LINEAR BORNOLOGIES AND ASSOCIATED TOPOLOGIES

by

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0. INTRODUCTION.

This paper is basically devoted to the study of the relation between a linear bornology and the associated M-closure topology, introduced in (3) and (18). This topology is invariant in translations and dilations, and it has a base of absorbing and balanced neighbourhoods of zero. In Section 1 we study the class of topologies on linear spaces having these properties, namely, quasilinear topologies.

In Section 2, we associate in a natural way a bornology to every quasilinear topology, and, in Section 3, we introduce, under the same point of view, the M-closure topology relative to a linear bornology. We study the quasilinear topologies which can be obtained as M-closure topologies (q-bornological topologies), and the linear bornologies obtained from a quasilinear topology by the procedure of Section 2 (infratopological bornologies), also considered by B. Perrot in (18).

In Section 4, we study the stability of the preceding classes of topologies and bornologies in passing to initial and final structures, and in forming spaces of bounded linear mappings. Finally, in Section 5, we see another way to obtain the M-closure topology.

We use, if the contrary is not specified, the usual terminology on bornologies, which can be found in (3), (9) and (10).

1. QUASILINEAR TOPOLOGIES.

Let E be a linear space over K (in the following, K = R or C), provided with a topology τ. We say that τ is quasilinear, or that $E(\tau)$ is a quasilinear topological space, when the following holds:

i) τ is translation and dilation-invariant.

ii) There is a base of balanced and absorbing zero neighbourhoods of τ.

Obviously, every topological vector space (TVS) is a quasilinear topological space (QTS), but the converse is not true, as can be seen in a lot of well-known examples. The QTS appeared in (12), named locally starlike ($T\alpha$)-spaces, but we won't use this terminology, seeming more natural the one introduced above. Note that the condition ii) can be reinforced as follows: *every quasilinear topology admits a base of open, balanced and absorbing* NZ. We quote now some known results about these topologies, and some other easy to check.

1.1. *PROPOSITION. Let* E *be a* QTS. *Then:*

i) *If* S *is a linear subspace of* E, *the closure of* S *is also a linear subspace of* E.

ii) *The intersection of all* NZ *in* E *is a closed linear subspace of* E, *which is trivial if and only if* E *is accesible (i.e.* E *satifies the axiom T_1 of separation).*

iii) *If* u *is a linear functional on* E, u *is continuous if and only if its kernel is closed.*

Proof: i) is valid for any topology which is invariant in translations and dilations (see (9), II.12, Proposition 1). For ii), we only point out that this intersection is the closure of the linear subspace $\{0\}$. iii) can be found in (12), where it is also shown that the hypothesis of E having a base of balanced NZ is essential. QED.

1.2. *PROPOSITION. Let* E *be a linear space over* K, $(E_i)_{i\in I}$ *a family of QTS over the same field, and let us suppose that for every* $i\in I$ *there is a linear mapping* f_i: $E_i \to E$. *Then the initial topology on* E *with respect to this system is quasilinear.*

Proof: Routinary.

1.3. *PROPOSITION. Let* E *be a* QTS, *and* S *a linear subspace of* E. *The quotient topology on* E/S *is quasilinear. It is accesible if and only if* S *is closed.*

Proof: (12), page 446.

2. BORNOLOGY ASSOCIATED TO A QUASILINEAR TOPOLOGY.

Let E be a QTS. We say that $B \subset E$ is bounded when every NZ absorbs B. It is easy to see that this definition gives us a bornology admitting a balanced base. We shall see some conditions for this bornology to be linear. Trivially, every continuous linear mapping between a couple of QTS is bounded with respect to their bornologies. We call the bornology defined in that way, canonical bornology of the QTS E.

2.1. *PROPOSITION. Let* E *be a* QTS, *and* $B \subset E$. *The following are equivalent:*

i) B *is bounded.*

ii) *Every countable subset of* B *is bounded.*

iii) *For every sequence* $(x_n)_n$ *in* B, *and every null sequence* $(t_n)_n$ *in* K, $(t_n x_n)_n$ *converges to zero in* E.

Proof: Just remember the proof for the linear case.

Note that, if $(x_n)_n$ converges to zero in a QTS E, and $(t_n)_n$ is bounded in K, $(t_n x_n)_n$ converges to zero. Then, by 2.1. iii), every null sequence in a QTS is bounded.

2.2. *PROPOSITION. Let* E *be a* QTS *with the following property (sequentially continuous sum): If* $(x_n)_n$ *and* $(y_n)_n$ *are sequences converging to* x *and* y *respectively,* $(x_n + y_n)_n$ *converges to* $x + y$. *Then, the following holds:*

i) *The canonical bornology of* E *is linear.*

ii) *Every convergent sequence in* E *is bounded.*

iii) *The mapping* $(t, x) \to tx$ *of* $K \times E$ *into* E *is sequentially continuous.*

iv) *If* E *is accesible, the limits and M-limits in* E *are unique.*

v) *Every sequentially compact subset of* E *is bounded.*

Proof: The first three are obvious. To prove iv), let's consider a sequence $(x_n)_n$ with two different limits x and y. Then the constant sequence zero converges to $x - y$, and E is not accesible. For the Mackey-convergence, do the same reasoning. To prove v), let $A \subset E$ sequentially compact, $(x_n)_n$ a sequence in A, and $(t_n)_n$ null in K. Every subsequence of $(x_n)_n$ has a subsequence converging in A, which must be bounded. Then every subsequence of $(t_n x_n)_n$ has a null subsequence, i.e. $(t_n x_n)_n$ is null. QED.

We look now at the bornologies obtained in passing to initial topologies.

2.3. *PROPOSITION. Let* E *be a linear space,* $(E_i)_{i \in I}$ *a family of* QTS *and let's suppose that for every* $i \in I$ *a linear mapping* $f_i \colon E \to E_i$ *is defined. If we provide* E *with the initial topology with respect to this system, the canonical bornology of* E *is initial with respect to the canonical bornologies of the* E_i's *and the* f_i's. *If every* E_i *has linear bornology, also has* E.

Proof: The same as for the linear case.

A similar result cannot be expected for quotient topologies, as we know for the locally convex case (see (7) for the classical counterexample), nor for final topologies, as can be seen in the case of compact bornologies in infinite dimensional Banach spaces, that, being convex, are inductive limits of normed bornologies.

3. QUASILINEAR TOPOLOGY ASSOCIATED TO A LINEAR BORNOLOGY.

Let E be a linear space, equipped with a linear bornology b, whose dual is E^x. We denote by τb the M-closure topology associated to b (see (9) or (18). τb is a quasilinear topology with sequentially continuous sum, and it is accesible if and only if b is separated. We see now other properties of this topology.

3.1. *PROPOSITION. Let* E(b) *be a linear bornological space* (LBS). *Then:*

i) τb *is the finest topology for which all the* b*-bounded sets are bounded.*

ii) τb *is the finest quasilinear topology for which all the M-null sequences of* b *are null.*

iii) *The topological dual of* E(τb) = τE *is* E^x.

iv) τb *has linear canonical bornology, which is separated if and only if* b *is separated.*

v) *The dual of the canonical bornology of* τb *is* E^x.

Proof: i)-iv) are routine. To see v), let F be the dual of the canonical bornology of τb. Being b finer than this bornology, $F \subset E^x$. Let's suppose now $u \epsilon E^x$, and let B be bounded in τb. If u(B) is unbounded, there is a sequence $(x_n)_n \subset B$ with $|ux_n| > n^2$. But we can take a subsequence $(x_{n_k})_k$ such that $(1/n_k \cdot x_{n_k})_k$ is bounded, and thus we are in contradiction. QED.

The following seems to be a natural question: If b is convex, must the canonical bornology of τb be convex? Arnold gives an affirmative answer in (1), but his proof is not valid, because it is based upon the following wrong fact: the convex hull of a subset B is contained in the balanced hull of B + B. Perrot in (18) shows that the canonical bornology of τb is linear if b is linear, and considers trivial the convexness if b is convex, assertion which can also be found in (9). We show here with a counterexample that the answer is, really, negative.

3.2. *EXAMPLE.* Let E be the space of Lebesgue p-integrable functions in [0,1] for $0 < p < 1$, with its usual topology. A base of neighbourhoods of zero is given by the dilations of:

$$U = \left\{ f \epsilon E : \int_0^1 |f|^p \, dm < 1 \right\}$$

The convex hull of U is E, being E' = 0, and U is bounded in E. Therefore, the bounded sets of E do not constitute a convex bornology. But we can take a base of bornology formed by the absolutely convex bounded subsets of E, which gives us a convex bornology b, strictly finer than the canonical bornology of E. The topology τb is precisely the topology of E (see (22), Chap. 2, Prop. 11), and, therefore, τb has nonconvex bornology.

Following B. Perrot in (18), we say that a linear bornology b is infratopological when b coincides with the canonical bornology of τb. If we denote by B the covariant functor which assigns to each quasilinear topology its canonical bornology, the identity of the definition above can be written $B\tau b = b$. It is easy to see that if E is a QTS with linear bornology, this one is infratopological. Then it follows:

3.3. *PROPOSITION. An* LBS E(b) *is infratopological if and only if there is an accesible quasilinear topology on* E *whose canonical bornology is* b.

We give now another characterization of infratopological bornologies. In (18), an LBS is called of type P when a set which is absorbed by all bornivorous sets is bounded. Obviously, an infratopological LBS is of type P. We prove here the converse.

3.4. *LEMMA. Let* E *be a separated* LBS, *and* F *be an* LBS *of type* P. *If* $f: E \to F$ *is linear and bounded on null sequences,* f *is bounded.*

Proof: If B is bounded and f(B) is unbounded, there is a balanced bornivorous set $U \subset F$, and a sequence $(x_n)_n \subset B$ with $f(x_n) \notin n^2 U$ for all n. Then $\left(\frac{1}{n} x_n\right)_n$ is M-null in E, and has unbounded image. QED.

3.5. *PROPOSITION. A separated* LBS *is of type* P *if and only if is infratopological.*

Proof: If E is of type P, it is of type b (i.e. a null sequence of τE is bounded in E. See (18) for more details) and M-convergences in E and $B\tau E$ are the same. Then the identity mapping $B\tau E \to E$ is bounded. QED.

In a similar context, we say that an accesible QTS is q-bornological, if it has linear canonical bornology, and its topology is, precisely, the M-closure topology relative to its canonical bornology. One can obtain easily:

3.6. *PROPOSITION. If* E *is a separated* LBS, τE *is a q-bornological* QTS.

4. STABILITY PROPERTIES

We are going to study now the stability of q-bornological QTS and infratopological LBS in passing to products, subspaces, direct sums, quotients and spaces of linear mappings. We see first:

4.1. *PROPOSITION. Let* E *be an* LBS, S $\subset$ E *a linear subspace with the induced bornology. We denote by* τ_1 *the topology induced in* S *by* τE *and by* τ_2 *the topology of* τS. *Then:*

i) τ_2 *is finer than* τ_1.

ii) *If* S *is M-closed in* E, $\tau_1 = \tau_2$.

iii) If E *has the M-closure property,* $\tau_1 = \tau_2$.

Proof: The first assertion is trivial. For the second one, it suffices to see that if A $\subset$ S is τ_2-closed, it is closed in the M-convergence of E. Finally, if E has the M-closure property, the τ_1-closedness of A $\subset$ S is equivalent to the coincidence of A with the intersection of S an the M-closure of A in E (see (9)), and this is the case when A is τ_2-closed. QED.

According to Webb ((23)), we call a topology C_1-sequential when every closure point of a set is the limit of a sequence contained in the set. Let's remember now that an LBS E has the M-closure property if and only if τE is C_1-sequential (see the second Chapter of (9)).

4.2. *COROLLARY. Let* E *be a q-bornological* QTS. *Then:*

i) *Every closed linear subspace of* E *is q-bornological.*

ii) *If* E *is* C_1*-sequential, every linear subspace of* E *is q-bornological.*

B. Perrot has proved in (18) that every separated C_1-sequential TVS is q-bornological. Therefore, if E is linear, the hypothesis of E being q-bornological is redundant in 4.2.ii). We see now that this is also the case when E is a QTS with sequentially continuous sum.

4.3. *LEMMA. Let* E *be a* QTS *with sequentially continuous sum,* C_1*-sequential. Then, every null sequence of* E *has an M-null subsequence.*

Proof: One can easily reproduce the proof given by Averbuck and Smolyanov in (2) for the linear case, with the obvious corrections. QED.

4.4. *PROPOSITION. Let* E *be an accesible* C_1*-sequential* QTS *with sequentially continuous sum. Then* E *is q-bornological.*

Proof: To prove the identity $E = B\tau E$, it suffices to see that both topologies have the same null sequences, being E C_1-sequential. If $(x_n)_n$ is null in E, every subsequence of $(x_n)_n$ has an M-null subsequence, and therefore is null in τBE. QED.

4.5. *EXAMPLE.* We cannot expect, for q-bornological QTS, beautiful results as those known for bornological locally convex spaces (LCS) (Mackey-Ulam theorem). The product of two q-bornological QTS can fail to be q-bornological, as we see next. Let E_1 be the space $\mathcal{D}(R)$ with its VN bornology, and E_2 be the space $\mathcal{D}'(R)$ with the corresponding equicontinuous bornology. The topologies $\beta(\mathcal{D}(R), \mathcal{D}'(R))$ and $\beta(\mathcal{D}'(R), \mathcal{D}(R))$ are Schwartz-Montel, and therefore have the Mackey-convergence property (see (7)). Then, M-convergent sequences in E_1 and E_2 coincide with convergent sequences in these topologies. T. Shirai and R. M. Dudley have shown, in (21) and (6) respectively, that these topologies do not coincide with those of τE_1 and τE_2. The M-convergent sequences in $E_1 \times E_2$ are those which converge coordinatewise, and, therefore, the bilinear mapping:

$$\begin{aligned} \mathcal{D}'(R) \times \mathcal{D}(R) &\longrightarrow C \\ (\psi, f) &\longrightarrow \langle \psi, f \rangle \end{aligned}$$

is continuous with respect to the topology of $\tau(E_1 \times E_2)$ ((19), III.XI, Th. 3). But this functional fails to be continuous with respect to $\tau E_1 \times \tau E_2$ (just follow the argument of (5), page 506). Now $\tau E_1 \times \tau E_2 \neq \tau(E_1 \times E_2)$, and these spaces have a non-q-bornological product.

4.6. *PROPOSITION. Let* E *and accesible* QTS *with sequentially continuous sum, and* S *a closed linear subspace of* E. *Then:*

$$\frac{\tau BE}{S} = \tau(BE/S).$$

Proof: Let p be the quotient mapping. As a mapping between BE and BE/S, p is bounded, and by virtue of the functorial properties of τ and B, we can pass to the quotient, obtaining the identity $\tau BE/S \to \tau(BE/S)$ as a continuous map. To see the continuity in the opposite sense, it suffices to see that every null sequence in $\tau(BE/S)$ is null in $\tau BE/S$ (the last is sequential), and this is a routine. QED.

4.7. *COROLLARY. Let* E *be a q-bornological* QTS, *and* S *a closed linear subspace of* E. *Then,* E/S *is q-bornological.*

The stability of infratopological LBS in the construction of initial bornologies follows directly from 2.3. Now, we can precise this a little more.

4.8. *PROPOSITION. Let* E *be a separated* LBS, *and* S *a linear subspace of* E. *The bornology induced by* $B\tau E$ *on* S *is the bornology of* $B\tau S$.

Proof: Let b_1 be the induced bornology, and b_2 the other. Of course b_2 is finer than b_1. An M-null sequence of b_1 is null in E and therefore b_2-bounded. Being b_1 infratopological, we have $b_1 = b_2$. QED.

The topology of τS can be strictly finer than the one induced by τE, but both topologies have the same canonical bornology. Nevertheless, τS is the q-bornological QTS associated to the topology induced by τE in S.

4.9. *PROPOSITION. Let* $(E_n)_n$ *be a sequence of separated* LBS. *Then:*

$$B\tau\left(\prod_{n=1}^{\infty} E_n\right) = \prod_{n=1}^{\infty} (B\tau E_n)$$

Proof: If $(E_i)_{i\in I}$ is an arbitrary family of separated LBS, the identity $B\tau(\prod_i E_i) \to \prod_i (B\tau E_i)$ is bounded. This follows from the boundedness of projections.

Now come to the countable case. Let A be bounded in $\prod_{n>0} (B\tau E_n)$. We must show that A is bounded in $B\tau(\prod_{n>0} E_n)$, i.e. that if $(x_m)_m$ is a sequence in A and $(t_m)_m$ is null in K, $(t_m x_m)_m$ has a subsequence which is bounded in $\prod_{n>0} E_n$. We take first a subsequence $(t_{m_j} x_{m_j})_j$ with bounded projection in E_1. Second, we can take another subsequence with bounded projection in E_2. We can proceed by induction, and then, by a Cantor diagonal process, obtain a subsequence whose projections on the factors E_n are all bounded. QED.

The preceding proof depends essentially upon the countability. It does not seem easy to find a proof for the general case, but the author does not know any counter-example. Finally, we examine direct sums.

4.10. *PROPOSITION. Let* $(E_i)_{i\in I}$ *be a family of separated* LBS. *Then:*

$$B\tau\left(\bigoplus_{i\in I} E_i\right) = \bigoplus_{i\in I} (B\tau E_i)$$

Proof: Let A be bounded in $\oplus (B\tau E_i)$. Then A is contained in a finite sum, and therefore bounded in $B\tau(\bigoplus_{i\in I} E_i)$. Conversely, if A is bounded in $B\tau(\bigoplus_{i\in I} E_i)$ and fails to be contained in a finite sum, A contains a sequence $(x_n)_n$ with the same property, and $({}^1/_n x_n)_n$ has a subsequence bounded in $\bigoplus_{i\in I} E_i$, which is contradictory. Then $A \subset \bigoplus_{j=1}^{n} E_{i_j}$, and, by 4.9., A is bounded in $\bigoplus_{i\in I} (B\tau E_i)$. QED.

4.11. *COROLLARY. The direct sum of an arbitrary family of separated infratopological* LBS *is infratopological.*

A similar result is not valid, as we shall see in another paper (25), for quotients.

Let E be an LBS and F a QTS over K. We denote by $L^x(E, F)$ the set of all linear mappings which are bounded on bounded sets of E. If BF is linear, $L^x(E,F)$ is a linear space. In (20), can be found a functorial approach to the study of this space when E is convex and F locally convex.

A sufficient condition for $L^x(E, F) \neq 0$ is $E^x \neq 0$ and $F \neq 0$. Indeed, we can identify $E^x \otimes F$ to a linear subspace of $L^x(E, F)$ in the usual way. Note that it is possible $E^x = 0$ and $L^x(E, F) \neq 0$, as can be seen in the case $E = F = L^p([0, 1]; m)$, $0 < p < 1$, with its VN bornology.

Suppose $L^x(E, F)$ nontrivial, and let L be a linear subspace. In order to define in L a topology "of uniform convergence on bounded subsets of E", we can think in considering, as in the linear case, the sets:

$$U(A, V) = \{f \in L: f(A) \subset V\},$$

where A, V are taken in a base of bornology of E, and in a base of NZ of F, respectively. But these sets do not seem to be manageable when F has nonlinear topology. We choose, then, another way. Let's suppose that F has sequentially continuous sum, a fact with some advantages: a) it covers the case when F is q-bornological, b) F has linear bornology, and c) $L^x(E, F)$ is a linear space.

If $(f_n)_n$ is a sequence in $L^x(E, F)$, we say that $(f_n)_n$ converges uniformly to zero on bounded sets of E (we use the word "uniformly" improperly, because do not suppose any uniformity defined on F) when for every bounded subset $A \subset E$ and every NZ $V \subset F$ there is an n_0 such that $n > n_0$ implies $f_n(A) \subset V$. We shall construct in $L^x(E, F)$ a quasilinear topology whose null sequences will be, precisely, the ones described above.

4.12. *PROPOSITION. Let* E, F *be in the conditions described above. A sequence* $(f_n)_n$ *converges to zero uniformly on bounded sets of* E *if and only for every bounded sequence* $(x_n)_n$ *of* E, $(f(x_n))_n$ *is null in* F.

Proof: It follows routinely from the definition.

As usual, the convergence to an element $f \in L^x(E, F)$ can be defined. We have now a convergence c in $L^x(E, F)$ whose properties are summarized in the following Proposition:

4.13. *PROPOSITION. In the conditions above,* c *is a linear convergence, which is separated if* F *is accesible. Furthermore,* c *is a topological convergence.*

Proof: The first part is easy to check. For the second, it suffices to use the well-known criterion of Kisynski (see (4), Theorem 2.1., or (10)).QED.

It is clear now, that the topology we are searching for is, precisely, the topologization of c. We call this topology, topology of bounded convergence in $L^x(E, F)$, and we denote by $L^x_\gamma(E, F)$ the corresponding QTS. Note that, if E and F are LCS, this topology induces on the space L(E, F) of continuous linear mappings a topology which is finer than the topology usually called "of bounded convergence". More precisely, it is the sequential topology (see (24)) associated to the usual one.

Now, we characterize the bounded sets of $L^x_\gamma(E, F)$.

4.14. *PROPOSITION. A subset* $H \subset L^x_\gamma(E, F)$ *is bounded in the topology of bounded convergence if and only if for every bounded subset* $A \subset E$, the set

$$H(A) = \bigcup_{h \in H} h(A)$$

is bounded in F.

Proof: We use 2.1. and the characterization of convergent sequences in $L^x_\gamma(E, F)$. If H is γ-bounded an $A \subset E$ is bounded, every sequence in H(A) has the form $(h_n(x_n))_n$, with $(h_n)_n \subset H$ and $(x_n)_n \subset A$. Being $(h_n(^1/_n\, x_n))_n$ null in F, $(h_n(x_n))_n$ is bounded. Conversely, if H(A) is bounded in F for all $A \subset E$ bounded, and $(h_n)_n \subset H$, for every bounded sequence $(x_n)_n$ of E, $(h_n(x_n))_n$ is bounded in F, and hence $(^1/_n\, h_n(x_n))_n$ is null. Then $(^1/_n\, h_n)_n$ converges to zero on bounded sets of E. QED.

Let E and G be LBS, and $L^x(E, G)$ the space of bounded linear mappings of E into G. Hogbé-Nlend considers in (8), in the case in which E, G are convex, the following "natural bornology": H is bounded when H(A) is bounded for $A \subset E$ bounded. This bornology appears also in (20), where convexity is not assumed (the functor Lcb of Chapter 1.4.), and in (22), where the naturally bounded sets are called equibounded. Provided with its natural bornology, we denote this space by $L^x_n(E, G)$. Now, 4.14. can be rewritten:

$$B(L^x_\gamma(E, F)) = L^x_n(E, BF).$$

Using this terminology, we can write now:

4.15. *COROLLARY. If* F *is an infratopological* LBS, *and* E *is an* LBS *with* $L^x(E, F) \neq 0$, *then* $L^x_n(E, F)$ *is infratopological.*

Proof: It follows from the identity $L^x_n(E, F) = B(L^x_\gamma(E, F))$. This result can also be proved directly, without constructing effectively a quasilinear topology whose canonical bornology is the natural bornology on $L^x(E, F)$. Indeed, let H be bounded in $B\tau(L^x_n(E, F))$ and $A \subset E$ be bounded. Being F infratopological, for H(A) to be bounded, the boundedness of its sequences is sufficient. Put $(x_n)_n \subset A$ and $(h_n)_n \subset H$. If $(t_n)_n$ is null in K, every subsequence of $(t_n h_n)_n$ admits a subsequence which is M-null in $L^x_n(E, F)$. If $(t_{n_j} h_{n_j})_j$ is such a subsequence, M-lim $t_{n_j} h_{n_j}(x_{n_j}) = 0$ in F. Hence $(t_n h_n(x_n))_n$ is null in F, and $(h_n(x_n))_n$ is bounded in F. QED.

We prove now the converse of 4.15. Let E, F be LBS and assume $E^x \neq 0$. If $u \in E^x$, $u \neq 0$, for every $y \in F$ we can indentify $u \otimes y \in E^x \otimes F$ with the map:

$$\begin{array}{ccc} E & \longrightarrow & F \\ x & \longrightarrow & (ux)y, \end{array}$$

and denote by $u \otimes F$ the linear subspace of $L^x(E, F)$ obtained by this procedure. Now:

4.16. *PROPOSITION. With the same terminology as above,* $u \otimes F$ *is an M-closed linear subspace of* $L^x_n(E, F)$.

Proof: It is pure routine. If wanted, it can be copied from the proof given in my paper (4) for the case of an equicontinuous bornology.

4.17. *PROPOSITION. Let* E *and* F *be* LBS, *with* $L^x(E, F) \neq 0$ *and* $E^x \neq 0$. *Then* $L^x_n(E, F)$ *is infratopological if and only if* F *is infratopological.*

5. ANOTHER WAY TO OBTAIN THE MACKEY-CLOSURE TOPOLOGY.

We close this paper with the study of certain questions related to reflexive CBS. If E is a reflexive CBS and ν is the natural topology on its dual (topology of convergence on bounded subsets of E in the usual sense), $(E^x_\nu)' = E$. This identity is a bornological one when the corresponding equicontinuous bornology is considered on the dual $(E^x_\nu)'$. We can look now at the topology ν^f (resp. ν^{lf}), i.e. the finest topology (resp. locally convex topology) which coincides with $\sigma(E, E^x)$ on every bounded subset of E.

5.1. *PROPOSITION. Let* E *be a reflexive* CBS *and* ν *the natural topology on* E^x. *Then the topology* ν^f *(resp.* ν^{lf}*) is finer than the topology of* τE *(resp.* TE).

Proof: For ν^f it is sufficient to prove that every M-null sequence $(x_n)_n$ of E is null in ν^f. We remark only that the set:

$$X = \{x_n : n \in N\} \cup \{0\}$$

is bounded in E, and, being $(x_n)_n$ $\sigma(E, E^x)$-null, converges to zero in ν^f.

For ν^{lf}, note that, being E^x_ν complete, the dual of $E(\nu^{lf})$ is E^x. Then ν^{lf} is compatible with the duality $< E, E^x >$, and therefore finer than the topology of TE. QED.

5.2. *PROPOSITION. Let* E *be a reflexive* CBS, *and* ν *the natural topology on* E^x. *Then:*

i) *If* E *is a Schwartz* CBS, *the topology of* τ E *is* ν^f.

ii) *If* E *is infra-Schwartz, the topology of* τ E *and* ν^f *have the same closed convex sets.*

Proof: Suppose E Schwartz, and A closed in τ E. If B $\subset$ E is bounded, absolutely convex and $\sigma(E, E^x)$-closed, there is a bounded absolutely convex subset D $\subset$ E such that B is compact in E_D. Then, B is closed in τ E, and, being A closed in τE, A $\cap$ E_D is closed in E_D, and therefore A $\cap$ B is compact in E_D. Finally, A $\cap$ B is weakly compact.

Now, let E be an infra-Schwartz CBS, and A closed and convex in τE. If B is bounded in E, absolutely convex and $\sigma(E, E^x)$-closed, there is an absolutely convex bounded subset D such that B is weakly compact in E_D. Being A $\cap$ E_D closed in E^x, it is weakly closed, and A $\cap$ B is weakly compact in E_D, and therefore $\sigma(E, E^x)$-compact. QED.

NOTE: If E has a countable base, E^x_ν is Fréchet. Then, the Krein-Smulian theorem applied to ii) implies that the closed convex sets of ν^f coincide with the $\sigma(E, E^x)$-closed ones, and we obtain thus the Proposition VII.5.2. of (9) as a particular case of 5.2.ii). In this situation, E^x_ν is ultracomplete in the sense of (17). The first part of 5.2. has been proved by Moscatelli in (17) in an undirect way.

5.3. *EXAMPLES.* If E has its maximal bornology, E = $E(\nu^f) \neq$ TE = $E(\nu^{lf})$, when E has uncountable dimension.

If E is a reflexive infinite dimensional Banach space, its VN bornology is clearly infratopological, but not Schwartz. Now, ν is the norm topology on E' and E = TE is the primitive space. ν^{lf} is the topology of convergence on compact sets of E', and it does not coincide with the norm topology, neither does ν^f. We see now that the assumption on the convexness of A is essential in 5.2.ii).

REFERENCES

(1) B. H. ARNOLD. Topologies defined by bounded sets. Duke Math. J. 18 (1951) 631-642.

(2) V. I. AVERBUCK and O. G. SMOLYANOV. The various definitions of the derivative in linear topological spaces. Russian Math. Surveys 23 (1968), 67-113.

(3) H. BUCHWALTER. Topologies, bornologies et compactologies, Thèse, Fac. Sci. Lyon (1968).

(4) M. A. CANELA. Bornología equicontinua en un espacio de aplicaciones lineales continuas. Rev. Univ. Santander 2, Parte 1 (1979), 111-121.

(5) R. M. DUDLEY. On sequential convergence. Trans. Amer. Math. Soc. 112 (1964), 483-507.

(6) R. M. DUDLEY. Convergence of sequences of distributions. Proc. Amer. Math. Soc. 27 (1971), 531-534.

(7) A. GROTHENDIECK. Sur les espaces (F) et (DF). Sum. Bras. Math. 3 (1952), 57-123.

(8) H. HOGBE-NLEND. Complétion, tenseurs et nucléarité en bornologie. J. Math. pures et appl. 49 (1970), 193-288.

(9) H. HOGBE-NLEND. Théorie des bornologies et applications. Springer Lecture Notes 213 (1971).

(10) H. HOGBE-NLEND. Bornology and Functional Analysis. North-Holland (1978).

(11) J. KISYNSKI. Convergence du type L. Colloq. Math. 7 (1959-60), 205-211.

(12) V. L. KLEE, Jr. Convex sets in linear spaces I. Duke Math. J. 18 (1951), 443-466.

(13) V. L. KLEE, Jr. Convex sets in linear spaces III. Duke Math. J. 20 (1953), 105-111.

(14) Y. KOMURA. Some examples in linear topological spaces. Math. Ann. 153 (1964), 150-162.

(15) G. KOTHE. Topological Vector Spaces I. Springer Verlag (1969).

(16) J. MIKUSINSKI. Convergence and locally convex spaces. Bull. Acad. Pol. des Sci. 23, no 11, (1975), 1171-1173.

(17) V. MOSCATELLI. Polar duality and separation properties in bornological spaces. Bolletino U. M. I. 4, 8 (1973), 79-90.

(18) B. PERROT. Sur la Mackey-Convergence. Thèse 3ème Cycle, Univ. Bordeaux (1970).

(19) L. SCHWARTZ. Théorie des distributions II. Hermann (1959).

(20) SEMINAIRE BANACH. Springer lecture Notes 277 (1972).

(21) T. SHIRAI. Sur les espaces de L. Schwartz. Proc. Japan Acad. 35 (1959), 31-36.

(22) L. WAELBROECK. Topological Vector Spaces and Algebras. Springer Lecture Notes 230 (1971).

(23) J. H. WEBB. Convergence in locally convex spaces. Proc. Camb. Phil. Soc. 64 (1968), 341-364.

(24) A. WILANSKY. Topics in Functional Analysis. Springer Lecture Notes 45 (1967).

(25) M. A. CANELA. Some classes of linear bornological spaces. Proc. Royal Soc. Edinb. 90 (1981), 155-161.

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