CONTINUOUS DEPENDENCE FOR IMPLICIT DIFFERENTIAL EQUATIONS IN BANACH SPACES

by

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ABSTRACT.

In this paper we derive an existence theorem for the implicit defferential equation

$$F(t, x, x') = 0 \quad ; \quad x(t_0) = x_0$$

where F is a β-Lipschitz or α-Lipschitz operator in the second variable. The existence of maximal and unlimited solution is studied and a continuous dependence theorem is proved.

1.— INTRODUCTION AND NOTATION.

Some existence theorems for an implicit differential equation in a Banach space B were given by ABIAN-BROWN [1] and CONTI [4] when B has finite dimension. This problem has been studied in the infinite-dimensional case by PULVIRENTI [11] for Lipschitz-continuous and completely continuous operators, CARMONA [3] for Lipschitz-continuous operators and DOMÍNGUEZ [7] for α-Lipschitz and β-Lipschitz operators.

For the explicit differential equation in a Banach space some existence theorems have appeared in the latter years. (See DEIMLING [6] pag. 29 for references). However, some fundamental properties of the solutions, especially the continuous dependence problem, have not been studied more than lightly. A continuous dependence theorem for Lipschitz-continuous operators has been stated by

LASOTA-YORKE [9]. This result has been questioned by VIDOSSICH [12], who thinks that it is necessary to use additional hypotheses.

This paper concerns the study of fundamental properties of the solution of the implicit differential equation

$$F(t, x, x') = 0 \quad ; \quad x(t_0) = x_0 \tag{I}$$

Most of these results were known for the explicit differential equation

$$x' = f(t, x) \quad ; \quad x(t_0) = x_0$$

but Corollary 3 had not been stated before. We derive an existence theorem for (I) and study the behavior of the unlimited solutions and the continuous dependence of the solutions on the initial values and the operator F. The contniuous dependence theorem, in the particular case of the explicit differential equation (Corollary 3), shows that LASOTA-YORKE'S hypotheses suffice and that it is possible to weaken them.

In the following B will be a Banach space with norm $\|\cdot\|$. When S is a subset of B, $B(S, r)$ will denote the set $\{x \in B : d(S, x) < r\}$ $(d(S, x) = \inf\{\|y - x\| : y \in S\})$ and $\bar{B}(S, r)$ will denote the closure of $B(S, r)$. Recall the definitions of non-compactness measures [10], [13]. Let X be a metric space, Ω a bounded subset of X, the β- and α-measure of non-compactness of Ω is defined by $\beta(\Omega) = \inf\{\varepsilon > 0 : \Omega$ can be covered by finitely many balls with centres in X and diameter $\varepsilon\}$, $\alpha(\Omega) = \inf\{\varepsilon > : \Omega$ can be covered by finitely many sets of diameter less than $\varepsilon\}$. (In [13] a comparison is made between these measures). We shall write $\beta_X(\Omega)$ when it is necessary to specify that the centres are in X.

Let T be a cotinuous mapping from X into itself which maps bounded sets into bounded sets. We say that T is α-Lipschitz (resp.: β-Lipschitz) if there exists a real constant k such that

$$\alpha(T(\Omega)) \leq k\alpha(\Omega) \qquad (\text{resp.; } \beta(T(\Omega)) \leq k\beta(\Omega))$$

for every bounded set Ω in X. In the case $k < 1$ we say that T is an α-contraction (resp.: β-contraction).

Let R be the set of real numbers, Ω a subset of $R \times B \times B$ and $F : \Omega \to B$ a continuous mapping. We say that F is locally α-Lipschitz (resp.: locally β-Lipschitz) in the second variable if for every point (t_0, x_0, u_0) in Ω there is a neighborhood $V \times W \times U$

of this point such that $f(t, \cdot, u)$ is α-Lipschitz (resp.: β-Lipschitz) on W for every point (t, u) in $V \times U$.

2. — Local and global existence of solution.

The following existence theorem was proved in [7]. Here we develop a much simpler proof than the one that was made in [7]. This improvement can be obtained because the integrodifferential operator used in [7] (the same as in [11]) is replaced by another integrodifferential operator which was introduced in [3].

Theorem 1. Let a, b, c be positive numbers, (t_0, x_0, u_0) a point in $R \times B \times B$. Let A be the set

$$A = \{(t, x, u) \in R \times B \times B: t_0 - t_0| \leq a, ||x - x_0|| \leq b, ||u - u_0|| \leq c\}$$

and $F : A \to B$ a uniformly continuous mapping that satisfies

$$F(t_0, x_0, u_0) = 0 \tag{1}$$

and the following condition:

(L) F is β-Lipschitz (resp.: α-Lipschitz) in the second variable with modulus h; there exists a continuous linear injective mapping $\mu : B \to B$ and a real constant $L, 0 \leq L < 1$ (resp.: $0 \leq L < 1/2$) such that

$$||u + \mu F(t, x, u) - u_0|| \leq c \tag{2}$$

$$|| u - v + \mu[F(t, x, u) - F(t, x, v)] || < L ||u - v|| \tag{3}$$

for every (t, x, u) and (t, x, v) in A.

Then there exists a positive number δ, $\delta = \min(a, (1 - L)/4h, b/(||u_0|| + c))$ (resp.: $\delta = \min(a, (1 - 2L)/2h, b/(||u_0|| + c))$ such that a solution of (I) is defined on $(t_0 - \delta, t_0 + \delta)$.

Proof. We define the mapping $g : A \to B$ by $g(t, x, u) = u + \mu F(t, x, u)$. This mapping satisfies

(a) $g(t_0, x_0, u_0) = u_0$

(b) $||g(t, x, u) - u_0 || \leq c$ for every (t, x, u) in A.

(c) $\|g(t, x, u) - g(t, x, v)\| \leq L\|u - v\|$ for every (t, x, u) and (t, x, v) in A.

(d) g is β-Lipschitz (resp.: α-Lipschitz) in x with constant $k = h\|\mu\|$.

Let V be the set $\bar{B}(u_0, c)$, W the set $\bar{B}(x_0, b)$ and J the interval $(t_0 - \delta, t_0 + \delta)$. We denote $C(J, V)$ the set of all continuous mapping from J into V. We will assume that this set is equipped with the topology of the uniform convergence on J. Let $\mathrm{I} : C(J, V) \to C(J, W)$ be the mapping defined by

$$Ix(t) = x_0 + \int_{t_0}^{t} x(s)\, ds$$

For every positive number σ define (as in [11]) the real valued function $\omega(\sigma) = \sup\{\|g(t_1, Ix(t_1), x(t_1)) - g(t_2, Ix(t_2), x(t_1))\| : t_1, t_2 \in I, x(\cdot) \in C(J, V), |t_1 - t_2| < \sigma\}$. By using $\|Ix(t_1) - Ix(t_2)\| \leq \leq |t_1 - t_2|(c + \|u_0\|)$ and the uniform continuity of g we obtain

$$\lim_{\sigma \to 0^+} \omega(\sigma) = 0 \tag{4}$$

Let K denote the subset of $C(J, V)$ of all mappings satisfying

$$x(t_0) = x_0 \quad ; \quad \|x(t_1) - x(t_2)\| \leq \omega(|t_1 - t_2|)/(1 - L)$$

It is easy to prove (as in [11]) that K is a non-empty, closed and convex set. Furthermore, condition (4) implies that K is equicontinuous. We define the operator $T : K \to K$ by

$$Tx(t) = g(t, Ix(t), x(t))$$

Consider the α-measure case. Let H be a subset of K. Since H and $TH = \{Tx(\cdot) : x((\cdot) \in H\}$ are equicontinuous sets, theorem 2,3 in [2] is satisfied. Hence

$$\alpha(H) = \sup\{\alpha(H(t)) : t \in J\}$$
$$\alpha(TH) = \sup\{\alpha(TH(t)) : t \in J\}$$

By using $TH(t) \subset g(t, IH(t), H(t))$ and lemma 3.2 in [7] we can write

$$\alpha(TH(t)) \leq \alpha(g(t, IH(t), H(t)) \leq k\alpha(IH(t)) + 2L\,\alpha(H(t))$$

Since I is a Lipschitz operator with constant $(1 - 2L)/2k$ we obtain

$$\alpha(TH(t)) \leq ((1 - 2L)/2 + 2L)\,\alpha(H(t)) = \chi\alpha(H(t)) \qquad (0 \leq \chi < 1)$$

and taking supremum

$$\alpha(TH) \leq \chi\alpha(H)$$

Darbo's Theorem [5] implies the existence of a fixed point of T which is solution of (I).

Whenever the β-measure is concerned we consider the metric space X of all mappings from J into B that are continuous unless for finitely many points where they have bounded jumps. Let $Y \supset X$ be the set of all mappings $x(\cdot) \in X$ satisfying $x(J) \supset V$. It can be proved as in [7]

$$\beta_\gamma(H) = \sup_{t \in J} \beta_V(H(t))$$

where $H \subset C(J, V)$ is an equicontinuous set. Lemma 3.2 in [7] implies $\beta_V(g(t, IH(t), H(t)) \leq 2\, k\beta_B(IH(t)) + L\beta_V\,(H(t)) \leq (2k\delta + L)\,\beta_V(H(t)) \leq \chi'\,\beta_V(H(t))$ where $0 \leq \chi' < 1$.

Taking again supremum we prove that T is a β_γ-contraction. Hence T has a fixed point ([7] Th. 2.3) which is solution of (I).

Corollary 1. Let Ω be an open subset of $R \times B \times B$, F a locally uniformly continuous mapping in $C(\Omega, B)$ (i, e.: for every point (t_0, x_0, u_0) in Ω there is a neighborhood U of this point such that F is uniformly continuous on U. This condition is satisfied by a continuous function when dim $B < +\infty$). Assume

(L') F is locally β-Lipschitz (resp.: locally α-Lipschitz) in the second variable and for every point (t_0, x_0, u_0) satisfying $F(t_0, x_0, u_0) = 0$ there is a neighborhood V of this point, a real constant L, $0 \leq L < 1$ (resp.: $0 \leq L < 1/2$) and a continuous linear injective mapping μ such that (3) is satisfied in V.

Then for every point (t_0, x_0, u_0) in Ω satisfying $F(t_0, x_0, u_0) = 0$ there exists a maximal solution $\phi(t)$ of (I). Furthermore this solution is unlimited, i.e.: $(t, \phi(t), \phi'(t))$ has no limits in Ω as $t \to \alpha^+$ or $t \to \beta^-$.

Proof. Let (t_0, x_0, u_0) be a point in Ω such that $F(t_0, x_0, u_0) = 0$. Let U be a neighborhood of (t_0, x_0, u_0) such that F is β-Lipschitz (resp.: α-Lipschitz), uniformly continuous and satisfies (3) in U.

Take positive numbers a$'$, b', c such that $\overline{B}(t_0, a') \times \overline{B}(x_0, b') \times \times \overline{B}(u_0, c)$ is contained in U. We can choose two positive numbers a, b $(a \leq a', b \leq b')$ such that $\|u_0 + \mu F(t, x, u_0)\| \leq (1 - L)\, c$ when $| t - t_0 | \leq a$, $\| x - x_0\| \leq b$. Then $\| u + F(t, x, u)\| \leq c$ and we can use Theorem 1 to obtain a local solution.

The partial ring of the solutions of (I) and Zorn's Lemma imply the existence of a maximal solution. This solution must be unlimited, because the condition $\lim_{t \to \beta} (t, \phi(t), \phi'(t)) = (\beta, x_\beta, u_\beta)$ in Ω implies $(F(\beta, x_\beta, u_\beta) = 0$. Then we can define a solution on a neighborhood of β contradicting that ϕ is a maximal solution. These arguments apply equally well to $\lim_{t \to \alpha} (t, \phi(t), \phi'(t))$.

In order to obtain a definition interval $J = (t_0 - \delta, t_0 + \delta)$ independent of the β-Lipschitz modulus h we introduce the following lemma.

Lemma 1. Let Ω be an open subset of $R \times B \times B$, (t_0, x_0, u_0) a point in Ω, F a mapping in $C(\Omega, B)$, a, b, c positive numbers such that

$$A = \overline{B}(t_0, a) \times \overline{B}(x_0, b) \times \overline{B}(u_0, c)$$

is contained in Ω and F is uniformly continuous on A. Assume that there exists a mapping μ and a real constant L as in Theorem 1 such that (2) and (3) are satisfied on A. Define $\eta = \min\,(a, b/(\| u_0\| + c))$, $I = (t_0 - \eta, t_0 + \eta)$, $V = \overline{B}(x_0, b)$ and

$$\omega(\sigma) = \sup \{\| g(t_1, Ix(t_1)\, x(t_1)) - g(t_2, Ix(t_2), x(t_1))\| : x(\cdot) \in C(I, V), \\ |t_1 - t_2| \leq \sigma\}.$$

Let δ be a positive number $\delta \leq \eta$ such that $\omega(\delta) < (1 - L(c,$ and ϕ and unlimited solution of (I). Then ϕ is defined on $(t_0 - \delta, t_0 + \delta)$.

Proof. Consider the interval $[t_0, t_0 + \delta)$. Assume that $\|\phi'(t) - u_0\| = c$ for some t in this interval. Then

$$c = \|\phi'(t) - u_0\| \leq \|g(t, \phi(t), \phi'(t)) - g(t_0, x_0, \phi'(t))\| + \\ + \|g(t_0, x_0, \phi'(t)) - u_0\| < c$$

Hence $\|\phi'(t) - u_0\| < c$ in $[t_0, t_0 + \delta)$. Furthermore we have

$$||\phi'(t) - \phi'(s)|| \leq (1 - L)^{-1}\, \omega(|t - s|) \tag{5}$$

and

$$||\phi(t) - \phi(s)|| \leq (c + ||u_0||)\, |t - s| \tag{6}$$

for every t and s in $[t_0, t_0 + \delta)$.

Let $[t_0, t_0 + \delta)$ be the maximal definition interval of ϕ, and assume $\beta < t_0 + \delta$. Then (5) and (6) are Cauchy's conditions that assure the existence of $\lim_{t \to \beta-} \phi(t)$, $\lim_{t \to \beta-} \phi'(t)$. These limits must be in $\bar{B}(x_0, b) \times \bar{B}(u_0, c)$ contradicting that ϕ is unlimited.

The following result is a straigthforward application of Corollary 1 and Lemma 1.

Corollary 2. Let Ω and F be as in Corollary 1; a, b, c positive numbers such that $A = \bar{B}(t_0, a) \times \bar{B}(x_0, b) \times \bar{B}(u_0, c)$ is contained in Ω. Assume that F is uniformly continuous on A and conditions (1) and (L) in Theorem 1 are satisfied. Let δ be a positive number defined as in Lemma 1. Then every solution of (I) can be defined on $J = (t_0 - \delta, t_0 + \delta)$.

3. — Continuous dependence

We will prove that the solutions of (I) continuously depend on the initial values and the operator F.

Theorem 2. Let Ω be an open subset of $R \times B \times B$, F_0 a mapping in $C(\Omega, B)$ as F in Corollary 1. Let $\{F_n\}$ be a sequence in $C(\Omega, B)$ which converges to F_0 uniformly on bounded sets in Ω. Assume
(C) For every point $(\bar{t}_0, \bar{x}_0, \bar{u}_0)$ in Ω such that $F_0(\bar{t}_0, \bar{x}_0, \bar{u}_0) = 0$ there is a neighborhood U of $(\bar{t}_0, \bar{x}_0, \bar{u}_0)$, a sequence $\{L_{F_n}\}$ in R^+ and a sequence $\{\mu_{F_n}\}$ of linear continuous injective mappings such that (3) is satisfied on U for every $n = 0, 1, \ldots$ Furthermore $\{L_{F_n}\} \to L_{F_0}$ and $\{\mu_{F_n}\} \to \mu_{F_0}$.

Let (t_0, x_0, u_0) be a point in Ω, $\{(t_n, x_n, u_n)\}$ a sequence in Ω that converges to (t_0, x_0, u_0) in Ω. Assume

(i) $F_n(t_n, x_n, u_n) = 0$

(ii) The implicit differential equation

$$F_n(t, x, x') = 0 \quad ; \quad x(t_n) = x_n \tag{II}$$

has unlimited solution ϕ_n $n = 1, 2, \ldots$

(iii) The solution of (II) is unique when $n = 0$.

Then for every compact interval J contained in the maximal interval of definition of ϕ_0 there exists a positive integer n_0 such that ϕ_n is defined on J for every $n \geq n_0$ and $\{\phi_n(t)\} \to \phi_0(t)$ uniformly on J.

Proof. We shall assume that F_0 is a locally α-Lipschitz mapping. The argument for this proof applies equally well to the β-Lipschitz case. (1) Local case. Let U be a neighborhood of (t_0, x_0, u_0) such that F is uniformly continuous and condition (C) is satisfied on U. Choose a real number L', $0 \leq L_{F_0} < L' < 1$ and positive numbers a, b, c as in Corollary 1 (replace L by L_{F_0}, μ by μ_{F_0}) so that (L) is satisfied when we replace (2) by

$$\|u + \mu_{F_0} F_0(t, x, u) - u_0\| < c' \tag{2'}$$

on $A = \overline{B}(t_0, a) \times \overline{B}(x_0, a) \times \overline{B}(u_0, c)$. (Here c' is a positive constant, $L'c < c' < c$). In order to do that it suffices to take a, b small enough so that $\|\mu_{F_0} F_0(t, x, u)\|$ is smaller than $c' - L'c$.

Let g_n be the mapping $u_n + \mu_{F_n} F_n(t, x, u)$ $(n = 0, 1, \ldots)$. Since $\{g_n\} \to g_0$ uniformly on bounded sets we have

$$\|u_0 - g_n(t, x, u)\| < c' \tag{4}$$

on A when n is large enough. Let J be the interval $(t_0 - \delta, t_0 + \delta)$ where $\delta = 4^{-1} \min\,(a, b/(c + \|u_0\|), (1 - 2L)/2k)$. Assume that $\omega(2\delta) < (1 - L')c'$ (otherwise we can replace δ by $\delta' < \delta$). Denote $V = \overline{B}(u_0, c)$ and K' the set of all mappings in $C(J, V)$ satisfying

$$\|x(t_1) - x(t_2)\| \leq \tilde{\omega}(|t_1 - t_2|)/(1 - L')$$

where ω is an increasing function, ω strictly smaller than ω, such that $\lim_{\sigma \to 0^+} \tilde{\omega}(\sigma) = 0$. Let N be a positive integer such that $L_{F_n} < L'$, $\|x_n - x_0\| < b/2$, $|t_n - t_0| < \delta$ and (4) is satisfied for every $n \geq N$. Define the operator $T_n : K' \to C(J, V)$ by

$$T_n\, x(t) = g_n(t, I_n x(t), x(t)) \qquad n = 0, N, N + 1, \ldots$$

where

$$I_n x(t) = x_n + \int_{t_n}^{t} x(s)\, ds$$

For every $n \geq N$ we define the function

$$\omega_n(\sigma) = \sup \{\|g_n(t, I_n x(t_1), x(t_1)) - g_n(t_2, I_n x(t_2), x(t_1))\| : \\ |t_1 - t_2| < \sigma, x(\cdot) \in C(J, V)\}$$

It is easy to prove that $\{\omega_n\} \to \omega$ uniformly on $[0, \delta]$. Hence, for N large enough we have $T_n(K') \subset K'$, ϕ_n is defined on J and the restriction of ϕ_n to J is in K' as soon as $n \geq N$. Indeed, since g_n satisfies (4) we can obtain

$$\|u_n - g_n(t, x, u)\| < c' \tag{4'}$$

and $\omega_n(2\delta) < (1 - L')\, c'$ in $\bar{B}(t_n, a/2) \times \bar{B}(x_n, b/2) \times \bar{B}(u_n, c') \subset A$. Lemma 1 implies that ϕ_n is defined on $(t_n - 2\delta, t_n + 2\delta)$. Hence ϕ_n is defined on J. Furthermore as soon as n is large enough one has

$$\|\phi(t_1) - \phi(t_2)\| \leq \omega_n(|t_1 - t_2|) + L_{F_n} \|\phi_n(t_1) - \phi_n(t_2)\| \leq \\ \leq \omega(|t_1 - t_2|) + L' \|\phi_n(t_1) - \phi_n(t_2)\|$$

and this inequality implies that ϕ_n is in K'.

We have proved that the restriction of ϕ_n to J is a fixed point of the operator $T_n (n = 0, N, N+1, \ldots)$. It is easy to check that $\{T_n\} \to T_0$ uniformly and that K' is an equicontinuous set.

Replace K by K', T by T_0 in the proof of Theorem 1. Then T_0 becomes an α-contraction with modulus $\chi < 1$. Let H be the equicontinuous set $\{\phi'_0, \phi'_N, \phi'_{N+1}, \ldots\}$ and assume $\alpha(H) > 0$. Choose $\varepsilon > 0$, $\varepsilon < (1 - \chi)\, \alpha(H)$. There exists an integer $m \geq N$ such that $\|T\phi - T_m\phi\| < \varepsilon/2$ for every ϕ in K'. Then one has

$$\alpha(H) = \alpha(\{\phi'_0, \phi'_m, \phi'_{m+1}, \ldots\}) = \alpha(\{T_0\, \phi'_0, T_m \phi'_m, \ldots\}) \leq \\ \leq \alpha(B(TH), \varepsilon/2) \leq \alpha(TH) + \varepsilon < \alpha(H)$$

Hence $\alpha(H) = 0$ that means H is a relatively compact set. From the uniqueness of the solution of (II) for $n = 0$ we derive $\{\phi'_n\} \to \phi'_0$ and $\{\phi_n\} \to \phi_0$ uniformly on J.

2. Global case. We shall study the problem on the right of t_0. Let $[t_0, \tau]$ be a compact interval which is contained in the maximal interval of definition of ϕ_0. Consider the set.

$S = \{t \geq t_0$: there is an integer N such that for every $n \geq N$ ϕ_n and ϕ_n' are defined on $[t_0, t]$ and converge to ϕ_0 and ϕ_0' uniformly on $[t_0, t]\}$ From the local case we know that S is non-empty set. Denote $\sigma = \sup S$ and assume $\tau \geq \sigma$. Then $(\sigma, \phi(\sigma), \phi'(\sigma))$ is in Ω and we can take a «security ball» $V = \bar{B}(\sigma, a) \times \bar{B}(\phi(\sigma), b) \times \bar{B}(\phi'(\sigma), c)$ contained in Ω such that condition (C), uniform continuity of F_0 and condition (L) (replacing (2) by (2')) are satisfied on V (replace (t_0, x_0, u_0) by $(\sigma, \phi(\sigma), \phi'(\sigma))$). Choose δ as in local case and take $\sigma' < \sigma$ such that $\sigma - \sigma' < \delta/2$ and $\omega(\sigma - \sigma') < (1 - L)(c - c')/3$. Consider the point $(\sigma', \phi(\sigma'), \phi'(\sigma')) \subset V$ and the set $W = B(\sigma', a/2) \times \times B(\phi(\sigma'), b/2) \times B(\phi'(\sigma'), c''')$ where $c''' = (c' + 2c)/3$. Since W is contained in V conditions (C) and (L) are satisfied in W replacing (2) by

$$||\phi'(\sigma') - g(t, x, u)|| > c''$$

where $c'' = (2c' + c)/3 < c'''$.

From local case in the point $(\sigma', \phi(\sigma'), \phi'(\sigma'))$ we get a solution of (II) in $(\sigma' - \delta', \sigma' + \delta')$ where $\delta' = 4^{-1} \min(a/2, b/(2c + ||u_0||), (1 - 2L)/2k)$ and $\omega(2\delta') < (1 - L')c''$. Then $\delta' \geq \delta/2$ contradicting the fact $\sigma = \sup S$. Hence $\tau < \sigma$ and Theorem 2 is proved.

When the differential equation is explicit Theorem 2 can be stated much more easily. This statement is more general than Lemma 2 of [9] and Lemma 3 of [8].

Corollary 3. Let Ω be an open subset of $R \times B$, f_0 a mapping in $C(\Omega, B)$ that satisfies the following condition

(E) f_0 is locally uniformly continuous and locally β-Lipschitz (resp.: α-Lipschitz) in the second variable.

Let $\{f_n\}$ be a sequence in $C(\Omega, B)$ that converges to f_0 uniformly on bounded sets in Ω. Let (t_0, x_0) be a point in Ω, $\{(t_n, x_n)\}$ a sequence in Ω that converges to (t_0, x_0) in Ω. Assume

(i) The explicit differential equation

$$f_n(t, x) = x' \quad ; \quad x(t_n) = x_n \tag{III}$$

has unlimited solutions ϕ_n $(n = 1, 2, \ldots)$

(ii) The solution of (III) is unique if $n = 0$.

Then, the assertion of Theorem 2 holds.

Proof. By putting $F(t, x, u) = u - f(t, x)$, conditions (L') and (C) in Theorem 2 are satisfied.

Remark. Condition (C) is satisfied in the implicit case when we assume

(i) There exists the partial derivative $D_3 F_n$ $(n = 0, 1, ...)$ in Ω.

(ii) The sequence $\{D_3 F_n\}$ converges to $D_3 F_0$ uniformly on bounded sets in Ω.

(iii) $D_3 F_0(t, x, u)$ has inverse for every (t, x, u) in Ω.

Indeed, let (t_0, x_0, u_0) be a point in Ω. There is an integer N such that $D_3 F_n(t_0, x_0, u_0)$ has an inverse $(n = 0, N, N + 1, ...)$. Putting

$$\mu_n = -(D_3 F_n(t_0, x_0, u_0)^{-1} \quad ; \quad g_n = u_n + \mu_n F_n$$

we obtain $\{\mu_n\} \to \mu_0$ and $D_3\, g_0\,(t_0,\, x_0,\, u_0) = 0$. Hence $||D_3 g_0(t_0, x_0, u_0)|| < L < 1$ on a neighborhood V of this point. Since $\{D_3 g_n\} \to D_3 g_0$ uniformly on bounded sets we get $||D_3 g_n(t, x, u)|| < L$ on V as soon as n is large enough. Condition (3) is a straightforward application of this result.

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