_n being equicontinuous on X; and

$\{y_n\} \subset Y, \{g(y_n)\} \in \delta^x$ for every $g \in Y^*$ such that

$$Tx = \sum_{n \geq 1} \alpha_n f_n(x) y_n, \qquad \forall x \text{ in } X.$$

The basic problem in the theory of δ-nuclear operators is to characterize operators on locally convex spaces which are δ-nuclear. In general this problem seems to be too complicated to yield any satisfactory solution. However, in special circumstances we do have a solution contained in

Theorem 4.2. Let λ and μ be two sequence spaces such that λ is monotone. Assume that δ is simple and $\delta = \mu^{xx}$ and that $T = [a_{ij}]: \lambda \to \mu$ is a matrix transformation. Then T is δ-nuclear if and only if

$$\{ (\sup_j [|a_{ij}|/y_j])_i \} \in \delta, \tag{4.3}$$

for some $y \in \lambda^x$, $y > 0$.

Proof: Let T be δ-nuclear. Then

$$Tx = \sum_{i > 1} \alpha_i < x, f^i > y^i, \quad \forall x \text{ in } \lambda$$

where $\{\alpha_i\} \in \delta$, $\{f^i\} \subset \lambda^x$ and is equicontinuous on λ and $\{y^i\} \subset \mu$ with $\{< y^i, u >\} \in \delta^x$ for each u in μ^x. By Proposition 3.5, there exists z in λ^x, $z > 0$ such that

$$\sup_{i,j} [|f^i_j|/z_j] < +\infty. \tag{4.4}$$

Let $a^i = \{a_{ij} : j \geq 1\}$. Since

$$(Tx)_i = \sum_{j \geq 1} a_{ij} x_j = \langle x, a^i \rangle; \text{ and}$$

$$Tx = \eta(\mu, \mu^x) - \lim_{n \to \infty} \sum_{i=1}^{n} (Tx)_i e^i = \sum_{i \geq 1} (Tx)_i e^i,$$

we find that

$$Tx = \sum_{i \geq 1} \langle x, a^i \rangle e^i.$$

Comparing the two representations of T, we get

$$\sum_{i \geq 1} \langle e^k, a^i \rangle e^i = \sum_{i \geq 1} \alpha_i \langle e^k, f^i \rangle y^i$$

$$= \sum_{i \geq 1} \alpha_i \langle e^k, f^i \rangle \sum_{j \geq 1} \langle y^i, e^j \rangle e^j$$

$$= \sum_{j \geq 1} \sum_{i \geq 1} \alpha_i \langle e^k, f^i \rangle \langle y^i, e^j \rangle e^j.$$

Hence

$$a^i_k = \sum_{n \geq 1} \alpha_n \langle e^k, f^n \rangle \langle y^n, e^i \rangle, \quad \forall k, i \geq 1$$

$$= \sum_{n \geq 1} \alpha_n f^n_k y^n_i, \quad \forall k, i \geq 1.$$

Therefore

$$|a^i_k| \leq \sum_{n \geq 1} |\alpha_n| \, |f^n_k| \, |y^n_i|$$

$$\leq M_z \sum_{n \geq 1} |\alpha_n| \, |z_k| \, |y^n_i|, \quad (\text{cf. } (4.4)).$$

Consequently

$$\sup_k [|a^i_k|/z_k] \leq M_z \sum_{n \geq 1} |\alpha_n| \, |y^n_i|, \quad \forall i \geq 1.$$

If $b_{ij} = y_j^i$, then for u in μ^x

$$([b_{ij}](u))_i = \sum_{j \geq 1} b_{ij}\, u_j = \langle y^i, u \rangle$$

and so $[b_{ij}]$ is a matrix transformation from μ^x to δ^x. Hence the transpose $[y_i^j]$ of $[b_{ij}]$ is a matrix transformation from δ^{xx} to μ^{xx} (cf. [3] for details). Now μ^{xx} is simple and $\{\alpha_n\} \in \delta^{xx}$ and so from Proposition 2.2, we find a $v \in \mu^{xx}$ such that

$$\sum_{j > 1} |y_j^i|\, |\alpha_j| \leq v_i, \quad \forall i \geq 1.$$

Therefore

$$\sup_k [|a_k^i|/z_k] \leq M_z\, v_i, \quad \forall i \geq 1.$$

Clearly δ is perfect ($\delta = \mu^{xx} \Rightarrow \delta^x = \mu^x$) and so it is normal. But $v \in \delta$, hence (4.3) follows.

Conversely, let (4.3) be true. Set

$$v_i = \sup_j [|a_{ij}|/y_j],$$

then $v \in \delta$ and we have

$$Tx = \sum_{i \geq 1} v_i \left\langle x, \frac{a^i}{v_i} \right\rangle e^i, \quad \forall x \in \lambda.$$

For each $u \in \mu^x$, it is clear that $\{\langle u, e^i \rangle\} \in \delta^x$. To show the δ-nuclearity of T, it is therefore sufficient to establish that $\{a^i/v_i\} \subset \lambda^x$ and is equicontinuous on λ.

Observe that $\lambda^x = \lambda^\beta$ (cf. [3], Proposition 2.2.7) where λ^β is the β-dual of λ. As

$$\sum_{j \geq 1} a_{ij}\, x_j$$

converges for each x in λ, $\{a^i\} \subset \lambda^x$. Thus

$$\{a^i/v_i\} \subset \lambda^x.$$

Next

$$|a_j^i|/|v_i|\ |y_j| = |a_{ij}|/|v_i|\ |y_j|$$
$$= |a_{ij}|/y_j\,(\sup\ |a_{ij}|/y_j)$$
$$\leq 1.$$

Hence

$$\sup\,[|a_j^i|/v_i\, y_j] < +\infty$$

and which in virtue of Proposition 3.5, establishes the requiered equicontinuity.

Combining Theorem 3.11 and 4.2, we derive

Theorem 4.5. Let λ and μ be two sequence spaces with λ being monotone and μ normal and simple. Suppose $T = [a_{ij}] : \lambda \to \mu$ is a matrix transformation. Then T is μ-nuclear if and only if T is simply bounded.

In the proof of this result, we need observe as mentioned earlier that μ is perfect.

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Department of Mathematics
Indian Institute of Technology,
Kanpur-208016, India.