ON THE THREE-SPACE-PROBLEM FOR TOPOLOGICAL VECTOR SPACES

by

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Dedicated to Professor Gottfried Köthe on the occasion of his 75^{th} birthday.

In this article we will investigate the following problem, sometimes called a three-space-problem:

Let X be a topological vector space (or a locally convex space) and $L \subset X$ a linear subspace such that L and the corresponding quotient space X/L possess a certain property P; does X also possess P?

For several properties P this problem has an easy solution even in a more general setting. For instance, let X be a topological space and R an equivalence relation on X such that all fibres $R[x]$ $(x \in X)$ and the quotient space X/R are T_1-spaces. Then also X is a T_1-space. The same statement holds for the properties separable, connected, or totally disconnected (see [25], 12.20).

Furthermore, if X is a uniform space and R an equivalence relation on X which is in a certain sense compatible with the uniform structure (see [25], 4.10), then the completeness (resp. precompactness) of all fibres $R[x]$ $(x \in X)$ and of X/R implies the completeness (resp. precompactness) of X (see [25], 12.1, 12.5). Thus a topological group X is complete (resp. precompact) if it contains a subgroup G such that G and X/G are complete (resp. precompact). The same is true for «locally compact» (see [25], 12.19).

On the other hand, Kalton has shown in [18] using very delicate methods that a topological vector space X need not be locally convex if it contains a straight line L such that X/L is locally convex. Another example of that kind has been given by Ribe [22].

In the present article we are going to investigate the above mentioned problem for several other well-established properties of topological vector spaces or locally convex spaces, such as certain completeness and barrelledness properties, semireflexivity, DF- and

Schwartz space, local boundedness and quasinormability. — We restrict ourselves to the case of linear spaces although some of our results allow a natural extension to topological groups.

0. Notations and terminology

For a topological space Z and $a \in Z$ we denote by $\mathfrak{U}_a(Z)$ the filter of all neighbourhoods of a in Z.

K always denotes one of the fields **R** or **C**. All linear spaces are supposed **K**-linear. For a subset A of a linear space X let ΓA denote its absolutely convex hull and $[A]$ its linear span. We do not tacitly assume topological vector spaces to be Hausdorff.

Let X, Y be topological vector spaces. We call a linear map $f: X \to Y$ almost continuous, if for every $U \in \mathfrak{U}_0(Y)$ the set $\overline{\overset{-1}{f}(U)}$ (the bar denoting the closure) belongs to $\mathfrak{U}_0(X)$.

Furthermore, we call a sequence $(x_n)_{n\in\mathbf{N}}$ in a topological vector space X a local null sequence (resp. a local Cauchy sequence) if there exists a normed space Y, a continuous linear map $g: Y \to X$, and a null sequence (resp. a Cauchy sequence) $(y_n)_{n\in\mathbf{N}}$ in Y such that $g(y_n) = x_n$ for all $n \in \mathbf{N}$. We call a topological vector space X locally complete if every local Cauchy sequence in X converges. A thorough investigation of these notions can be found in P. Dierolf [8], where the terminology «au sens de Mackey» instead of «local» is used.

Let X be a topological vector space. A sequence $(T_n)_{n\in\mathbf{N}}$ of closed balanced and absorbing subsets of X such that $T_{n+1} + T_{n+1} \subset T_n$ for all $n \in \mathbf{N}$, is called an ultrabarrel in X. An ultrabarrel $(T_n)_{n\in\mathbf{N}}$ in X is called bornivorous if every T_n is bornivorous ($n \in \mathbf{N}$). X is called ultrabarrelled (resp. quasi-ultrabarrelled) if for every ultrabarrel (resp. bornivorous ultrabarrel) $(T_n)_{n\in\mathbf{N}}$ in X the set T_1 is a zero-nbhd. in X (see W. Robertson [23], p. 249, and Iyahen [16], p. 293 and p. 300). — According to Iyahen [16], p. 298, a topological vector space X is called ultrabornological if for every sequence $(T_n)_{n\in\mathbf{N}}$ of balanced bornivorous subsets of X such that $T_{n+1} + T_{n+1} \subset T_n$ ($n \in \mathbf{N}$) the set T_1 belongs to $\mathfrak{U}_0(X)$. — According to Bourbaki [3], Ch. III, § 3, exercice 11, a locally convex space X is called ultrabornologique if every convex subset of X which absorbs all absolutely convex, sequentially complete, and bounded subsets of X, belongs to $\mathfrak{U}_0(X)$.

1. Completeness properties

1.1 *Lemma.* Let X, Y be topological vector spaces and let $f: X \to Y$ be a linear and open map, such that $N := \ker f$ is complete. Then f satisfies the filter condition of *W.* Robertson [23], p. 243, i.e., for every Cauchy filter $\mathfrak{F}$ on X such that $f(\mathfrak{F})$ converges to a point $y \in Y$ there is $x \in \overset{-1}{f}(y)$ such that $\mathfrak{F}$ converges to x in X.

Proof. $\mathfrak{G} := \{F + U : F \in \mathfrak{F}, U \in \mathfrak{U}_0(X)\}$ is a Cauchy filter on X. For every $U \in \mathfrak{U}_0(X)$ there is $F \in \mathfrak{F}$ such that $f(F) \subset f(U) + y$ whence $F \subset U + \overset{-1}{f}(y)$. Consequently, $\mathfrak{G} | \overset{-1}{f}(y) := \{G \cap \overset{-1}{f}(y) : G \in \mathfrak{G}\}$ is a filter, hence a Cauchy filter on $\overset{-1}{f}(y)$. By the completeness of $\overset{-1}{f}(y)$ there is $x \in \overset{-1}{f}(y)$ such that $\mathfrak{G} | \overset{-1}{f}(y)$ converges to x. Thus x is an adherent point of $\mathfrak{G}$, which implies that $\mathfrak{G}$ and hence $\mathfrak{F}$ converge to x.

As an immediate consequence we obtain

1.2 *Proposition.* Let X be a topological vector space, $L \subset X$ a complete linear subspace, and let $q : X \to X/L$ denote the quotient map. Assume that a subset $A \subset X$ satisfies one of the following two conditions

(a) A is closed (resp. sequentially closed) in X and every Cauchy filter (resp. Cauchy sequence) in $q(A)$ converges in X/L;

(b) A is closed (resp. sequentially closed) in $A + L$ and $q(A)$ is complete (resp. sequentially complete).

Then A is a complete (resp. sequentially complete) subset of X.

1.3 *Proposition* (cf. Bourbaki [2], Ch. III, § 3, exercice 9; cf. also [25], 12.2).

Let X be a topological vector space and $L \subset X$ a complete linear subspace. If X/L has one of the properties:
complete, quasicomplete, sequentially complete, locally complete, convex compactness property [1]),

then X has the same property.

1) i.e., the closed absolutely convex hull of every compact subset is complete.

Proof. For the first four properties we observe that the quotient map $q : X \to X/L$ maps Cauchy filters (bounded sets, Cauchy sequences, local Cauchy sequences) onto Cauchy filters (bounded sets, ...) and apply 1.1.

For the convex compactness property let $K \subset X$ be compact. Then $\Gamma\overline{q(K)}$ is complete; thus, by 1.1, every Cauchy filter on ΓK converges in X which implies the completeness of $\overline{\Gamma K}$ by Bourbaki [2], Ch. II, § 3, Prop. 9.

In order to give an example which shows that the completeness assumption for L in 1.3 is essential, we need

1.4 *Lemma*. We consider two triples (X_1, H_1, a_1), (X_2, H_2, a_2), where X_i is a Hausdorff topological vector space, $H_i \subset X_i$ a dense hyperplane, $a_i \in X_i \setminus H_i$, and $q_i : X_i \to X_i/[a_i]$ denotes the quotient map $(i = 1,2)$.

In the topological product $X_1 \times X_2$ we define the one dimensional linear subspace A generated by $a := (a_1, a_2)$, provide $Y := H_1 \times H_2 + A \subset X_1 \times X_2$ with the relative topology and $X := Y/A$ with its quotient topology. Then clearly the isomorphism $H_1 \times H_2 \to X$ induced by the quotient map $q : Y \to Y/A$, is continuous but not open. Moreover the following statements are valid.

(i) $j_1 : H_1 \to X, x \mapsto q((x, 0))$, is a topological isomorphism onto its image $L := j_1(H_1)$.

(ii) There is a linear surjection $p_2 : X \to X_2/[a_2]$ such that $p_2 \circ q = q_2 \circ pr_2|Y$ (where $pr_2 : X_1 \times X_2 \to X_2$ denotes the canonical projection). p_2 is continuous and open, and $\ker p_2 = L$.

Consequently, L is a closed linear subspace of X topologically isomorphic to H_1 such that X/L is topologically isomorphic to $X_2/[a_2]$.

By symmetry the corresponding maps $j_2 : H_2 \to X$ and $p_1 : X \to X_1/[a_1]$ behave analogously.

Proof.

(i) The restriction $q|(H_1 \times \{0\} + A)$ induces a topological isomorphism $\tilde{q}$ from $(H_1 \times \{0\} + A)/A$ onto L; as $H_1 \times \{0\}$ is a closed hyperplane in $H_1 \times \{0\} + A$ we obtain that $\tilde{q}|(H_1 \times \{0\}) : H_1 \times \{0\} \to L$ is a topological isomorphism. This proves (i).

(ii) As $\ker q = A \subset \ker(q_2 \circ pr_2)$, p_2 is well-defined; moreover $\ker p_2 = q(\ker(q_2 \circ pr_2|Y)) = q(H_1 \times \{0\} + A) = L$.

The relation $p_2 \circ q = q_2 \circ pr_2|Y$ immediately implies the continuity of p_2. Moreover, in order to show that p_2 is open, it suffices to prove that $pr_2|Y : Y \to X_2$ is open. Let $U_i \in \mathfrak{U}_0(X_i)$, $i = 1,2$, and let $x_2 \in U_2$. Then there are $z_2 \in H_2$ and $\lambda \in \mathbf{K}$ such that $x_2 = z_2 + \lambda a_2$. As H_1 is dense in X_1 there is $z_1 \in H_1 \cap (U_1 - \lambda a_1)$.

Now $y := (z_1, z_2) + \lambda a \in Y \cap (U_1 \times U_2)$ and $pr_2(y) = x_2$. Thus $U_2 \subset pr_2(Y \cap (U_1 \times U_2))$ which proves that $pr_2|Y$ is open. Finally, L is closed in X, as $X_2/[a_2]$ is Hausdorff.

1.5. *Example.* Let X_1 be a Hausdorff locally convex space containing the incomplete Montel space H_1 constructed by I. Amemiya, Y. Kōmura [1], § 2, as a dense hyperplane and choose $a_1 \in X_1 \setminus H_1$. Moreover let X_2 denote the product space $\omega := \mathbf{K}^{\mathbf{N}}$ (see Köthe [19], p. 151), and choose a dense hyperplane $H_2 \subset X_2$ and $a_2 \in X_2 \setminus H_2$ arbitrarily.

We apply Lemma 1.4 to the triples (X_i, H_i, a_i) $(i = 1,2)$ and obtain a Hausdorff locally convex space X containing a closed linear subspace L with the following properties:

(a) L is a Montel space, hence quasicomplete and reflexive;

(b) X/L is topologically isomorphic to $\omega/[a_2]$, hence to ω and is thus a Fréchet Montel space;

(c) X is not even locally complete, as it contains the closed metrizable and incomplete linear subspace $j_2(H_2)$ (see P. Dierolf [8], Thm. 1).

2. Barrelled and bornological spaces

We begin this section with a simple proof of a result of N.J. Vilenkin (see Graev [12], p. 17, Thm. 6, and Hewitt, Ross [14], p. 47, (5.38) (e); cf. also [25], 12.12 and 12.13). The method developed by us in this proof will turn out to be an essential tool in the rest of this section.

2.1. *Proposition.* Let X be a topological vector space, $L \subset X$ a pseudometrizable linear subspace, and let $q : X \to X/L$ denote the quotient map. Then for every subset $A \subset X$ and every $x \in A$ such that $\mathfrak{U}_{q(x)}(q(A))$ has a countable basis, also the nbhd.-filter $\mathfrak{U}_x(A)$ has a countable basis.

In particular, if X/L is pseudometrizable, then also X is pseudometrizable.

Proof. Let $A \subset X$, $x \in A$, and suppose that $\mathfrak{U}_{q(x)}(q(A))$ has a countable basis. Then we find a sequence $(V_n)_{n \in \mathbf{N}}$ in $\mathfrak{U}_0(X)$ such that

a) $V_{n+1} - V_{n+1} \subset V_n$ $(n \in \mathbf{N})$;
b) $(L \cap V_n)_{n \in \mathbf{N}}$ is a basis of $\mathfrak{U}_0(L)$;
c) $(q(A) \cap q(x + V_n))_{n \in \mathbf{N}}$ is a basis of $\mathfrak{U}_{q(x)}(q(A))$.

Let $U \in \mathfrak{U}_0(X)$ be given and choose $W \in \mathfrak{U}_0(X)$ satisfying $W + W \subset U$. There is $n \in \mathbf{N}$ such that $L \cap V_n \subset W$, and there is $m \in \mathbf{N}$, $m > n$, such that $q(A) \cap q(x + V_m) \subset q(x + W \cap V_{n+1})$. Then $A \cap (x + V_m) \subset$ $\subset x + W \cap V_{n+1} + L$ whence

$A \cap (x + V_m) \subset$[2] $x + W \cap V_{n+1} + L \cap (V_m - V_{n+1}) \subset x + W + L \cap (V_{n+1} - V_{n+1}) \subset x + W + L \cap V_n \subset x + W + W \subset x + U$. Thus $(A \cap (x + V_m))_{m \in \mathbf{N}}$ is a basis of $\mathfrak{U}_x(A)$ which is countable.

Before we start with the main subject of this section we insert some auxiliary statements about the «lifting of local null sequences».

2.2 *Remark.* Let X, Y be topological vector spaces and $f: X \to Y$ a linear map. If f satisfies the following condition

(*) For every local null sequence $(y_n)_{n \in \mathbf{N}}$ in Y there is a bounded set $B \subset X$ such that $\{y_n : n \in \mathbf{N}\} \subset f(B)$

then for every bornivorous set T in X the set $f(T)$ absorbs every local null sequence in Y and is hence bornivorous in Y. Analogously, if f satisfies

$(\overline{*})$ For every local null sequence $(y_n)_{n \in \mathbf{N}}$ in Y there is a bounded set $B \subset X$ such that $\{y_n : n \in \mathbf{N}\} \subset \overline{f(B)}$

then for every bornivorous set T in X the set $\overline{f(T)}$ is bornivorous in Y.

2.3 *Lemma.* Let X, Y be topological vector spaces, let $f: X \to Y$ be a linear open map, and assume that $N := \ker f$ is pseudometri-

2) If $v \in V_m$ is such that $x + v \in A$, then there are $w \in W \cap V_{n+1}$, $y \in L$ such that $x + v = x + w + y$, whence $y = v - w \in V_m - W \cap V_{n+1}$.

zable. Then for every local null sequence $(y_n)_{n\in\mathbf{N}}$ in Y there exists a local null sequence $(x_n)_{n\in\mathbf{N}}$ in X such that $f(x_n)=y_n$ for all $n\in\mathbf{N}$. — In particular, f satisfies condition (*) and hence condition ($\bar{*}$).

Proof. Let $(V_n)_{n\in\mathbf{N}}$ be a sequence in $\mathfrak{U}_0(X)$ such that $(N\cap V_n)_{n\in\mathbf{N}}$ is a basis of $\mathfrak{U}_0(N)$. We may assume that $V_{n+1}+V_{n+1}\subset V_n$ for all $n\in\mathbf{N}$. Let $(y_n)_{n\in\mathbf{N}}$ be a local null sequence in Y. Then there exists a sequence $(\alpha_n)_{n\in\mathbf{N}}$ of positive reals converging to ∞ such that the sequence $(\alpha_n y_n)_{n\in\mathbf{N}}$ converges to zero in Y. We may choose a strictly increasing map $h:\mathbf{N}\to\mathbf{N}$ such that $\alpha_k y_k\in f(V_n)$ for all $n\in\mathbf{N}$ and $k\geq h(n)$. Then, for every $n\in\mathbf{N}$ and every $k\in\mathbf{N}$ such that $h(n)\leq k<h(n+1)$, we fix $x_k\in\overset{-1}{f}(y_k)$ satisfying $\alpha_k x_k\in V_n$.

It remains to show that the sequence $(\alpha_k x_k)_{k\geq h(1)}$ converges to zero in X. Let $U\in\mathfrak{U}_0(X)$. Then there is $j\in\mathbf{N}$ such that $N\cap V_j\subset U$, and there is $k>j$ such that $\alpha_m y_m\in f(U\cap V_{j+1})$ for all $m\geq k$. Let $n_0\in\mathbf{N}$ satisfy $h(n_0)\geq k$. Then for every $m\geq h(n_0)$ one has $\alpha_m x_m\in V_{h(n_0)}\cap(U\cap V_{j+1}+N)\subset V_{j+1}\cap(U\cap V_{j+1}+N)$ whence $\alpha_m x_m\in U+N\cap(V_{j+1}+V_{j+1})\subset U+N\cap V_j\subset U+U$.

2.4 *Proposition.* Let X, Y be topological vector spaces, let $f:X\to Y$ be a linear open and almost continuous map, and put $N:=\ker f$.

(a) If N and Y are ultrabarrelled[3], then also X is ultrabarrelled.

(b) If N is quasi-ultrabarrelled[4] and Y is ultrabarrelled, then X is quasi-ultrabarrelled.

(c) If N and Y are quasi-ultrabarrelled and if f satisfies condition (*) in 2.2, then also X is quasi-ultrabarrelled.

Proof. Let $(T_n)_{n\in\mathbf{N}}$ be an ultrabarrel or, in the cases (*b*), (*c*), a bornivorous ultrabarrel in X. Then, in all three cases, $N\cap T_n$ belongs to $\mathfrak{U}_0(N)$ for all $n\in\mathbf{N}$. Thus we may choose a sequence $(U_n)_{n\in\mathbf{N}}$ of balanced zero-nbhds. in X such that $U_{n+1}+U_{n+1}\subset U_n$ and $N\cap(U_n+U_n+U_n)\subset T_n$ for all $n\in\mathbf{N}$. Since f is surjective, the sequence $(\overline{f(T_n\cap U_n)})_{n\in\mathbf{N}}$ is an ultrabarrel in Y.

Moreover, in case (*c*), f satisfies (*) whence the ultrabarrel $(\overline{f(T_n\cap U_n)})_{n\in\mathbf{N}}$ is bornivorous in Y. Consequently, in any of the three cases, $\overline{f(T_n\cap U_n)}$ belongs to $\mathfrak{U}_0(Y)$ for every $n\in\mathbf{N}$. Since f is open,

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3) In the sense of W. Robertson [23], p. 249.

4) In the sense of Iyahen [16], p. 300.

one has $\overline{f(T_2 \cap U_2)} \subset f(\overline{T_2 \cap U_2 + N})$; thus, by the almost continuity of f, the set $V := U_2 \cap \overline{T_2 \cap U_2 + N}$ belongs to $\mathfrak{U}_0(X)$. From $V \subset U_2 \cap \bigcap_{W \in \mathfrak{U}_0(X)} (W \cap U_2 + T_2 \cap U_2 + N)$ we obtain that $V \subset U_2 \cap \bigcap_{W \in \mathfrak{U}_0(X)} (W \cap U_2 + T_2 \cap U_2 + N \cap (U_2 + U_2 + U_2)) \subset \overline{T_2 \cap U_2} + T_2 \subset T_1$. Thus $T_1 \in \mathfrak{U}_0(X)$ which completes the proof.

A straightforward «locally convex simplification» of the above proof yields.

2.5 *Proposition.* Let X, Y be locally convex spaces, let $f: X \to Y$ be a linear open and almost continuous map, and put $N := \ker f$.

(a) If N and Y are barrelled, then also X is barrelled.
(b) If N is quasibarrelled and Y is barrelled, then X is quasibarrelled.
(c) If N and Y are quasibarrelled and if f satisfies condition ($\bar{*}$) in 2.2, then also X is quasibarrelled.

From 2.4 and 2.5 we obtain with the help of 2.3

2.6 *Theorem.* Let X be a locally convex space (resp. a topological vector space) and let $L \subset X$ be a linear subspace.

(a) If L and X/L are barrelled (resp. ultrabarrelled), then X is barrelled (resp. ultrabarrelled).
(b) If L is quasibarrelled (resp. quasi-ultrabarrelled) and X/L is barrelled (resp. ultrabarrelled), then X is quasibarrelled (resp. quasi-ultrabarrelled).
(c) If L is pseudometrizable and X/L is quasibarrelled (resp. quasi-ultrabarrelled), then X is quasibarrelled (resp. quasi-ultrabarrelled).

Next we prove an analogue to 2.5 for the properties «countably barrelled» and «countably quasibarrelled» introduced by Husain in [15], which will have an application to DF-spaces in section 3. [A locally convex space X is called countably (quasi)barrelled if for every sequence $(U_n)_{n \in \mathbf{N}}$ of closed and absolutely convex zero-nbhds. in X the intersection $T := \bigcap_{n \in \mathbf{N}} U_n$ belongs to $\mathfrak{U}_0(X)$ whenever T is absorbing (resp. bornivorous).] We mention that the above two properties coincide with «infrabarrelled» and «infraevaluable», respectively, introduced by De Wilde, Houet in [7], p. 257.

2.7 *Proposition.* Let X, Y be locally convex spaces, let $f: X \to Y$ be a linear open and almost continuous map, and put $N := \ker f$.

(a) If N and Y are countably barrelled, then also X is countably barrelled.

(b) If N is countably quasibarrelled and Y is countably barrelled, then X is countably quasibarrelled.

(c) If N and Y are countably quasibarrelled and if f satisfies condition (*) in 2.2, then also X is countably quasibarrelled.

PROOF. Let $(U_n)_{n\in\mathbf{N}}$ be a sequence of closed and absolutely convex zero-nbhds. in X, whose intersection $T := \bigcap_{n\in\mathbf{N}} U_n$ is absorbing or, in the cases (b), (c), bornivorous in X.

Then, in all three cases, $N \cap T \in \mathfrak{U}_0(N)$, and we find $U = \Gamma U \in \mathfrak{U}_0(X)$ satisfying $N \cap 3U \subset T$.

As f is open, the set $f(U \cap U_n)$ belongs to $\mathfrak{U}_0(Y)$ for every $n \in \mathbf{N}$. Moreover, $\overline{f(U \cap T)} \subset \bigcap_{n\in\mathbf{N}} \overline{f(U \cap U_n)} \subset \bigcap_{n\in\mathbf{N}} 2f(U \cap U_n)$. $f(U \cap T)$ is clearly absorbing; in case (c), f satisfies (*) whence the barrel $\overline{f(U \cap T)}$ is even bornivorous in Y. Consequently, in any of the three cases, $\bigcap_{n\in\mathbf{N}} \overline{f(U \cap U_n)}$ and hence $2 \bigcap_{n\in\mathbf{N}} f(U \cap U_n)$ belong to $\mathfrak{U}_0(Y)$. As f is almost continuous, $V := U \cap \bigcap_{n\in\mathbf{N}} \overline{(U \cap U_n + N)}$ is a zero-nbhd. in X. Now, for every $n \in \mathbf{N}$, one has $V \subset U \cap \bigcap_{W\in\mathfrak{U}_0(X)} (U \cap W + U \cap U_n + N)$ whence $V \subset U \cap \bigcap_{W\in\mathfrak{U}_0(X)} (U \cap W + U \cap U_n + N \cap 3U) \subset \overline{U \cap U_n} + T \subset 2U_n$. Thus $V \subset \bigcap_{n\in\mathbf{N}} 2U_n = 2T$, which proves that $T \in \mathfrak{U}_0(X)$.

2.8 *Remarks.*

(*a*) Given an infinite cardinal number c, let us call a locally convex space X c-(quasi)barrelled if for every family $(U_\iota)_{\iota\in I}$ in $\mathfrak{U}_0(X)$ such that $U_\iota = \Gamma\overline{U_\iota}$ $(\iota \in I)$ and card (I) $\leq c$, the intersection $T := \bigcap_{\iota\in I} U_\iota$ belongs to $\mathfrak{U}_0(X)$ whenever T is absorbing (resp. bornivorous) in X. The proof of 2.7 shows that all the statements in 2.7 remain valid if «countably (quasi)barrelled» is replaced by «c-(quasi)barrelled» for a fixed infinite cardinal number c.

(b) Proposition 2.7 has an (obvious) analogue for topological vector spaces X, Y and the properties «countably (quasi-)ultrabarrelled» introduced by Iyahen in [17], p. 610, which is proved by combining the methods of the proofs of 2.4 and 2.7. We do not explicitly formulate this analogue.

Our next aim is to give an example which shows that the «additional» assumptions of (b) and (c) in 2.5 to 2.7 are essential. To do this, we need a lemma whose simple proof is omitted (cf. also [10], (2)).

2.9 *Lemma.* Let $(X,\mathfrak{T})$ be a topological vector space, let $L \subset X$ be a linear subspace, and let $\mathfrak{S}$ be a linear topology on X/L which is finer than the quotient topology $\mathfrak{T}/L$. Let $\mathfrak{Z}$ denote the initial topology on X w.r. to the identity map id: $X \to (X, \mathfrak{T})$ and the quotient map $q : X \to (X/L, \mathfrak{S})$. Then the linear topology $\mathfrak{Z}$ satisfies $\mathfrak{Z} \supset \mathfrak{T}$, the relative topologies $\mathfrak{T}|L$ and $\mathfrak{Z}|L$ coincide, and the quotient topology $\mathfrak{Z}/L$ on X/L is equal to $\mathfrak{S}$.

If $\mathfrak{T}$ and $\mathfrak{S}$ are locally convex, then also $\mathfrak{Z}$ is locally convex.

2.10 *Example.* Let $(X, \mathfrak{T})$ be a barrelled Hausdorff locally convex space, containing a dense linear subspace L of countably infinite codimension, such that every bounded set in $(X, \mathfrak{T})$ has finite dimensional linear span. (See for example, I. Amemiya, Y. Kōmura [1], § 2.) Then L with its relative topology $\mathfrak{T}|L$ is barrelled by Valdivia [27], Thm. 3. Thus $(L, \mathfrak{T}|L)$ is even a Montel space.

Now fix an arbitrary norm topology $\mathfrak{S}$ on X/L and provide X with the initial topology $\mathfrak{Z}$ w.r. to id: $X \to (X, \mathfrak{T})$ and the quotient map $q : X \to (X/L, \mathfrak{S})$. By 2.9, $(X,\mathfrak{Z})$ is a Hausdorff locally convex space satisfying $\mathfrak{Z}|L = \mathfrak{T}|L$ and $\mathfrak{Z}/L = \mathfrak{S}$. From this we obtain

(a) $(L,\mathfrak{Z}|L)$ is a Montel space;
(b) $(X/L,\mathfrak{Z}/L)$ is normable, hence quasibarrelled;
(c) $(X, \mathfrak{Z})$ is not even countably quasibarrelled. In fact, as $(X/L, \mathfrak{Z}/L)$ is not countably barrelled by De Wilde, Houet [7], Cor. 3 (recall that $\dim(X/L)$ is countably infinite and that $\mathfrak{S}$ is not the finest locally convex topology on X/L), also $(X, \mathfrak{Z})$ is not countably barrelled. As all bounded sets in $(X, \mathfrak{T})$ and hence in $(X, \mathfrak{Z})$ are finite dimensional, the last statement implies that $(X, \mathfrak{Z})$ is not countably quasibarrelled.

Before we turn to bornological spaces, we insert the following statements about weak and Mackey topologies.

2.11 *Proposition.* Let X, Y be locally convex spaces, let $f : X \to Y$ be a linear open and almost continuous map, and put $N := \ker f$.

(a) If N and Y carry weak topologies, then also X carries a weak topology.

(b) If N and Y are Mackey spaces, then also X is a Mackey space.

PROOF. Let $q : X \to X/N$ denote the quotient map. As f is surjective, there exists an isomorphism j from X/N onto Y such that $f = j \circ q$. It is easy to see that j is open and almost continuous.

If Y carries a weak topology, then also X/N $(= j^{-1}(Y))$ carries a weak topology by Bourbaki [3], Ch. II, § 6, exercice 1. Next suppose that Y is a Mackey space and let $U \subset X/N$ be a zero-nbhd. for the Mackey topology of X/N. We may assume that U is closed in the original topology of X/N. Since j is open, $j(U)$ belongs to $\mathfrak{U}_0(Y)$, and since j is almost continuous we obtain that $U = \overline{j^{-1}(\overline{j(U)})}$ is a zero-nbhd. for the original topology of X/N. Thus X/N is a Mackey space.

Let $\mathfrak{T}$ denote the original topology of X, σ its weak topology, and τ its Mackey topology. It follows from Köthe [19], p. 276 (1) and p. 277 (3), that the relative topology $\sigma|N$ on N and the quotient topology σ/N on X/N coincide with the weak topologies corresponding to $\mathfrak{T}|N$ and $\mathfrak{T}/N$, respectively. From Köthe [19], p. 277 (3), we deduce that τ/N equals the Mackey topology corresponding to $\mathfrak{T}/N$. Finally, if $(N, \mathfrak{T}|N)$ is a Mackey space, then $\tau|N = \mathfrak{T}|N$ holds by Köthe [19], p. 277, line 23 to 25.

Now the statements (a) and (b) follow from a special case of Lemma 1 in [11]:

Two comparable linear topologies $\mathfrak{S}$ and $\mathfrak{T}$ on a linear space X coincide if X contains a linear subspace L such that $\mathfrak{S}|L = \mathfrak{T}|L$ and $\mathfrak{S}/L = \mathfrak{T}/L$.

2.12 *Proposition.* Let X, Y be topological vector spaces and let $f : X \to Y$ be a continuous linear map satisfying condition (*) in 2.2.

If $N := \ker f$ and Y are ultrabornological[5], then also X is ultrabornological.

5) In the sense of Iyahen [16], p. 298.

PROOF. Let $(T_n)_{n\in\mathbf{N}}$ be a sequence of balanced bornivorous sets in X such that $T_{n+1}+T_{n+1}\subset T_n$ $(n\in\mathbf{N})$.

Since N is ultrabornological, there exists a sequence $(U_n)_{n\in\mathbf{N}}$ of balanced zero-nbhds. in X such that $N\cap(U_n+U_n)\subset T_n$ and $U_{n+1}+U_{n+1}\subset U_n$ for all $n\in\mathbf{N}$.

As f satisfies (*), the sets $f(T_n\cap U_n)$ are bornivorous in Y $(n\in\mathbf{N})$. Y being ultrabornological, we obtain that in particular $f(T_2\cap U_2)\in\mathfrak{U}_0(Y)$, whence $V:=U_2\cap((T_2\cap U_2)+N)\in\mathfrak{U}_0(X)$. Because of $V\subset U_2\cap(T_2\cap U_2+N\cap(U_2+U_2))\subset T_2+T_2\subset T_1$, we have that T_1 belongs to $\mathfrak{U}_0(X)$, which finishes the proof.

Similarly as in 2.4/2.5 a «locally convex simplification» of the above proof yields.

2.13 *Proposition.* Let X, Y be locally convex spaces and let $f: X\to Y$ be a continuous linear map satisfying condition (*) in 2.2.
If $N:=\ker f$ and Y are bornological, then also X is bornological.

In particular, by 2.3, a locally convex space X is bornological if it contains a pseudometrizable linear subspace L such that X/L is bornological.

2.14 *Remark.* Proposition 2.13 has the following analogue for the property «ultrabornologique» introduced in Bourbaki [3], Ch. III, § 3, exercice 11.

Let X, Y be locally convex spaces and let $f: X\to Y$ be a continuous linear map satisfying the following condition

(+) For every sequence $(y_n)_{n\in\mathbf{N}}$ in Y which is «très convergente» in the sense of De Wilde [5], p. 55, there exists a bounded Banach disk[6] B in X such that $\{y_n: n\in\mathbf{N}\}\subset f(B)$.
(Cf. the «Théorème de Relèvement» in [5], p. 61).

Then X is ultrabornologique if $N:=\ker f$ and Y are ultrabornologique.

We omit the proof which is very much like that of 2.13.

The following example shows that in 2.12, 2.13, and 2.14 the hypotheses about the conditions (*) and (+), respectively, cannot be dropped.

6) i.e, $B=\Gamma B$, and $[B]$ provided with the Minkowski functional p_B is a Banach space.

2.15 *Example.* Let $X_1 := \{(x_r)_{r\in\mathbf{R}} \in \mathbf{K}^{\mathbf{R}}$: there is $\alpha \in \mathbf{K}$ such that the set $\{r \in \mathbf{R}: x_r \neq \alpha\}$ is countable$\} \subset \mathbf{K}^{\mathbf{R}}$ be provided with the relative topology induced by the product topology of $\mathbf{K}^{\mathbf{R}}$. X_1 is not bornological, as $H_1 := \{(x_r)_{r\in\mathbf{R}} \in \mathbf{K}^{\mathbf{R}}: \{r \in \mathbf{R}: x_r \neq 0\}$ is countable$\}$ is a dense hyperplane in X_1 which is sequentially complete. Thus also $X_1/[a_1]$ is not bornological with $a_1 := (1)_{r\in\mathbf{R}} \in X_1\setminus H_1$. — On the other hand, H_1 is ultrabornological in the sense of Iyahen [16], p. 298, according to [9], 2. Prop., hence also ultrabornologique in the sense of Bourbaki [3], Ch. III, § 3, exercice 11, since H_1 is locally convex and sequentially complete (cf. loc. cit. b)).

Let X_2 be the Hilbert space $\mathbf{l}_2$, and fix a dense hyperplane H_2 in X_2 and $a_2 \in X_2\setminus H_2$.

We apply Lemma 1.4 to the triples (X_i, H_i, a_i) $(i = 1, 2)$ and obtain a Hausdorff locally convex space X containing a closed linear subspace L with the following properties:

(a) L is topologically isomorphic to H_1, hence ultrabornological and ultrabornologique;
(b) X/L is a Hilbert space, hence also ultrabornological and ultrabornologique;
(c) X is not bornological, as it has (via p_1) a quotient space which is topologically isomorphic to $X_1/[a_1]$, hence not bornological.

3. Locally bounded spaces and DF-spaces

We recollect that a topological vector space is called locally bounded if it contains a bounded zero-nbhd.

3.1 *Proposition.* Let X, Y be topological vector spaces and let $f: X \to Y$ be an open linear map such that $N := \ker f$ is locally bounded.

(a) For every bounded set $B \subset Y$ there is a bounded set $C \subset X$ such that $B = f(C)$. (Cf. De Wilde [6], Cor. 2.)
(b) If, in addition, f is almost continuous, then for every bounded zero-nbhd. B in Y there is a bounded zero-nbhd. V in X such that $B \subset f(V)$.

Proof. Let $U \in \mathfrak{U}_0(X)$ be balanced and open such that $N \cap (U + U)$ is bounded. Let $B \subset Y$ be bounded. Since f is open, there is $n \in \mathbf{N}$

such that $B \subset n f(U)$. The set $C := n U \cap \overset{-1}{f}(B)$ clearly satisfies $f(C) = B$.

We first show that C is bounded. Given $W \in \mathfrak{U}_0(X)$, W balanced, there is $m \in \mathbf{N}$ such that $N \cap (U + U) \subset m W$ and there is $k \in \mathbf{N}$, $k \geq n$, such that $B \subset k f(W \cap U)$. From $C \subset n U \cap (k(W \cap U) + N)$ we get $C \subset n U \cap (k(W \cap U) + N \cap (n U + kU)) \subset k W + N \cap \cap (k(U + U)) \subset k W + km W \subset km(W + W)$. This proves that C is bounded.

If f is almost continuous and if $B \in \mathfrak{U}_0(Y)$, then $V := \overline{n U \cap \overset{-1}{f}(B)}$ belongs to $\mathfrak{U}_0(X)$ and $f(V) \supset B$; moreover, V is bounded in X since $V \subset \overline{n U \cap \overset{-1}{f}(B)} = \bar{C}$ by the openness of U.

By Köthe [19], p. 433, 5., there exists a Fréchet Montel space admitting the Banach space l_1 as a quotient space. This example shows that 3.1(a) fails if N is assumed to be metrizable instead of locally bounded. Cf. however 2.3.

From 3.1(b) we immediately obtain

3.2 *Theorem.* A locally convex space (resp. a topological vector space) X is seminormable (resp. locally bounded) if it contains a linear subspace L such that L and X/L are seminormable (resp. locally bounded).

3.3 *Proposition.* Let X, Y be locally convex spaces and let $f: X \to Y$ be a linear open and almost continuous map such that $N := \ker f$ is seminormable.

(a) If, in addition, f is continuous and if X/L has a fundamental sequence of bounded sets[7], then also X has a fundamental sequence of bounded sets.

(b) If Y is a DF-space, then also X is a DF-space.

[Recall that a locally convex space is a DF-space if and only if it is countably quasibarrelled and has a fundamental sequence of bounded sets.]

PROOF. (a) Let $(B_n)_{n \in \mathbf{N}}$ be a fundamental sequence of bounded sets in Y, and choose an absolutely convex zero-nbhd. U in X such that $N \cap U$ is bounded. It follows from the proof of 3.1 that $C_n := n U \cap$

7) In the sense of Köthe [19], p. 392.

$\cap \overset{-1}{f}(B_n)$ is bounded in X for every $n \in \mathbf{N}$. On the other hand, let $C \subset X$ be bounded. Using the continuity of f we find $m, n \in \mathbf{N}$ such that $f(C) \subset B_m$ and $C \subset n\,U$. Thus $C \subset C_{\max\{m,n\}}$.

We have proved that $(C_n)_{n\in\mathbf{N}}$ is a fundamental sequence of bounded sets in X.

(b) Let $\mathfrak{T}$ denote the given topology of Y and let $\mathfrak{S}$ denote the final topology on Y w.r.t. to $f: X \to Y$. As f is linear and surjective, $(Y, \mathfrak{S})$ is a locally convex space and the map $f: X \to (Y, \mathfrak{S})$ is continuous and open. Moreover, one verifies easily that $\mathfrak{T} \supset \mathfrak{S}$ and that the $\mathfrak{S}$-closure of every zero-nbhd. in $(Y, \mathfrak{T})$ belongs to $\mathfrak{U}_0(Y, \mathfrak{S})$. Let $(B_n)_{n\in\mathbf{N}}$ be a fundamental sequence of bounded sets in $(Y, \mathfrak{T})$, and let A_n denote the closed absolutely convex hull of B_n in $(Y, \mathfrak{S})$ $(n \in \mathbf{N})$. By $\mathfrak{Z}$ we denote the finest locally convex topology on Y such that $\mathfrak{Z}|A_n = \mathfrak{S}|A_n$ for all $n \in \mathbf{N}$. From Grothendieck [13], p. 69, Cor. 1, we get that $\mathfrak{T} \supset \mathfrak{Z} \supset \mathfrak{S}$. We first show that $\mathfrak{Z} = \mathfrak{S}$. In fact, let $U \in \mathfrak{U}_0(Y, \mathfrak{Z})$. By [24], p. 58, Lemma 1, we may assume that U is closed in $(Y, \mathfrak{S})$. Since in particular, $U \in \mathfrak{U}_0(Y, \mathfrak{T})$, we obtain that the $\mathfrak{S}$-closure of U and hence U belong to $\mathfrak{U}_0(Y, \mathfrak{S})$. — Now [24], p. 64, Thm. 4, implies that $(A_n)_{n\in\mathbf{N}}$ is a fundamental sequence of bounded sets in $(Y, \mathfrak{Z}) = (Y, \mathfrak{S})$.

Statement (a) now implies that X has a fundamental sequence of bounded sets; and from 2.3, 2.7(c) we obtain that X is countably quasibarrelled.

REMARK. Let $X := \{(x_n)_{n\in\mathbf{N}} \in \mathbf{K}^{\mathbf{N}}: \{n \in \mathbf{N}: x_n \neq 0\}$ is finite$\}$, let $\mathfrak{S}$ be its relative topology induced by the product topology of $\mathbf{K}^{\mathbf{N}}$, and let $\mathfrak{T}$ denote the weak topology $\sigma(X, X^*)$, where X^* denotes the algebraic dual of X. Then id: $(X, \mathfrak{S}) \to (X, \mathfrak{T})$ is open and almost continuous, as can easily be verified; moreover, $(X, \mathfrak{T})$ has a fundamental sequence of bounded sets, and $(X, \mathfrak{S})$ has not. — This example shows that the continuity-assumption in 3.3(a) is essential.

The following two examples show that 3.3,(a) resp. (b), fail if we only assume that L has a fundamental sequence of bounded sets resp. that L is a DF-space instead of assuming that L is seminormable.

3.4 *Example.* Let L be a linear space of countably infinite dimension. Let L^* denote its algebraic dual, and let X denote the algebraic dual of L^*. Then X provided with the weak topology $\sigma := \sigma(X, L^*)$ is topologically isomorphic to the product space $\mathbf{K}^{\dim L^*} \cong \mathbf{K}^{\mathbf{R}}$ and contains L as a dense linear subspace. The relative topology $\sigma|L$ equals $\sigma(L, L^*)$, whence every bounded set in $(L, \sigma|L)$ has finite dimensional linear span. Since $\dim L$ is countable, $(L, \sigma|L)$ has a fundamental sequence of bounded sets. — Fix an arbitrary norm topology $\mathfrak{S}$ on X/L (we can even choose $\mathfrak{S}$ such that $(X/L, \mathfrak{S})$ is a Banach space as $\dim X/L = \operatorname{card}(2^{\mathbf{R}})$). Let $\mathfrak{Z}$ denote the initial topology on X w.r. to id: $X \to (X, \sigma)$ and the quotient map $q: X \to (X/L, \mathfrak{S})$. By 2.9, $(X, \mathfrak{Z})$ is a Hausdorff locally convex space satisfying $\mathfrak{Z}|L = \sigma|L$ and $\mathfrak{Z}/L = \mathfrak{S}$. From this we get

(a) $(L, \mathfrak{Z}|L)$ has a fundamental sequence of bounded sets;
(b) $(X/L, \mathfrak{Z}/L)$ is a normed space;
(c) $(X, \mathfrak{Z})$ does not have a fundamental sequence of bounded sets. In fact, because of $\mathfrak{Z} \supset \sigma$ it suffices to show that $\mathbf{K}^{\mathbf{R}}$ is not a union of a sequence of bounded sets. But this is clearly true, as $\mathbf{K}^{\mathbf{R}}$ is a nonnormable Baire space.

3.5 *Example.* Let Y be an infinite dimensional Fréchet Montel space admitting continuous norms, and let $X := Y'$ be its topological dual. Then X provided with the Mackey topology $\tau := \tau(X, Y)$ is a nonnormable complete bornological DF-space containing a total bounded set. Let $(B_n)_{n \in \mathbf{N}}$ be a fundamental sequence of bounded sets in (X, τ) such that $[B_1]$ is dense in (X, τ). Since $[B_n] \neq X$ for all $n \in \mathbf{N}$, we may assume that $[B_n] \subsetneq [B_{n+1}]$ $(n \in \mathbf{N})$. Then we can easily construct a dense linear subspace L of countably infinite codimension in (X, τ) such that $\dim(L + [B_n])/L$ is finite for every $n \in \mathbf{N}$.

By Valdivia [27], Thm. 3, and [28], Cor. 1.3, the subspace $(L, \tau|L)$ is a barrelled and bornological DF-space.

Fix an arbitrary norm topology $\mathfrak{S}$ on X/L and provide X with the initial topology $\mathfrak{Z}$ w.r. to id: $X \to (X, \tau)$ and the quotient map $q: X \to (X/L, \mathfrak{S})$. By 2.9, the Hausdorff locally convex space $(X, \mathfrak{Z})$ satisfies $\mathfrak{Z}|L = \tau|L$ and $\mathfrak{Z}/L = \mathfrak{S}$, and we obtain

(a) $(L, \mathfrak{Z}|L)$ is a barrelled and bornological DF-space;
(b) $(X/L, \mathfrak{Z}/L)$ is a normed space;
(c) $(X, \mathfrak{Z})$ is not a DF-space. In fact, assume that $(X, \mathfrak{Z})$ is a DF-space and let B be a bounded zero-nbhd. in $(X/L, \mathfrak{Z}/L)$. Then

by Köthe [19], p. 401, line 36-38, there is $n \in \mathbf{N}$ such that $B \subset \subset \overline{q(B_n)}$, whence $[q(B_n)]$ is dense in $(X/L, \mathfrak{Z}/L)$. Thus $[B_n] + L$ is dense in $(X, \mathfrak{Z})$. But this is a contradiction since, by (b), L is closed in $(X, \mathfrak{Z})$ and $\dim(L + [B_n])/L$ is finite.

The following example, which is of similar type, shows that a locally convex space X need not be quasinormable[8] if it contains a closed linear subspace L such that L and X/L are quasinormable. (Cf. however Proposition 4.5.)

3.6 *Example.* Let $(X, \mathfrak{T})$ be a Hausdorff locally convex space with the following properties: $\mathfrak{T}$ is a weak topology, every bounded set in $(X, \mathfrak{T})$ has finite dimensional linear span, and $(X, \mathfrak{T})$ contains a dense linear subspace L of infinite codimension. (For example, the incomplete Montel space constructed by *I.* Amemiya, *Y.* Komura [1], § 2, satisfies all these conditions). Fix a norm topology $\mathfrak{S}$ on X/L and provide X with the initial topology $\mathfrak{Z}$ w.r. to id: $X \to (X, \mathfrak{T})$ and the quotient map $q : X \to (X/L, \mathfrak{S})$. Then, by 2.9, the Hausdorff locally convex space $(X, \mathfrak{Z})$ has the following properties:

(a) $\mathfrak{Z}|L = \mathfrak{T}|L$ is a weak topology, whence in particular $(L, \mathfrak{Z}|L)$ is a Schwartz space and therefore quasinormable by Grothendieck [13], p. 116, Prop. 17;

(b) $(X/L, \mathfrak{Z}/L)$ is a normed space, hence quasinormable;

(c) $(X, \mathfrak{Z})$ is not quasinormable. In fact, since every bounded set in $(X, \mathfrak{Z})$ is finite dimensional, $(X, \mathfrak{Z})$ is quasinormable if and only if $(X, \mathfrak{Z})$ is a Schwartz space by [13], p. 116, Prop. 17. On the other hand, the quotient space $(X/L, \mathfrak{Z}/L)$ is infinite dimensional and normed, hence not a Schwartz space. Now [13], p. 118, Prop. 18, implies that $(X, \mathfrak{Z})$ is not quasinormable.

3.7 *Proposition.* Let X, Y be locally convex spaces, let $f: X \to Y$ be a linear open and almost continuous map, and put $N := \ker f$. Let N and Y be Schwartz spaces. Then also X is a Schwartz space. In particular, a locally convex space X is a Schwartz space if it contains a linear subspace L such that L and X/L are Schwartz spaces.

Proof. Let $q : X \to X/N$ denote the quotient map. As in the proof of 2.11 there is an open and almost continuous isomorphism $j : X/N \to Y$ such that $f = j \circ q$. We first show that X/N is a Schwartz space.

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8) In the sense of Grothendieck [13], p. 106, Déf. 4.

Let $U := \Gamma U \in \mathfrak{U}_0(X/N)$. Then $j(U) \in \mathfrak{U}_0(Y)$. As Y is a Schwartz space, there is $V \in \mathfrak{U}_0(Y)$ such that V is a precompact subset of the seminormed space $(Y, p_{j(U)})$ (where $p_{j(U)}$ denotes the Minkowski functional of $j(U)$). Let W denote the closure of V in $(Y, p_{j(U)})$. Clearly W is precompact in $(Y, p_{j(U)})$, whence also $j^{-1}(W)$ is precompact (and closed) in $(X/N, p_U)$. Moreover, $j^{-1}(W)$ is in particular closed in X/N w.r. to its original topology, and $j^{-1}(W) \supset j^{-1}(V)$. Thus, by the almost continuity of $j, j^{-1}(W)$ belongs to $\mathfrak{U}_0(X/N)$. We have proved that X/N is a Schwartz space.

Let $\mathfrak{T}$ denote the given topology of X and let $\mathfrak{T}_0$ denote the Schwartz space topology on X associated with $\mathfrak{T}$. Then $\mathfrak{T}_0 \subset \mathfrak{T}$. It follows from Swart [26], p. 268, 3.4, and p. 269, 3.7, that $\mathfrak{T}_0|N = \mathfrak{T}|N$ and that $\mathfrak{T}_0/N = \mathfrak{T}/N$, since $(N, \mathfrak{T}|N)$ and $(X/N, \mathfrak{T}/N)$ are Schwartz spaces. Consequently $\mathfrak{T}_0 = \mathfrak{T}$ (see the end of the proof of 2.11), whence X is a Schwartz space.

3.8 *Proposition.* Let X be a locally convex space and $L \subset X$ a linear subspace such that L and X/L are nuclear. Then also X is nuclear.

Proof. Given a seminormed space (Y, p), we call a sequence $(y_n)_{n\in\mathbf{N}}$ in (Y, p) (absolutely) summable if $(y_n + \overline{\{0\}})_{n\in\mathbf{N}}$ is (absolutely) summable[9] in the associated normed space $(Y, p)/\overline{\{0\}}$.

Let $U = \Gamma U \in \mathfrak{U}_0(X)$. As L is nuclear, we find $V = \Gamma V \in \mathfrak{U}_0(X)$, $V \subset U$, such that every summable sequence in $(L, p_{L\cap V})$ is absolutely summable in $(L, p_{L\cap U})$. As X/L is nuclear, we find $W = \Gamma W \in \mathfrak{U}_0(X)$, $W \subset V$, such that every summable sequence in $(X/L, p_{q(W)})$ is absolutely summable in $(X/L, p_{q(V)})$, where $q : X \to X/L$ denotes the quotient map.

Let $(x_n)_{n\in\mathbf{N}}$ be summable in (X, p_W). Then $(q(x_n))_{n\in\mathbf{N}}$ is summable in $(X/L, p_{q(W)})$, hence absolutely summable in $(X/L, p_{q(V)})$.

As $p_{q(V)}(z) = \inf\{p_V(x): x \in q^{-1}(z)\}$ for all $z \in X/L$, we find a sequence $(y_n)_{n\in\mathbf{N}}$ in X such that $\sum_{n\in\mathbf{N}} p_V(y_n) < \infty$ and such that $q(y_n) = q(x_n)$ for every $n \in \mathbf{N}$. $(y_n)_{n\in\mathbf{N}}$ is absolutely summable, hence summable in (X, p_V). Because of $W \subset V$, the sequence $(x_n)_{n\in\mathbf{N}}$ is also summable in (X, p_V). Consequently, $(x_n - y_n)_{n\in\mathbf{N}}$ is summable in $(L, p_{L\cap V})$, which implies that $(x_n - y_n)_{n\in\mathbf{N}}$ is absolutely summable

9) In the sense of Pietsch [21], p. 25 and p. 27.

in $(L, p_{L \cap U})$. Since $p_U|L = p_{L \cap U}$, we obtain that $\sum_{n \in \mathbf{N}} p_U(x_n - y_n) < \infty$. Because of $V \subset U$ we have $\sum_{n \in \mathbf{N}} p_U(y_n) < \infty$, whence $\sum_{n \in \mathbf{N}} p_U(x_n) < \infty$. Thus $(x_n)_{n \in \mathbf{N}}$ is absolutely summable in (X, p_U), which proves the nuclearity of X.

We do not know if 3.8 can be extended to the more general case of an open and almost continuous linear map $f: X \to Y$ (instead of a quotient map $q: X \to X/L$).

4. Some three-space-problems for Fréchet spaces

Example 1.5 in particular showed that a locally convex space X need not be semireflexive if it contains a linear subspace L such that L and X/L are semireflexive. Our first aim in this section is to contrast this example with a positive statement. Before we do so we will prove a slight generalization of a result of Grothendieck [13], p. 75, Prop. 4.

For a dual pair $\langle X, Y \rangle$ let $\sigma(X, Y)$, $\beta(X, Y)$, $\tau(X, Y)$ denote the weak, the strong, and the Mackey topology on X, respectively. For a locally convex space X we denote by X' its topological dual and define the polar A^0 of a subset $A \subset X$ by $A^0 := \{f \in X' : |f(x)| \leq 1$ for all $x \in A\}$.

4.1 *Lemma.* Let X be a locally convex space and $L \subset X$ a countably quasibarrelled linear subspace, whose strong dual is bornological. Then on the quotient space X'/L^0 the strong topology $\beta(X'/L^0, L)$ coincides with the quotient topology $\beta(X', X)/L^0$.

Proof. By Köthe [19], p. 278, line 27, one has $\beta(X'/L^0, L) \subset \subset \beta(X', X)/L^0$. We may identify L' and X'/L^0 ([19], p. 275, (1) (a)). As $(X'/L^0, \beta(X'/L^0, L))$ is bornological by hypothesis, it suffices to show that every $\beta(X'/L^0, L)$-bounded sequence in X'/L^0 is $\beta(X'/X)/L^0$-bounded. As L is countably quasibarrelled, every $\beta(X'/L^0, L)$-bounded sequence $(f_n)_{n \in \mathbf{N}}$ is equicontinuous; by [19], p. 275, (1) (b), there is an equicontinuous subset A in X' such that $\{f_n : n \in \mathbf{N}\} \subset q(A)$, where $q: X' \to X'/L^0$ denotes the quotient map. This finishes the proof since $q(A)$ is clearly bounded with respect to $\beta(X', X)/L^0$.

4.2 *Proposition.* Let X be a locally convex space and $L \subset X$ a linear subspace such that $\beta(X'/L^0, L) = \beta(X', X)/L^0$. Assume that L and X/L are semireflexive. Then also X is semireflexive.

PROOF. We first show that $\bar{L} \subset L + \overline{\{0\}}$ (where the bar denotes the closure in X). Let $x \in \bar{L}$. Because of

$$\sigma(X'/L^0, L) \subset \beta(X'/L^0, L) \subset \beta(X', X)/L^0 = \beta(X'/L^0, L)$$

there is a bounded set $B = \Gamma B$ in L such that $x \in \bar{B}$. Since $\bar{B} \cap L$ is $\sigma(L, L')$-compact hence $\sigma(X, X')$-compact, we obtain that $\bar{B} \subset \bar{B} \cap L + \overline{\{0\}}$. Thus $x \in L + \overline{\{0\}}$. – Hence we may assume that L is closed in X and that X is Hausdorff.

Let F be a continuous linear form on $(X', \beta(X', X))$. As $\beta(L^0, X/L)$ is finer than the relative topology $\beta(X', X)|L^0$ (Köthe [19], p. 277, line 23-25), the restriction $F|L^0$ is a continuous linear form on $(L^0, \beta(L^0, X/L))$. By [19], p. 276, (2) (a), we may identify $(X/L)'$ and L^0. As X/L is semireflexive, we therefore find $x \in X$ such that $F(f) = f(x)$ for all $f \in L^0$.

$G : (X', \beta(X', X)) \to \mathbf{K}$, $G(f) := F(f) - f(x)$, is a continuous linear form which vanishes on L^0. Thus G induces a continuous linear form $\tilde{G}$ on the quotient space $(X'/L^0, \beta(X', X)/L^0)$. By hypothesis we have that $\beta(X', X)/L^0 = \beta(X'/L^0, L)$; and by [19], p. 275, (1) (a), we may identify L' and X'/L^0. As L is semireflexive we thus find $y \in L$ such that $G(f) = \tilde{G}(f + L^0) = f(y)$ for all $f \in X'$.

It follows that $F(f) = f(x + y)$ for all $f \in X'$. This proves the semireflexivity of X.

Remarks.

(a) The proof of 4.2 has been inspired by the proof given by Kreĭn and Šmulian of their Thm. 14 in [20], p. 575.

(b) DF-spaces L obviously satisfy the hypotheses of 4.1. Further, a reflexive Fréchet space L is countably quasibarrelled and its strong dual $(L', \beta(L', L))$ is bornological by Köthe [19], p. 400, (4). Thus 4.1 and 4.2 immediately yield

4.3 *Proposition.* Let X be a locally convex space and $L \subset X$ a linear subspace which is a DF-space or a Fréchet space. If L and X/L are semireflexive, then also X is semireflexive.

4.4 *Proposition.* Let X be a Fréchet space and $L \subset X$ a linear subspace such that L and X/L are Montel spaces. Then also X is a Montel space.

PROOF. Let $B \subset X$ be bounded. As X/L is a Montel space, the set $q(B)$ is compact in X/L (where $q : X \to X/L$ denotes the quotient map). By Köthe [19], p. 279, (7), there is a compact set C in X satisfying $q(C) = \overline{q(B)}$. From $B \subset C + L$ we get $B \subset C + L \cap \overline{(B - C)}$. As L is a Montel space and $L \cap (B - C)$ is bounded in L, the set $L \cap \overline{(B - C)}$ is compact. Thus B is contained in a compact subset of X. We have proved that X is a Semimontel space and hence a Montel space.

The following proposition contrasts Example 3.6.

4.5 *Proposition.* Let X be a Fréchet space and $L \subset X$ a linear subspace such that L and X/L are quasinormable. Then also X is quasinormable.

PROOF. Let Y be a product of a sequence of Banach spaces containing X as a (closed) linear subspace. Then Y is a quasinormable Fréchet space. On account of Cholodovskiĭ [4], p. 159, Thm.[10)], it suffices to show that for every bounded set $B \subset Y/X$ there is a bounded set $A \subset Y$ such that $q_1(A) \supset B$, where $q_1 : Y \to Y/X$ denotes the quotient map.

Let $q_2 : Y \to Y/L$ and $q_3 : Y/L \to Y/X$ denote the canonical continuous and open surjections, and let $B \subset Y/X$ be bounded. Since $\ker q_3$ is topologically isomorphic to X/L, which is a quasinormable Fréchet space, we obtain from De Wilde [6], Prop. 2, that there is a bounded set $C \subset Y/L$ such that $q_3(C) \supset B$. As L is also a quasinormable Fréchet space, [6], Prop. 2, again provides us with a bounded set $A \subset Y$ such that $q_2(A) \supset C$. Now clearly, $q_1(A) = q_3(q_2(A)) \supset \supset B$.

10) Let Z be a quasinormable and metrizable locally convex space, $M \subset Z$ a linear subspace, and let $q : Z \to Z/M$ denote the quotient map. Assume that for every bounded set $B \subset Z/M$ there is a bounded set $A \subset Z$ such that $q(A) \supset B$. Then M is quasinormable.

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