A NOTE ON THE LIFTING OF LINEAR AND LOCALLY CONVEX TOPOLOGIES ON A QUOTIENT SPACE

by

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Abstract. Let $(X, \mathfrak{T})$ be a Hausdorff locally convex space, let $L \subset X$ be a closed linear subspace, and let $\mathfrak{S}$ be a Hausdorff locally convex topology on the quotient space X/L which is coarser than the quotient topology $\mathfrak{T}/L$. We prove that there exists a Hausdorff locally convex topology $\check{\mathfrak{T}}$ on X which is coarser than $\mathfrak{T}$ such that the corresponding quotient topology $\check{\mathfrak{T}}/L$ coincides with $\mathfrak{S}$. This proves a statement of G. Köthe [6].

The above conclusion may fail if $(X, \mathfrak{T})$ is a topological vector space which is not necessarily locally convex. Moreover, even if $(X, \mathfrak{T})$ is a Banach space, $\check{\mathfrak{T}}$ cannot be chosen such that, in addition, the relative topologies $\check{\mathfrak{T}}|L$ and $\mathfrak{T}|L$ induced on L coincide.

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Notations. For a topological vector space $(X, \mathfrak{T})$ and a linear subspace $L \subset X$, let $\mathfrak{T}|L$ denote the relative topology induced by $\mathfrak{T}$ on L and let $\mathfrak{T}/L$ denote the quotient topology on the quotient space X/L. By $\mathfrak{U}_0(X, \mathfrak{T})$ we denote the filter of all 0-neighbourhoods in $(X, \mathfrak{T})$ and by $(X, \mathfrak{T})'$ (*or* X') its topological dual. Given a dual pair $\langle X, Y\rangle$, let $\sigma(X, Y)$ and $\tau(X, Y)$ denote the weak topology and the Mackey topology on X, respectively. For a subset $A \subset X$, let $A^0 := \{y \in Y: |\langle x, y\rangle| \leqslant 1 \text{ for all } x \in A\}$.

Introduction. Let L be a linear subspace of a topological vector space $(X, \mathfrak{T})$.

(1) If $\mathfrak{Z}$ is a linear topology on L which is coarser than $\mathfrak{T}|L$ then the triple $((X, \mathfrak{T}), L, \mathfrak{Z})$ has the following «extension property», which is well known and can be verified immediately:

> The set $\{U + V : U \in \mathfrak{U}_0(X, \mathfrak{T}),\ V \in \mathfrak{U}_0(L, \mathfrak{Z})\}$ is a O-nbhd-basis for a linear topology $\hat{\mathfrak{T}}$ on X which satisfies $\hat{\mathfrak{T}} \subset \mathfrak{T}$, $\hat{\mathfrak{T}}/L = \mathfrak{T}/L$, and $\hat{\mathfrak{T}}|L = \mathfrak{Z}$ (in fact, $\hat{\mathfrak{T}}$ is the finest linear topology on X with these three properties). Thus, if $\mathfrak{Z}$ is Hausdorff and L is closed in $(X, \mathfrak{T})$, then also $\hat{\mathfrak{T}}$ is Hausdorff. Moreover, $\hat{\mathfrak{T}}$ is locally convex, whenever $\mathfrak{T}$ and $\mathfrak{Z}$ are locally convex.

On the other hand, if $\mathfrak{Z}$ is supposed to be finer than $\mathfrak{T}|L$, an analogous extension property does not hold, as the following example shows:

Let $(X, \mathfrak{T})$ by a Mackey space such that $(L, \mathfrak{T}|L)$ is not a Mackey space. Assume that there exists a locally convex topology $\hat{\mathfrak{T}}$ on X such that $\hat{\mathfrak{T}} \supset \mathfrak{T}$, $\hat{\mathfrak{T}}/L = \mathfrak{T}/L$, and such that $\hat{\mathfrak{T}}|L$ coincides with the Mackey topology $\tau(L, (L, \mathfrak{T}|L)') \supsetneq \mathfrak{T}|L$. Then the corresponding weak topologies $\sigma(X, (X, \hat{\mathfrak{T}})')$ and $\sigma(X, (X, \mathfrak{T})')$ satisfy the hypotheses of [2; Lemma 1] (i.e., they are comparable and induce the same topologies on L and on X/L) and hence coincide. Since $\mathfrak{T}$ is a Mackey topology, we obtain that $\hat{\mathfrak{T}} \subset \mathfrak{T}$, which is a contradiction.

(2) If $\mathfrak{S}$ is a linear topology on X/L which is finer than $\mathfrak{T}/L$, then the triple $((X, \mathfrak{T}), X/L, \mathfrak{S})$ has the following «lifting property», the proof of which is again routine and is therefore omitted:

> The set $\{U \cap q^{-1}(V) : U \in \mathfrak{U}_0(X, \mathfrak{T}),\ V \in \mathfrak{U}_0(X/L, \mathfrak{S})\}$ (where $q : X \to X/L$ denotes the quotient map) is a O-nbhd-basis for a linear topology $\check{\mathfrak{T}}$ on X which satifies $\check{\mathfrak{T}} \supset \mathfrak{T}$, $\check{\mathfrak{T}}|L = \mathfrak{T}|L$, and $\check{\mathfrak{T}}/L = \mathfrak{S}$ (in fact, $\check{\mathfrak{T}}$ is the coarsest topology on X with these three properties). — $\check{\mathfrak{T}}$ is locally convex, whenever $\mathfrak{T}$ and $\mathfrak{S}$ are locally convex.

Finally, suppose that $\mathfrak{S}$ is coarser than $\mathfrak{T}/L$. The aim of this note is to investigate the lifting properties of such a triple $((X, \mathfrak{T}), X/L, \mathfrak{S})$.

Main results. The following proposition has been stated by G. Köthe in [6; p. 190]; however, the proof contained a gap.

Proposition. Let $(X, \mathfrak{T})$ be a Hausdorff locally convex space over the field $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$, let $L \subset X$ be a closed linear subspace and let $\mathfrak{S}$ be a locally convex topology on X/L which is coarser than $\mathfrak{T}/L$. Then there exists a locally convex topology $\check{\mathfrak{T}}$ on X with the following properties: $\check{\mathfrak{T}} \subset \mathfrak{T}$, $\check{\mathfrak{T}}/L = \mathfrak{S}$, and $\check{\mathfrak{T}}|L = \sigma(L, L')$ (where $L' := (L, \mathfrak{T}|L)')$.

In particular, $\check{\mathfrak{T}}$ is Hausdorff if $\mathfrak{S}$ is Hausdorff.

Proof. Let $M \subset X'$ be a linear subspace such that $L^0 \cap M = \{0\}$ and $L^0 + M = X'$. Then the initial topology $\check{\mathfrak{T}}$ on X w.r. to the quotient map $q : X \to (X/L, \mathfrak{S})$ and all linear forms $f : X \to \mathbf{K}$ with $f \in M$, is locally convex and coarser than $\mathfrak{T}$. Moreover, $\check{\mathfrak{T}}|L = \sigma(L, M) = \sigma(L, L')$ by G. Köthe [7; p. 275, (1) a] and $q : (X, \check{\mathfrak{T}}) \to (X/L, \mathfrak{S})$ is continuous. Thus it remains to prove that $q : (X, \check{\mathfrak{T}}) \to (X/L, \mathfrak{S})$ is open. — Let $U \in \mathfrak{U}_0(X, \check{\mathfrak{T}})$. Then there are $V \in \mathfrak{U}_0(X/L, \mathfrak{S})$ and $f_1, \ldots, f_n \in M$ such that $U \supset q^{-1}(V) \cap \bigcap_{1 \leq i \leq n} \ker f_i$, whence $q(U) \supset V \cap q(\bigcap_{1 \leq i \leq n} \ker f_i)$.

We show that $N := L + \bigcap_{1 \leq i \leq n} \ker f_i$ is equal to X. In fact, let $f \in X'$ satisfy $N \subset \ker f$. Then $f \in L^0$. On the other hand, f belongs to the linear span of $\{f_1, \ldots, f_n\}$ by Kelley, Namioka [5; p. 7, 1.3]. Thus $f \in M \cap L^0 = \{0\}$. Since N is closed in $(X, \mathfrak{T})$, we obtain that $N = X$. Now $q(\bigcap_{1 \leq i \leq n} \ker f_i) = q(N) = X/L$, whence $q(U) \supset V$ and consequently $q(U) \in \mathfrak{U}_0(X/L, \mathfrak{S})$.

Remark. Since there are many choices of M in X', the topology $\check{\mathfrak{T}}$ in the above proposition is not unique. Moreover, L^0 may have two algebraically complementary spaces M_1, M_2 in X' such that $M_1 + M_2 = X'$. In this case, the supremum of the corresponding two topologies $\check{\mathfrak{T}}_1$ and $\check{\mathfrak{T}}_2$ is finer than $\sigma(X, X')$ and hence may fail to induce the topology $\mathfrak{S}$ on X/L. Consequently, if $(X, \mathfrak{T})$, L, $\mathfrak{S}$ are given as in the above proposition, there is i.g. no finest (linear or locally convex) topology $\check{\mathfrak{T}}$ on X satisfying $\check{\mathfrak{T}} \subset \mathfrak{T}$ and $\check{\mathfrak{T}}/L = \mathfrak{S}$.

This last statement had been observed by V. Eberhardt already in 1972.

Moreover, in the above proposition «$\check{\mathfrak{T}}|L = \sigma(L, L')$» cannot be replaced by «$\check{\mathfrak{T}}|L = \mathfrak{T}|L$», as the following example shows.

Example 1. Let $(X, \mathfrak{T})$ be a Hausdorff locally convex space containing a linear subspace L such that $(L, \mathfrak{T}|L)$ is a Banach space and such that L is not topologically complemented in $(X, \mathfrak{T})$. (Thus $(X, \mathfrak{T})$ may be any Banach space which is not a Hilbert space). Let $\mathfrak{S}$ be a weak topology on X/L which is coarser than $\mathfrak{T}/L$. Then there is no locally convex topology $\check{\mathfrak{T}}$ on X such that $\check{\mathfrak{T}} \subset \mathfrak{T}$, $\check{\mathfrak{T}}|L = \mathfrak{T}|L$, and $\check{\mathfrak{T}}/L = \mathfrak{S}$.

Proof. Let us assume that such a topology $\check{\mathfrak{T}}$ exists. Then there exists an absolutely convex O-neighbourhood U in $(X, \check{\mathfrak{T}})$ such that $U \cap L$ is a bounded O-neighbourhood in $(L, \check{\mathfrak{T}}|L) = (L, \mathfrak{T}|L)$. The seminormed topology $\mathfrak{T}_U$ generated by the Minkowski functional p_U on X clearly satisfies $\mathfrak{T}_U \subset \check{\mathfrak{T}} \subset \mathfrak{T}$ and $\mathfrak{T}_U|L = \mathfrak{T}|L$.

Since $\mathfrak{S}$ is a weak topology, there is an $\mathfrak{S}$-closed linear subspace $M \subset X/L$ of finite codimension such that $q(U) \supset M$ (where $q : X \to X/L$ denotes the quotient map). Let $N := q^{-1}(M)$. Since $q(\varepsilon\, U \cap N) = \varepsilon\, q(U \cap q^{-1}(M)) = \varepsilon(q(U) \cap M) = M$ for every $\varepsilon > 0$, we obtain that $(\mathfrak{T}_U|N)/L$ equals the coarsest topology on N/L. Consequently, L is dense in $(N, \mathfrak{T}_U|N)$. As $\mathfrak{T}_U|L = \mathfrak{T}|L$ is a Banach space topology, we obtain that $(N, \mathfrak{T}_U|N)$ is the topologically direct sum of L and the $(\mathfrak{T}_U|N)$-closure P of $\{0\}$. Hence, because of $\mathfrak{T}|N \supset \mathfrak{T}_U|N$ and $\mathfrak{T}|L = \mathfrak{T}_U|L$, also $(N, \mathfrak{T}|N) = L \oplus P$. Now finally, N being closed and of finite codimension in $(X, \mathfrak{T})$, we get that L is topologically complemented in $(X, \mathfrak{T})$, a contradiction.

A slight modification of the above proof shows that there is not even a locally pseudoconvex topology $\check{\mathfrak{T}}$ on X satisfying $\check{\mathfrak{T}} \subset \mathfrak{T}$, $\check{\mathfrak{T}}|L = \mathfrak{T}|L$, and $\check{\mathfrak{T}}/L = \mathfrak{S}$.

A Hausdorff topological vector space $(X, \mathfrak{T})$ is called minimal, if there is no Hausdorff linear topology on X which is strictly coarser than $\mathfrak{T}$. It is an open problem, whether all Hausdorff quotients of a minimal topological vector space are again minimal. If an analogue to the above proposition for linear (instead of locally convex) topologies $\mathfrak{T}$ and $\mathfrak{S}$ were valid, one would get a positive solution of this problem immediately. The following example will show that there is no hope to solve the problem in this way.

Let us first recall some well known facts about the so-called «associated locally convex topology» which we will use in Example 2. For a topological vector space $(Z, \mathfrak{Z})$ let $\mathfrak{Z}_{lc}$ denote the finest locally convex topology on Z which is coarser than $\mathfrak{Z}$. Then $(\mathfrak{Z}/Y)_{lc} = \mathfrak{Z}_{lc}/Y$ for every linear subspace $Y \subset Z$, $(\mathfrak{Z}|Y)_{lc} = \mathfrak{Z}_{lc}|Y$ for every *dense* linear subspace $Y \subset (Z, \mathfrak{Z})$, and «$\cdot_{lc}$» commutes with the formation of finite products.

Example 2. Let $\mathfrak{P}$ denote the product topology on $\omega := \mathbf{K}^{\mathbf{N}}$, where $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$. Since ω has the same dimension as the space of all measurable functions $f: [0, 1] \to \mathbf{K}$, there exists a metrizable linear topology $\mathfrak{Q}$ on ω satisfying $(\omega, \mathfrak{Q})' = \{0\}$. The linear infimum $\mathfrak{Q} \wedge \mathfrak{P}$ is a weak topology on ω without nontrivial continuous linear forms, hence equal to the coarsest topology $\mathfrak{G}$ on ω. Thus, by [1; Lemma 1(a)], the diagonal $\Delta := \{(x, x) \in \omega \times \omega : x \in \omega\}$ is a dense linear subspace of the product space $(\omega, \mathfrak{Q}) \times (\omega, \mathfrak{P})$.

Next we choose a non-zero element $a \in \omega$ and define the one-dimensional linear subspace $L := \mathbf{K}\, a \times \{0\} \subset \omega \times \omega$.

Then $X := L + \Delta$ provided with the relative topology $\mathfrak{T}$ induced by the product topology $\mathfrak{Q} \times \mathfrak{P}$ is a metrizable topological vector space, which is dense in $(\omega, \mathfrak{Q}) \times (\omega, \mathfrak{P})$. Hence $\mathfrak{T}_{lc} = (\mathfrak{Q} \times \mathfrak{P})_{lc}|X = (\mathfrak{G} \times \mathfrak{P})|X$. From this we obtain that $\mathfrak{T}_{lc}|L$ is the coarsest topology on L; moreover, again by [1; Lemma 1(*a*)], X/L provided with the locally convex topology $\mathfrak{S} := \mathfrak{T}_{lc}/L = (\mathfrak{T}/L)_{lc}$ is topologically isomorphic to $(\omega, \mathfrak{P})$.

We show that there is no Hausdorff linear topology $\check{\mathfrak{T}}$ on X satisfying $\check{\mathfrak{T}} \subset \mathfrak{T}$ and $\check{\mathfrak{T}}/L = \mathfrak{S}$.

In fact, it follows from Kalton, Peck [4; 3.4 and 3.5] that for any such topology $\check{\mathfrak{T}}$ the space $(X, \check{\mathfrak{T}})$ would be topologically isomorphic to $(\omega, \mathfrak{P})$ (see Drewnowski [3; p. 99, section 4.]). Hence $\check{\mathfrak{T}}$ would be locally convex and thus coarser than $\mathfrak{T}_{lc}$. But this is impossible since $\mathfrak{T}_{lc}$ is not Hausdorff.

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